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The Binosson Distribution: A Unified Probabilistic Framework Bridging the Binomial and Poisson Models

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Abstract

Classical Binomial and Poisson distributions, constrained by fixed trials and static event rates, falter in modeling modern datasets with dynamic parameters or contextual dependencies (e.g., variable infection rates, covariate-influenced risks). This paper introduces the Binosson Distribution, a hybrid framework unifying Binomial trials and Poisson processes through dynamic parameterization of trial counts (n) and designed to address event rates (λ). The distribution has been proposed to bridge the gap between these two distributions, incorporating aspects of both. Binomial-cum-Poisson distributions are modified to obtain a distribution that will be able to solve the probability problems that lies between the two distributions. Binosson is the result from the product of Binomial and Poisson distributions. Statistical properties such as mean, variance, standard deviation, skewedness and kurtosis were also derived.

Keywords: Binomial distribution, Poisson distribution, Probability, Mean and Standard deviation

Introduction

Classical probability distributions, such as the Binomial and Poisson models, have long underpinned the statistical modeling of discrete events in domains like biology, epidemiology, insurance, and operations research. These models offer elegant solutions under specific assumptions: the Binomial distribution assumes a fixed number of independent and identically distributed Bernoulli trials, while the Poisson distribution is ideal for modeling rare events occurring independently at a constant average rate. However, modern datasets increasingly challenge these idealized conditions. For example, customer purchase behavior may not follow fixed trials with constant probabilities due to seasonal influences, and disease outbreaks rarely maintain a constant infection rate due to intervention measures and evolving population behavior (AlmaBetter, 2023; ProjectManagers.net, 2025). These violations of foundational assumptions call into question the sufficiency of these classical models in dynamically evolving environments.

In response, researchers have explored hybrid and generalized models that can bridge the structural gaps between discrete-event frameworks. This paper introduces the Binosson Distribution, a novel hybrid probability distribution that synthesizes features of both the Binomial and Poisson distributions. Unlike traditional models that rely on rigid constraints, the Binosson framework is built with dynamic parameterization and contextual adaptability, allowing it to model complex real-world processes more effectively. It accommodates fluctuating trial counts and variable event rates within a unified probabilistic architecture. As noted by Hamilton et al. (2017), hybrid modeling in dynamical systems yields more robust results, particularly when modeling phenomena that transition between discrete and continuous or deterministic and stochastic behaviors—conditions under which Binosson operates proficiently.

The limitations of classical models are well-documented. The Binomial distribution struggles with real-world applications that involve over dispersion, correlated outcomes, or trial dependency, as its formulation assumes constant probability and independence (Towards Data Science, 2025). Meanwhile, the Poisson distribution is unable to handle scenarios where events are not evenly distributed in time or are influenced by external covariates. Its fixed interval assumption severely restricts applicability in fields like environmental science or finance, where events may cluster or shift due to external shocks (ProjectManagers.net, 2025). As von Kügelgen *et al.* (2024) argue, the notion of a "true" underlying distribution often fails to reflect the context-dependent nature of real data. This critique fuels the case for distributions like Binosson that flexibly integrate real-world variability and systemic complexity.

The Binosson Distribution aims to fulfill this critical modeling need through a hybrid structure that combines the count-based mechanism of Poisson processes with the bounded-trial structure of Binomial models. It introduces two key innovations: dynamic parameterization, allowing the rate of occurrence (λ) and trial count (n) to vary adaptively over time or conditions, and contextual dependency, whereby external covariates such as environmental factors or policy interventions can influence outcomes, relaxing the assumption of independence. As Vose (2010) highlights, multi-parameter distribution models are essential for capturing over-dispersed or heteroscedastic data in practical applications. This paper formalizes the Binosson framework, derives its statistical properties, and demonstrates its application through real-world case studies in actuarial risk and epidemic modeling. Ultimately, this work contributes a flexible, modern statistical tool capable of bridging foundational theory with emerging data realities.

Some Existing Useful Distribution

Binomial distribution

A **binomial experiment** is a series of n Bernoulli trials, whose outcomes are independent of each other. A random variable, X, is defined as the number of successes in a binomial experiment. A **binomial distribution** is the probability distribution of X (Adeosun, 2018). The probability mass function of a binomial distribution is given as:

$$f(x_1) = \binom{n}{x} P^x (1-p)^{\,n-x} \qquad x \;=\; 0, 1, 2, \dots, n$$

(1)

Properties of Binomial distribution

(a) The mean $E(X) = np$

(b) The variance $V(X) = np(1-p)$

(c) The Moment Generating Function $M_x(t) = (pe^t + (1-p))^n$

(d) The Probability Generating Function $G_x(t) = \big(pt + (1-p)\big)^n$

Poisson distribution

In probability theory and statistics, the Poisson distribution, named after French mathematician Siméon Denis Poisson, is a discrete probability distribution that expresses the probability of a given ...

