NIPES - Journal of Science and Technology Research
www.nipes.org

An Efficient Intervalwise Partitioning Technique for the Solution of Ordinary Differential Systems

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Abstract* — An efficient two-point intervalwise partitioning technique (IPT) is formulated in this study for the solution of ordinary differential systems. The technique consists of two methods namely two-step adaptive hybrid block Adams method (TSAHBAM) and two-step adaptive block backward differentiation formula (TSAHBBDF). To solve a problem, the IPT will initially treat such problem as non-stiff and solve it with the aid of TSAHBAM. If a failure step is encountered as a result of stiffness, the IPT automatically switches to TSAHBBDF to handle the stiffness. Summary of analysis of the IPT was presented and the technique was applied in solving some problems. The results obtained showed that the IPT is computationally efficient and more accurate than some existing methods.

***Keywords*—Adams method, Adaptive method, block backward differentiation formula, differential system, intervalwise partitioning, stiffness**

https://doi.org/10.37933/nipes/7.4.2025.SI342
eISSN-2682-5821| pISSN-2734-2352 © 2025 NIPES Pub

I. Introduction

Ordinary differential systems find applications in many fields of human endeavours. These systems may be non-stiff or stiff in nature. The non-stiff systems are easier to solve than their stiff counterparts. Stiff differential systems equations are a class of differential equations that exhibit varying timescales. These systems possess fast-changing components that require very small time-steps to be accurately captured, alongside slow-changing components that evolve more gradually. This disparity can cause severe numerical instability when using traditional explicit methods in solving them. Thus stiff systems are better solved using implicit methods like the backward differentiation formula (BDF).

In this research, an efficient IPT shall be formulated for the solutions of first-order differential system of the form,

$$y_i' = f_i\left(t,\overline{Y}\right), \overline{Y}(a) = \overline{\eta},\ i = 1,2,...,s \tag{1}$$

where $\overline{Y}^T(t) = \left(y_1, y_2, ..., y_s\right)$ and $\overline{\eta}^T(t) = \left(\eta_1, \eta_2, ..., \eta_s\right)$.

The IPT shall be formulated using partitioning strategy. This is a procedure where a system such as (1) is divided into two subsystems, that is the non-stiff subsystem and the stiff subsystem, when instability occurs, [1]. The non-stiff parts of the system are solved using Adams-type methods while the stiff parts are solved using BDF-type methods.

A lot of researches have been carried out in formulating different partitioning methods. The authors in [2] were the first to introduce the partitioning strategy which was later extended by [3]. Ref. [4] formulated partitioning techniques in Runge-Kutta type methods. Refs. [5] and [1] developed 2-point and 3-point intervalwise block partitioning methods respectively for the solutions of differential systems of the form (1). The following authors also developed different methods for the solutions of non-stiff and stiff differential equations, [6], [7], [8], [9], [10] and [11].

II. MATHEMATICAL FORMULATION OF THE IPT

In this section, an efficient IPT shall be formulated for the solutions of differential systems of the form (1). The IPT shall consist of two methods, namely the TSAHBAM and the TSAHBBDF. Consider the general k-step linear multistep method (LMM),

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j}, \tag{2}$$

where α_j's and β_j's are real constants. The proposed IPT shall be implemented in block mode by developing a system of LMMs at the discrete points t_{n+1}, $t_{n+3/2}$ and t_{n+2}. The step size between t_n and t_{n+1} is taken as rh, where r is the stepsize ratio. In order to obtain an efficient IPT, r is selected at three points. These are $r=1/2$ (corresponding to doubling the stepsize), $r=1$ (corresponding to maintaining the stepsize), and $r=2$ (corresponding to halving the stepsize). See [12], [13], [14] and [15] for more details on adaptive stepsize and intervalwise partitioning.