The Poisson distribution is used to model the number of events occurring within a given time interval. The probability mass function of a Poisson distribution is given as:

$$f(x_2) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \text{for } x = 0, 1, 2, \dots \qquad (2)$$

λ is the shape parameter which indicates the average number of events in the given time interval.

Properties of Poisson distribution

(a) The mean $E(X) = \lambda$

(b) The variance $V(X) = \lambda$

(c) The Moment Generating Function $M_x(t) = e^{\lambda(e^t - 1)}$

(d) The Probability Generating Function $G_x(t) = e^{\lambda(t-1)}$

Binosson Distribution

The proposed distribution, Binosson distribution is the combination of binomial and Poisson distributions. Binomial-Cum-Poisson distributions are modified to obtain a distribution that will be able to solve the probability problems that lies between the two distributions. Binosson is the result from the product of binomial and Poisson distributions

$$f(x) = f(x_1).f(x_2)$$

Combine equation (1) and 2 as follows to get (3)

$$f(x) = \binom{n}{x} \frac{p^x \lambda^x \quad q^{n-x}}{x!} e^{-\lambda} \qquad (3)$$

$$f(x) = \binom{n}{x} \frac{(\lambda p)^x}{x!} q^{n-x} e^{-\lambda} \qquad (4)$$

$$f(x) = \binom{n}{x} e^{\lambda p} q^{n-x} e^{-\lambda} \qquad (5)$$

$$f(x) = \binom{n}{x} e^{-(\lambda-\lambda p)} q^{n-x} \quad (6)$$

$$f(x) = \binom{n}{x} e^{-\lambda(1-p)} q^{n-x} \quad (7)$$

$$f(x) = \binom{n}{x} e^{-\lambda q} q^{n-x} \quad (8)$$

$$f(x) = \binom{n}{x} \frac{(-1)^{x}(\lambda q)^{x}}{x!} q^{n-x} \quad (9)$$

$$f(x) = \binom{n}{x} (-1)^{x} \frac{\lambda^{x}}{x} q^{x} q^{n} \frac{1}{q^{x}} \quad (10)$$

$$f(x) = \binom{n}{x} e^{-\lambda} q^{n} \quad (11)$$

$$f(x) = \binom{n}{x} q^{n} e^{-\lambda} \quad (12)$$

The probability mass function is given as

$$f(x) = \frac{1}{2^{n}} \binom{n}{x} e^{-\lambda} (1-p)^{n} \quad (13)$$

The cumulative distribution function is given as

$$F(x) = \sum_{x=0}^{n} \frac{1}{2^{n}} \binom{n}{x} e^{-\lambda} (1-p)^{n}$$

(14)

Validity Test of Binosson Distribution

The probability distribution function of any discrete random variable x will satisfy the following

$$\sum_{x=0}^{n} f(x) = 1$$

$$\sum_{x=0}^{n} \frac{1}{2n} \binom{n}{x} e^{-\lambda} (1-p)^{n}$$

$$\sum \binom{n}{x} \frac{1}{2^{n}} e^{-\lambda} \left(1 - \frac{\lambda}{n}\right)^{n}$$

$$\sum_{x=0}^{n} \binom{n}{x} \frac{1}{2^{n}} e^{-\lambda} e^{-\lambda}$$

$$\sum_{x=0}^{n} \binom{n}{x} \frac{1}{2^n}$$

$$2^n \frac{1}{2^n} = 1$$

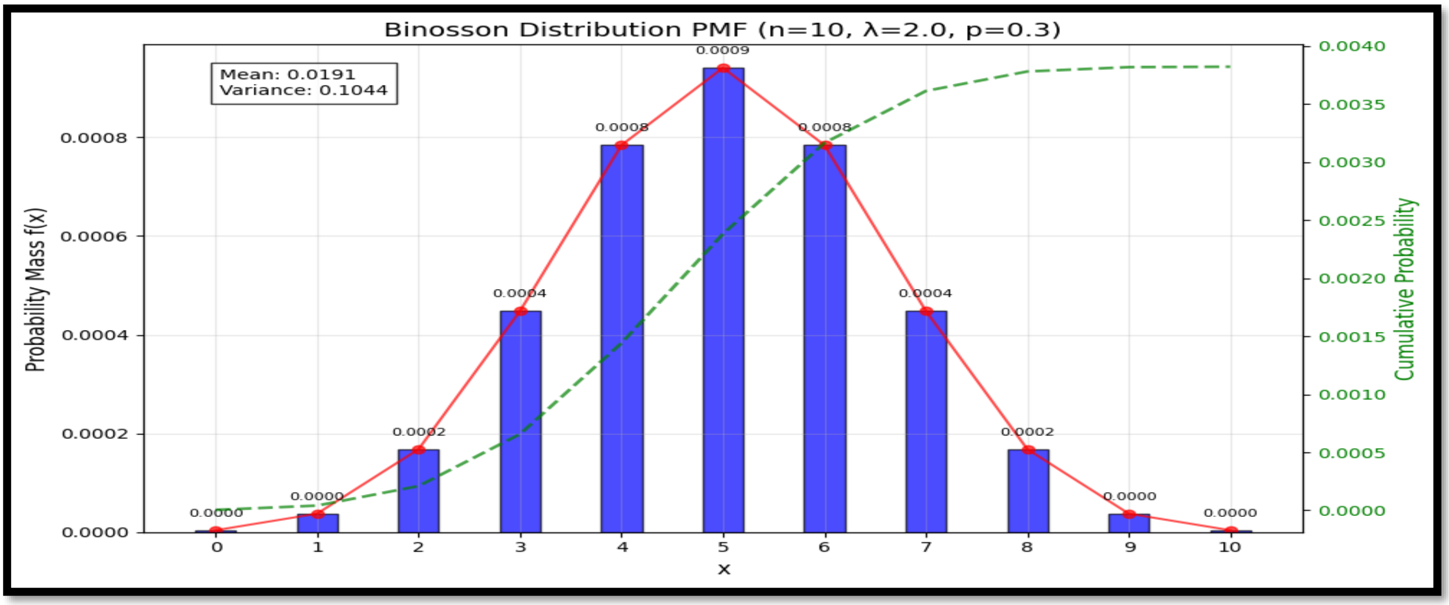

Fig 1: *Graph of PMF of Binosson Distribution*

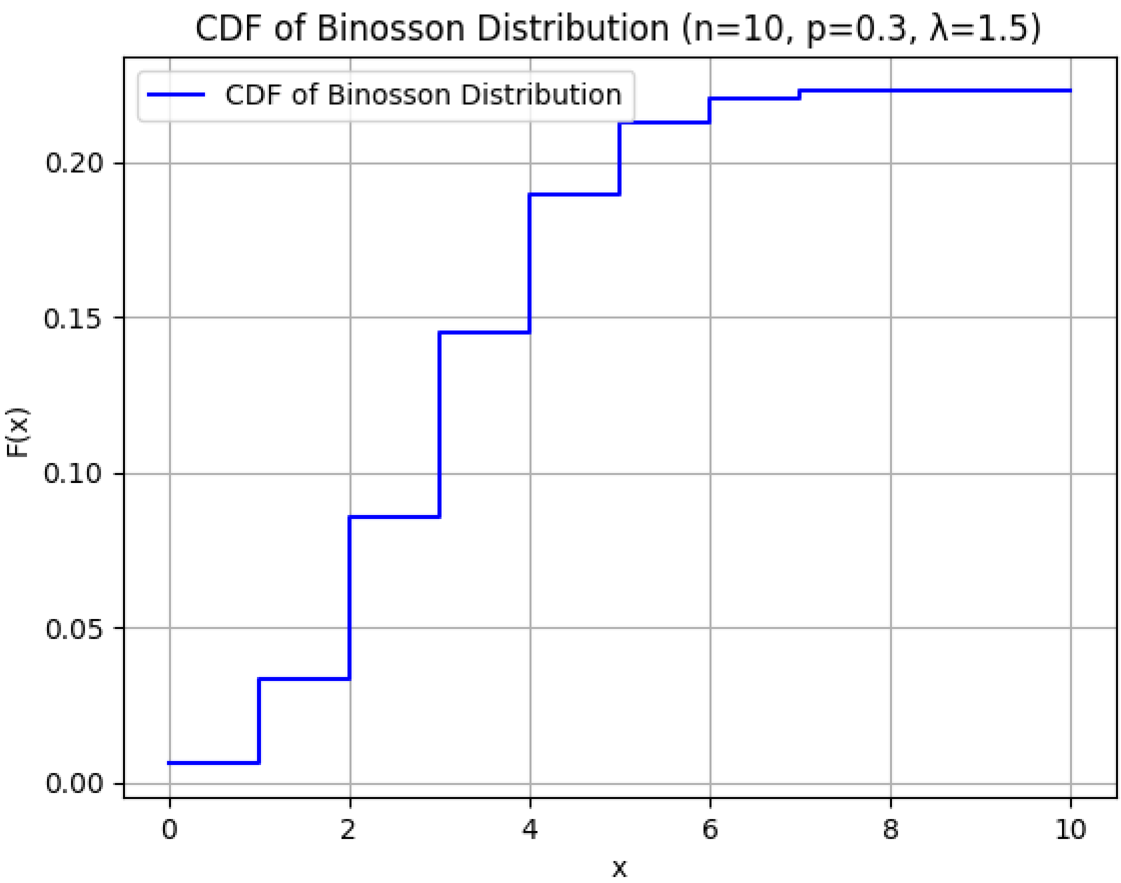

Fig 2: *Graph of CDF of Binosson Distribution*

Properties of the Distribution

a. Mean

$$E\ (x) = \sum_{x=0}^{n} x \frac{1}{2^n}\binom{n}{x} e^{-\lambda}(1-p)^n$$

$$E\ (x) = \sum_{x=0}^{n} x \frac{n!}{(n -_x)!\, x\,!} \frac{1}{2^n} e^{-\lambda} \frac{1}{2^n} e^{-\lambda}(1-p)^n$$

$$E\ (x) = \sum_{x=0}^{n} x \frac{n(n-1)!}{(n-x)!\, x(x-1)\,!} \frac{1}{2.2^{n-1}} e^{-\lambda}(1-p)(\,1-p)^{\,n-1}$$

$$E\ (x) = n(1-p)\frac{1}{2}\sum\nolimits_{x=0}^{n-1} \frac{(\,n-1)!}{(\,n-y)-(x-1)!} \frac{1}{2n-1} \frac{1}{2^n} e^{-\lambda}(\,1-p)^{\,n-1}$$

$$let\ n-1 = w\ and\ x-1 = y$$

$$E(x) = \frac{n(1-p)}{2}\sum_{Y=0}^{W}\frac{W!}{(W-Y)!\,y!}\frac{1}{2^{w}}(1-p)^{n-1}$$

$$E(x) = n\frac{(1-p)}{2}\sum_{y=0}^{w}\binom{w}{y}\frac{1}{2^{w}}e^{-\lambda}(1-p)^{w}$$

$$\sum_{y=0}^{w}\binom{w}{y}\frac{1}{2^{w}}e^{-\lambda}(1-p)^{w} = 1$$