A. *Derivation of the TSAHBAM*

The TSAHBAM is derived by integrating the differential system (1) over the interval $(t_n, t_{n+\varepsilon})$, for $\varepsilon = 1, 3/2$ and 2. This gives,

$$\int_{t_n}^{t_{n+\varepsilon}} y_i'(t)dt = \int_{t_n}^{t_{n+\varepsilon}} f_i\left(t,\overline{Y}(t)\right)dt. \tag{3}$$

The function $f_i\left(t,\overline{Y}(t)\right)$ in equation (1) is approximated at the interpolation points (t_n, y_n), (t_{n+1}, y_{n+1}), $(t_{n+3/2}, y_{n+3/2})$ and (t_{n+2}, y_{n+2}) using the Lagrange interpolating polynomial,

$$\begin{aligned} P(t) = {} & \frac{(t-t_n)(t-t_{n+1})(t-t_{n+3/2})}{(t_{n+2}-t_n)(t_{n+2}-t_{n+1})(t_{n+2}-t_{n+3/2})} y_{n+2} \\ & + \frac{(t-t_n)(t-t_{n+1})(t-t_{n+2})}{(t_{n+3/2}-t_n)(t_{n+3/2}-t_{n+1})(t_{n+3/2}-t_{n+2})} y_{n+\frac{3}{2}} \\ & + \frac{(t-t_n)(t-t_{n+3/2})(t-t_{n+2})}{(t_{n+1}-t_n)(t_{n+1}-t_{n+3/2})(t_{n+1}-t_{n+2})} y_{n+1} \\ & + \frac{(t-t_{n+1})(t-t_{n+3/2})(t-t_{n+2})}{(t_n-t_{n+1})(t_n-t_{n+3/2})(t_n-t_{n+2})} y_n. \end{aligned} \tag{4}$$

The integral equation (3) is evaluated with respect to s for $s=(t-t_{n+2})/h$. The integration limits are carefully chosen as (-2, -1), (-2, -1/2) and (-2, 0) and substituting hds for dt gives the TSAHBAM,

$$\begin{aligned} y_{n+1} = y_n & + \left[\left(\frac{2h}{r(2r+1)(r+1)}\right)\right] f_n \\ & + \left[\left(\frac{h}{6}\right)\left(\frac{19r-12}{r}\right)\right] f_{n+1} \\ & - \left[\left(\frac{2h}{3}\right)\left(\frac{10r-7}{2r+1}\right)\right] f_{n+\frac{3}{2}} \\ & + \left[\left(\frac{h}{6}\right)\left(\frac{7r-5}{r+1}\right)\right] f_{n+2}, \end{aligned} \tag{5}$$

$$\begin{aligned} y_{n+\frac{3}{2}} = y_n & + \left[\left(\frac{63h}{32}\right)\left(\frac{1}{r(2r+1)(r+1)}\right)\right] f_n \\ & + \left[\left(\frac{9h}{32}\right)\left(\frac{12r-7}{r}\right)\right] f_{n+1} \\ & - \left[\left(\frac{3h}{8}\right)\left(\frac{16r-13}{2r+1}\right)\right] f_{n+\frac{3}{2}} \\ & + \left[\left(\frac{9h}{32}\right)\left(\frac{4r-3}{r+1}\right)\right] f_{n+2}, \end{aligned} \tag{6}$$

$$y_{n+2} = y_n + \left[\left(\frac{2h}{r(2r+1)(r+1)}\right)\right] f_n + \left[\left(\frac{2h}{3}\right)\left(\frac{5r-3}{r}\right)\right] f_{n+1} - \left[\left(\frac{16h}{3}\right)\left(\frac{r-1}{2r+1}\right)\right] f_{n+\frac{3}{2}} + \left[\left(\frac{2h}{3}\right)\left(\frac{2r-1}{r+1}\right)\right] f_{n+2}. \tag{7}$$

At $r=1$, equations (5)-(7) give,

$$\left.\begin{aligned} y_{n+1} &= y_n + h\left(\frac{1}{3}f_n + \frac{7}{6}f_{n+1} - \frac{2}{3}f_{n+\frac{3}{2}} + \frac{1}{6}f_{n+2}\right), \\ y_{n+\frac{3}{2}} &= y_n + h\left(\frac{21}{64}f_n + \frac{45}{32}f_{n+1} - \frac{3}{8}f_{n+\frac{3}{2}} + \frac{9}{64}f_{n+2}\right), \\ y_{n+2} &= y_n + h\left(\frac{1}{3}f_n + \frac{4}{3}f_{n+1} + (0)f_{n+\frac{3}{2}} + \frac{1}{3}f_{n+2}\right). \end{aligned}\right\} \tag{8}$$

At $r=2$, equations (5)-(7) give,

$$\left.\begin{aligned} y_{n+1} &= y_n + h\left(\frac{1}{15}f_n + \frac{13}{6}f_{n+1} - \frac{26}{15}f_{n+\frac{3}{2}} + \frac{1}{2}f_{n+2}\right), \\ y_{n+\frac{3}{2}} &= y_n + h\left(\frac{21}{320}f_n + \frac{153}{64}f_{n+1} - \frac{57}{40}f_{n+\frac{3}{2}} + \frac{15}{32}f_{n+2}\right), \\ y_{n+2} &= y_n + h\left(\frac{1}{15}f_n + \frac{7}{3}f_{n+1} - \frac{16}{15}f_{n+\frac{3}{2}} + \frac{2}{3}f_{n+2}\right). \end{aligned}\right\} \tag{9}$$

At $r=1/2$, equations (5)-(7) give,

$$\left.\begin{aligned} y_{n+1} &= y_n + h\left(\frac{4}{3}f_n - \frac{5}{6}f_{n+1} + \frac{2}{3}f_{n+\frac{3}{2}} - \frac{1}{6}f_{n+2}\right), \\ y_{n+\frac{3}{2}} &= y_n + h\left(\frac{21}{16}f_n - \frac{9}{16}f_{n+1} + \frac{15}{16}f_{n+\frac{3}{2}} - \frac{3}{16}f_{n+2}\right), \\ y_{n+2} &= y_n + h\left(\frac{4}{3}f_n - \frac{2}{3}f_{n+1} + \frac{4}{3}f_{n+\frac{3}{2}} + (0)f_{n+2}\right). \end{aligned}\right\} \tag{10}$$

Equations (8)-(10) form the TSAHBAM. This method can be written in general form as,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \hat{\beta}_1 & \hat{\beta}_2 & \hat{\beta}_3 \\ \bar{\beta}_1 & \bar{\beta}_2 & \bar{\beta}_3 \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{bmatrix}. \tag{11}$$

Equation (11) can be rewritten in matrix finite difference equation,

$$I\hat{Y}_m = A\hat{Y}_{m-1} + hB\hat{F}_m \tag{12}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{Y}_m = \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \end{bmatrix}, \quad \hat{Y}_{m-1} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \hat{\beta}_1 & \hat{\beta}_2 & \hat{\beta}_3 \\ \bar{\beta}_1 & \bar{\beta}_2 & \bar{\beta}_3 \end{bmatrix}, \quad \hat{F}_m = \begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{bmatrix}.$$

B. Derivation of the TSAHBBDF

To derive the TSAHBBDF, we let $s = (t - t_{n+2})/h$, so that on substituting $t = t_{n+2} + sh$ into the interpolating polynomial (4), we obtain

$$P(t) = P(t_{n+2} + sh) = (s+1)(2s+1)\left(\frac{r+s+1}{r+1}\right)y_{n+2} - 8s(s+1)\left(\frac{r+s+1}{2r+1}\right)y_{n+\frac{3}{2}} + s(2s+1)\left(\frac{r+s+1}{r}\right)y_{n+1} - s(s+1)\left(\frac{2s+1}{r(r+1)(2r+1)}\right)y_n. \tag{13}$$