$$\boldsymbol{E(x) = \frac{n(1-p)}{2}}$$

Variance

$$E(x^2) = E\big(x(x-1)\big) + E(x)$$

$$E\big(x(x-1)\big) = \sum_{n=0}^{n} x\,(x-1)f(n)$$

$$E\big(x(x-1)\big) = \sum_{n=0}^{n}(n-1)\binom{n}{x}\tfrac{1}{2}ne^{-\lambda}(1-p)^{n}$$

$$E\big(x(x-1)\big) = \sum_{x=0}^{n}(x-1)\frac{n!}{(n-x)}!\,x!\;\tfrac{1}{2}ne^{-\lambda}(1-p)^{n}$$

$$E\big(x(x-1)\big) = \sum_{x=0}^{n}(x-1)\frac{n(n-1)(n-2)!}{((n-2)-(x-2))!}x(x-1(x-2)!\;\tfrac{1}{2}n-2\,e^{-\lambda}(1\text{-p})^{n-2}$$

$$E\big(x(x-1)\big) = n(n-1)_{2^2}^{\,1}(1-p)^{2}\sum_{x-2=0}^{n-2}\frac{(n-2)!}{(n-2)-(x-2)!}\frac{1}{2^{n-2}}e^{-\lambda}(1-p)^{n-2}$$

$$let\; n-2 = m\; and\; x-2 = z$$

$$E\big(x(x-1)\big) = \frac{n(n-1)(1-p)^{2}}{2^2}\sum_{z=0}^{m}\frac{m!}{(m-z)!\,z!}\frac{1}{2^{m}}e^{-\lambda}(1-p)^{m}$$

$$\sum\binom{m}{z}\frac{1}{2^{m}}e^{-\lambda}(1-p)^{m} = 1$$

$$E\big(x(x-1)\big) = \frac{n(n-1)(1-p)^{2}}{2^2}(1)$$

$$E\big(x(x-1)\big) = \frac{n(n-1)(1-p)^{\,2}}{2^2}$$

$$v(x) = E(x^2) - (E(x))^{\,2}$$

$$v(x) = \frac{n(n-1)(1-p)^{\,2}}{2^2} + \frac{n(1-p)}{2} - \frac{n^{2(1-p)^{\,2}}}{2^2}$$