Differentiating equation (13) with respect to s, gives

$$hP'\left(t_{n+2}+sh\right)=\left(\frac{3r+10s+4rs+6s^2+4}{(r+1)}\right)y_{n+2}-\left(\frac{8\left(r+4s+2rs+3s^2+1\right)}{(2r+1)}\right)y_{n+\frac{3}{2}}+\left(\frac{r+6s+4rs+6s^2+1}{r}\right)y_{n+1}-\left(\frac{\left(6s^2+6s+1\right)}{r\left(2r^2+3r+1\right)}\right)y_n. \quad (14)$$

Substituting $s=-1,-1/2$ and $s=0$ in equation (14) respectively gives,

$$\left.\begin{aligned}
hf_{n+1}&=-\left(\frac{r}{r+1}\right)y_{n+2}+\left(\frac{8r}{2r+1}\right)y_{n+\frac{3}{2}}-\left(\frac{3r-1}{r}\right)y_{n+1}-\left(\frac{1}{r\left(2r^2+3r+1\right)}\right)y_n,\\
hf_{n+\frac{3}{2}}&=\left(\frac{r+(1/2)}{r+1}\right)y_{n+2}+\left(\frac{8}{2r+1}\right)y_{n+\frac{3}{2}}-\left(\frac{r+(1/2)}{r}\right)y_{n+1}+\left(\frac{1}{2r\left(2r^2+3r+1\right)}\right)y_n,\\
hf_{n+2}&=\left(\frac{3r+4}{r+1}\right)y_{n+2}-\left(\frac{8(r+1)}{2r+1}\right)y_{n+\frac{3}{2}}+\left(\frac{r+1}{r}\right)y_{n+1}-\left(\frac{1}{r\left(2r^2+3r+1\right)}\right)y_n.
\end{aligned}\right\} \quad (15)$$

Solving equation (15) for y_{n+1}, $y_{n+3/2}$ and y_{n+2} respectively leads to,

$$\left.\begin{aligned}
y_{n+1}&=\frac{-\left(\frac{r}{r+1}\right)y_{n+2}+\left(\frac{8r}{2r+1}\right)y_{n+\frac{3}{2}}-\left(\frac{1}{r\left(2r^2+3r+1\right)}\right)y_n-hf_{n+1}}{\left(\frac{3r-1}{r}\right)},\\
y_{n+\frac{3}{2}}&=\frac{\left(\frac{r+(1/2)}{r+1}\right)y_{n+2}-\left(\frac{r+(1/2)}{r}\right)y_{n+1}+\left(\frac{1}{2r\left(2r^2+3r+1\right)}\right)y_n-hf_{n+\frac{3}{2}}}{-\left(\frac{2}{2r+1}\right)},\\
y_{n+2}&=\frac{\left(\frac{-8(r+1)}{2r+1}\right)y_{n+\frac{3}{2}}+\left(\frac{r+1}{r}\right)y_{n+1}-\left(\frac{1}{r\left(2r^2+3r+1\right)}\right)y_n-hf_{n+2}}{-\left(\frac{3r+4}{r+1}\right)}.
\end{aligned}\right\} \quad (16)$$

At $r=1$, equation (16) is

$$\left.\begin{aligned}
y_{n+1}&=-\frac{1}{12}y_n+\frac{4}{3}y_{n+\frac{3}{2}}-\frac{1}{4}y_{n+2}-\frac{1}{2}hf_{n+1},\\
y_{n+\frac{3}{2}}&=-\frac{1}{8}y_n+\frac{9}{4}y_{n+1}-\frac{9}{8}y_{n+2}+\frac{3}{2}hf_{n+\frac{3}{2}},\\
y_{n+2}&=\frac{1}{21}y_n-\frac{4}{7}y_{n+1}+\frac{32}{21}y_{n+\frac{3}{2}}+\frac{2}{7}hf_{n+2}.
\end{aligned}\right\} \quad (17)$$

At $r=2$, equation (16) is given by

$$\left.\begin{aligned}
y_{n+1}&=-\frac{1}{75}y_n+\frac{32}{25}y_{n+\frac{3}{2}}-\frac{4}{15}y_{n+2}-\frac{2}{5}hf_{n+1},\\
y_{n+\frac{3}{2}}&=-\frac{1}{24}y_n+\frac{25}{8}y_{n+1}-\frac{25}{12}y_{n+2}+\frac{5}{2}hf_{n+\frac{3}{2}},\\
y_{n+2}&=\frac{1}{100}y_n-\frac{9}{20}y_{n+1}+\frac{36}{25}y_{n+\frac{3}{2}}+\frac{3}{10}hf_{n+2}.
\end{aligned}\right\} \quad (18)$$

At $r=1/2$, equation (16) becomes

$$\left.\begin{aligned} y_{n+1} &= -\frac{2}{3}y_n + 2y_{n+\frac{3}{2}} - \frac{1}{3}y_{n+2} - hf_{n+1}, \\ y_{n+\frac{3}{2}} &= -\frac{1}{3}y_n + 2y_{n+1} - \frac{2}{3}y_{n+2} + hf_{n+\frac{3}{2}}, \\ y_{n+2} &= \frac{2}{11}y_n - \frac{9}{11}y_{n+1} + \frac{18}{11}y_{n+\frac{3}{2}} + \frac{3}{11}hf_{n+2}. \end{aligned}\right\} \quad (19)$$

Equations (17)-(19) collectively form the TSAHBBDF. This method can be written in general form as,

$$\left.\begin{aligned} y_{n+1} &= \theta_1 y_{n+\frac{3}{2}} + \varphi_1 y_{n+2} + \alpha_1 hf_{n+1} + \psi_1, \\ y_{n+\frac{3}{2}} &= \theta_2 y_{n+1} + \varphi_2 y_{n+2} + \alpha_2 hf_{n+\frac{3}{2}} + \psi_{\frac{3}{2}}, \\ y_{n+2} &= \theta_3 y_{n+1} + \varphi_3 y_{n+\frac{3}{2}} + \alpha_3 hf_{n+2} + \psi_2. \end{aligned}\right\} \quad (20)$$

where ψ_1, $\psi_{3/2}$ and ψ_2 are back values. In matrix-vector form, equation (20) is equivalent to,

$$(I - A)Y_m = hBF_m + \xi, \quad (21)$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \theta_1 & \varphi_1 \\ \theta_2 & 0 & \varphi_2 \\ \theta_3 & \varphi_3 & 0 \end{bmatrix}, \quad Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \end{bmatrix},$$

$$B = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \; F_m = \begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{bmatrix}, \; \xi = \begin{bmatrix} \psi_1 \\ \psi_{3/2} \\ \psi_2 \end{bmatrix}.$$

The TSAHBAM in equations (8)-(10) and TSAHBBDF in equations (17)-(19) collectively form the IPT.

III. IMPLEMENTATION STRATEGY OF THE IPT

To implement the IPT, the differential system (1) is initially treated as non-stiff and solved using the newly derived TSAHBAM in equations (8)-(10). Once a failure step is noticed due to suspected presence of stiffness, a stronger test is carried out. This test involves computing the trace of the Jacobian, that is $\partial f / \partial y$. If the trace is negative, the system (1) is treated as stiff and then solved using the newly derived TSAHBBDF in equations (17)-(19). However, if the trace is positive, the iteration is continued using the TSAHBAM with half of the step-size.

According to [16], the error control algorithm is as follows:

- if the error control is less than tolerance limit, then the step-size h is increased to gain computation speed,
- in the case of step failure, the step-size h is halved and the step repeated.

For more on the implementation strategies of the partitioning technique, see the works of [1], [5] and [6].

IV. ANALYSIS OF THE IPT

In this section, summary of analysis of basic properties of the two methods that made up the IPT shall be investigated.