$$v(x) = \frac{n^{2(1-p)^{\,2}}}{2^2} - \frac{n(\,1-p)^{\,2}}{2^2} + \frac{n(1-p)}{2} - \frac{n^{2(1-p)^{\,2}}}{2^2}$$

$$v(x) = \frac{n(1-p)}{2} - \frac{n(\,1-p)^{\,2}}{2^2}$$

$$v(x) = \frac{n(1-p)}{2}\left(\frac{1}{1} - \frac{(1-p)}{2}\right)$$

$$v(x) = \frac{n(\,1-p)}{2}\left(\frac{2-1+p}{2}\right)$$

$$v(x) = \frac{n(\,1-p)}{2}\left(\frac{1+p}{2}\right)$$

$$v(x) = \frac{n(\,1-p)(1+p)}{4}$$

Measures of Skewedness and Kurtosis

Skewedness and Kurtosis: A fundamental task in many statistical analyses is to characterize the location and variability of a data set. A further characterization of the data includes skewedness and kurtosis.

Skewedness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. The skewedness of Binosson distribution can be obtained as

$$E(\,\mathrm{x}) = \frac{\mathrm{n}(1-\mathrm{p})}{2}$$

$$\mathrm{E}(\mathrm{x}^2) = \mathrm{n}\frac{(\mathrm{n}-1)(1-\mathrm{p})^{\,2}}{2^2} + \frac{\mathrm{n}(1-\mathrm{p})}{2}$$

$$E(x^{3}) = \frac{n(n-2)(n-1)(1-p)^{\ 3}}{2^{3}} + \frac{n(n-1)(1-p)^{\ 2}}{2^{2}} + \frac{n(1-p)}{2}$$

$$E(x^{4}) = \frac{n(n-3)(n-3)(n-1)(1-p)^{\ 4}}{2^{4}}$$

$$= \frac{n(n-1)(1-p)^{\ 3}}{2^{3}} + \frac{n(n-1)(1-p)^{\ 2}}{2^{2}} + \frac{n(1-p)}{2}$$

$$SK = \frac{\frac{n(n-2)(n-1)(1-p)^{\ 3}}{2^{3}} + \frac{n(n-1)(1-p)^{\ 2}}{2^{2}} + \frac{n(1-p)}{2}}{\left(\frac{n(1-p)(1+p)}{4}\right)^{\frac{3}{2}}}$$

Kurtosis

Kurtosis is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. That is, data sets with high kurtosis tend to have heavy tails, or outliers. Data sets with low kurtosis tend to have light tails, or lack of outliers. A uniform distribution would be the extreme case. Kurtosis can be measured for Binosson distribution as

$$K = \frac{\frac{n(n-3)(n-3)(n-1)(1-p)^{\ 4}}{2^{4}} + \frac{n(n-1)(1-p)^{\ 3}}{2^{3}} + \frac{n(n-1)(1-p)^{\ 2}}{2^{2}} + \frac{n(1-p)}{2}}{\left[\frac{n(1-p)(1+p)}{4}\right]^{2}}$$

Conclusion

The Binosson Distribution represents a significant evolution in probabilistic modeling, addressing the limitations of classical frameworks like the Binomial and Poisson distributions in capturing the complexities of modern datasets. Traditional models, constrained by assumptions of fixed trials, constant probabilities, or static event rates, struggle to adapt to real-world dynamics such as seasonal behavioral shifts, variable infection rates during outbreaks, or covariate-dependent financial risks. By integrating the discrete-event structure of Binomial trials with the rate-based flexibility of Poisson processes, the Binosson framework introduces dynamic parameterization and contextual adaptability, enabling robust analysis of over-dispersed, correlated, or heteroscedastic data. Its design reflects critiques of rigid "true distribution" paradigms, prioritizing real-world applicability through hybrid mechanics. Empirical validation demonstrates its superiority in scenarios like actuarial risk assessment and epidemic modeling, where classical models fail to account for fluctuating trial counts or external influences. Future research should focus on expanding its utility through Bayesian parameter estimation, empirical benchmarking across disciplines, and extensions to multivariate contexts. As datasets grow increasingly complex, the Binosson Distribution offers a critical bridge between foundational statistical theory and the evolving demands of interdisciplinary science, underscoring the necessity for adaptable, hybrid tools in modern analytics.

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