Definition 4.1: The general k-step LMM (2) and its associated linear difference operator L given by

$$L\{y(t);h\} = \sum_{j=0}^{k}\left[\alpha_j y(t+jh) - h\beta_j y'(t+jh)\right] \quad (22)$$

are said to be of order p if $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_p = 0,\ \bar{c}_{p+1} \neq 0$, [17]. The term $\bar{c}_{p+1} \neq 0$ is called the error constant. The constants c_p are defined as

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k}\alpha_j \\ c_1 &= \sum_{j=0}^{k}\left(j\alpha_j - \beta_j\right) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ c_p &= \sum_{j=0}^{k}\left[\frac{1}{p!}j^p\alpha_j - \frac{1}{(p-1)!}j^{p-1}\beta_j\right], p = 2,3,\ldots \end{aligned}\right\} \quad (23)$$

Specifically, for the TSAHBAM at $r = 1$,

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T, \quad (24)$$

with error constant

$$\bar{c}_5 = \begin{bmatrix} -1.0764 \times 10^{-2} \\ -9.9609 \times 10^{-3} \\ -1.1111 \times 10^{-2} \end{bmatrix}^T. \quad (25)$$

On the other hand, for the TSAHBBDF at $r = 1$,

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (26)$$

with error constant

$$\bar{c}_4 = \begin{bmatrix} 1.0417\times10^{-2} \\ 2.3438\times10^{-2} \\ -1.1905\times10^{-2} \end{bmatrix}^T . \quad (27)$$

This implies that the TSAHBAM is of order four while the TSAHBBDF is of the third order. Similarly, the stability polynomial of the TSAHBAM at $r=1$, $r=2$ and $r=1/2$ are respectively given by,

$$R_1(t,H) = -t^3\left(\frac{1}{8}H^3 - \frac{13}{24}H^2 + \frac{9}{8}H - 1\right) - t^2\left(\frac{1}{24}H^3 + \frac{7}{24}H^2 + \frac{7}{8}H + 1\right), \quad (28)$$

$$R_2(t,H) = -t^3\left(\frac{67}{240}H^3 - \frac{1273}{1440}H^2 + \frac{169}{120}H - 1\right) - t^2\left(\frac{1}{120}H^3 + \frac{193}{1440}H^2 + \frac{71}{120}H + 1\right), \quad (29)$$

$$R_{1/2}(t,H) = t^3\left(\frac{5}{48}H^3 - \frac{77}{288}H^2 - \frac{5}{48}H + 1\right) - t^2\left(\frac{1}{6}H^3 + \frac{247}{288}H^2 + \frac{91}{48}H + 1\right). \quad (30)$$

Substituting $H=0$ in equations (28)-(30), we obtain

$$R_1(t,0) = R_2(t,0) = R_{1/2}(t,0) = t^3 - t^2. \quad (31)$$

Solving equations (31) gives $0,0$ and 1, which shows that the TSAHBAM is zero-stable. For the TSAHBBDF, the stability polynomials at $r=1$, $r=2$ and $r=1/2$ are respectively given by

$$R_1(t,H) = \frac{3}{14}t^3H^3 - \frac{13}{28}t^3H^2 + \frac{1}{28}t^2H^2 + \frac{9}{14}t^3H + \frac{3}{14}t^2H - \frac{3}{7}t^3 + \frac{3}{7}t^2, \quad (32)$$

$$R_2(t,H) = \frac{3}{10}t^3H^3 - \frac{37}{100}t^3H^2 + \frac{1}{100}t^2H^2 + \frac{3}{10}t^3H + \frac{3}{50}t^2H - \frac{3}{25}t^3 + \frac{3}{25}t^2, \quad (33)$$

$$R_{1/2}(t,H) = \frac{3}{11}t^3H^3 - t^3H^2 + \frac{2}{11}t^2H^2 + \frac{24}{11}t^3H + \frac{12}{11}t^2H - \frac{24}{11}t^3 + \frac{24}{11}t^2. \quad (34)$$

Substituting $H=0$ into equations (32)-(34), we obtain

$$R_1(t,0) = -\frac{3}{7}t^3 + \frac{3}{7}t^2, \quad (35)$$

$$R_2(t,0) = -\frac{3}{25}t^3 + \frac{3}{25}t^2, \quad (36)$$

$$R_1(t,0) = -\frac{24}{11}t^3 + \frac{24}{11}t^2. \quad (37)$$

Solving each of the polynomials in equations (35)-(37) give $0,0$ and 1, which also shows that the TSAHBBDF is zero-stable, [18].

Tables 1 and 2 present the summary of basic properties of the IPT made up of TSAHBAM and TSAHBBDF. Similarly Fig. 1 and Fig. 2 show the stability regions of the TSAHBAM and TSAHBBDF at the three step-size ratios.

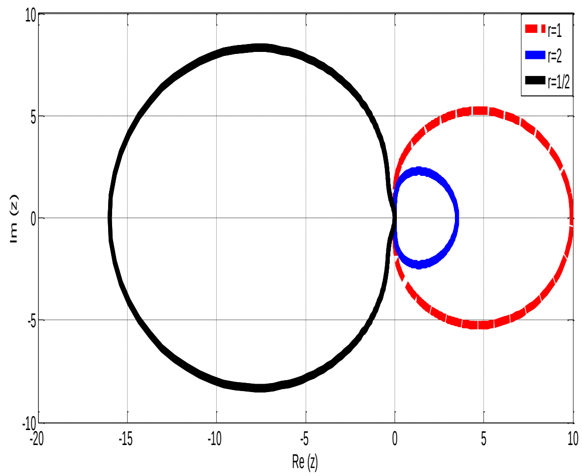

Fig. 1. Regions of absolute stability of the TSAHBAM

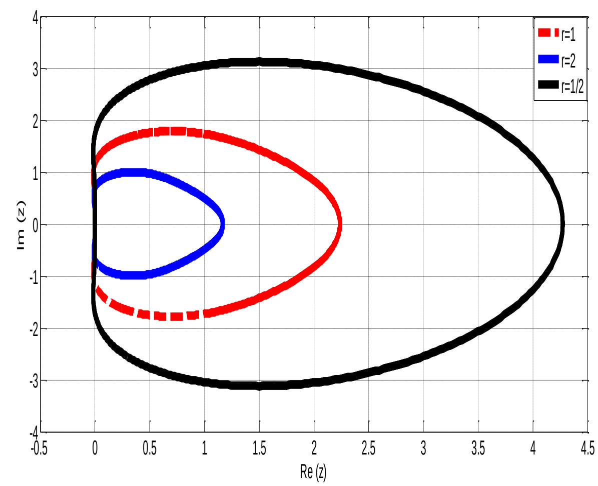

Fig. 2. Regions of absolute stability of the TSAHBBDF

From Fig. 1, the exterior part of the blue- and red-coloured contours are the stability regions of the TSAHBAM at $r=2$

and $r=1$ respectively. On the other hand, the interior part of the black-coloured contour is the stability region of the TSAHBAM at $r=1/2$. This implies that the TSAHBAM at $r=2$ has the largest region of stability, followed by TSAHBAM at $r=1$ and then $r=1/2$ has the smallest stability region. The instability intervals of the TSAHBAM are given in Table 2. In Fig. 2, the regions of absolute stability of the TSAHBBDF are the exterior of the blue-, red-, and black-coloured contours for step-size ratios $r=2$, $r=1$ and $r=1/2$ respectively. This implies that the TSAHBBDF at $r=2$ has the largest stability region, then the TSAHBBDF at $r=1$ and then TSAHBBDF at $r=1/2$ has the smallest region of stability. Table 2 also shows the intervals of instability of the TSAHBBDF. It is clear that the TSAHBAM is A-stable at $r=1$ and $r=2$ but is not A-stable at $r=1/2$. On the other hand, the TSAHBBDF is A-stable at all the three step-size ratios.

V. NUMERICAL EXPERIMENTS

The newly formulated IPT shall be adopted in solving some test problems. This is aimed at testing its efficiency as well as accuracy over existing methods. The following notations shall be used in presenting the results.

h_{init} : Initial step-size
TOL: Tolerance level/limit
TS: Total number of steps taken
TAS: Total number of accepted steps
TFS: Total number of failure steps
MAXE: Maximum absolute error
TIME: Execution time (in seconds)
ode15s: Matlab inbuilt variable order solver based on BDF
IBP: 3-point intervalwise block partitioning method developed by [1]
IPT: newly formulated intervalwise partitioning technique

Problem 5.1
Consider the nonlinear differential system,

$$\left.\begin{aligned} y_1' &= -2y_1 + y_2 + 2\sin t,\ y_1(0)=2,\\ y_2' &= 998y_1 - 999y_2 + 999(\cos t - \sin t),\ y_2(0)=3, \end{aligned}\right\} \quad (38)$$

defined over $t\in[0,5]$. The exact solution of the system is

$$\left.\begin{aligned} y_1(t) &= 2e^{-t} + \sin t,\\ y_2(t) &= 2e^{-t} + \cos t. \end{aligned}\right\} \quad (39)$$

Source: [12]

Problem 5.2
Consider the linear differential system

$$\left.\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3,\ y_1(0)=1,\\ y_2' &= 19y_1 - 21y_2 + 20y_3,\ y_2(0)=0,\\ y_3' &= 40y_1 - 40y_2 - 40y_3,\ y_3(0)=-1, \end{aligned}\right\} \quad (40)$$

defined for $t\in[0,10]$, with the exact solution

$$\left.\begin{aligned} y_1(t) &= (1/2)\left[e^{-2t} + e^{-40t}(\cos 40t + \sin 40t)\right],\\ y_2(t) &= (1/2)\left[e^{-2t} - e^{-40t}(\cos 40t + \sin 40t)\right],\\ y_3(t) &= (1/2)\left[2e^{-40t}(\sin 40t - \cos 40t)\right]. \end{aligned}\right\} \quad (41)$$

The Jacobian matrix of the system (40) has Eigen values $\lambda_1=-2$, $\lambda_2=-40+40i$ and $\lambda_3=-40-40i$.
Source: [19]

Table 1. Summary of analysis of basic properties of TSAHBAM and TSAHBBDF

Method	**Order**	**Zero-stability**	**Consistence**	**Convergence**
TSAHBAM	4	Zero-stable	Consistent	Convergent
TSAHBBDF	3	Zero-stable	Consistent	Convergent

Table 2. Summary of analysis of TSAHBAM and TSAHBBDF showing roots stability polynomials and intervals of instability at different step-size ratios

Method	**Step-size ratio**	**Roots of Stability Polynomial**	**Intervals of Instability**
TSAHBAM	$r=1$	$0,0,1$	$(0,9.941075)$
	$r=2$	$0,0,1$	$(0,3.500023)$
	$r=1/2$	$0,0,1$	$(-\infty,-15.927177)\cup(0,\infty)$
TSAHBBDF	$r=1$	$0,0,1$	$(0,2.238034)$
	$r=2$	$0,0,1$	$(0,1.168105)$
	$r=1/2$	$0,0,1$	$(0,4.273441)$

Table 3. Numerical results for Problem 5.1 at $h_{init}=10^{-1}$

TOL	Method	TS	TAS	TFS	MAXE	TIME
10^{-3}	ode15s	29	28	1	$2.40e-03$	0.7938
	IPT	21	19	2	$1.2314e-09$	0.0021
10^{-4}	ode15s	38	38	0	$2.41e-04$	2.5669
	IPT	27	25	2	$7.0025e-10$	0.0033
10^{-5}	ode15s	55	54	1	$5.30e-05$	52.4463
	IPT	45	41	4	$4.5119e-12$	0.0078

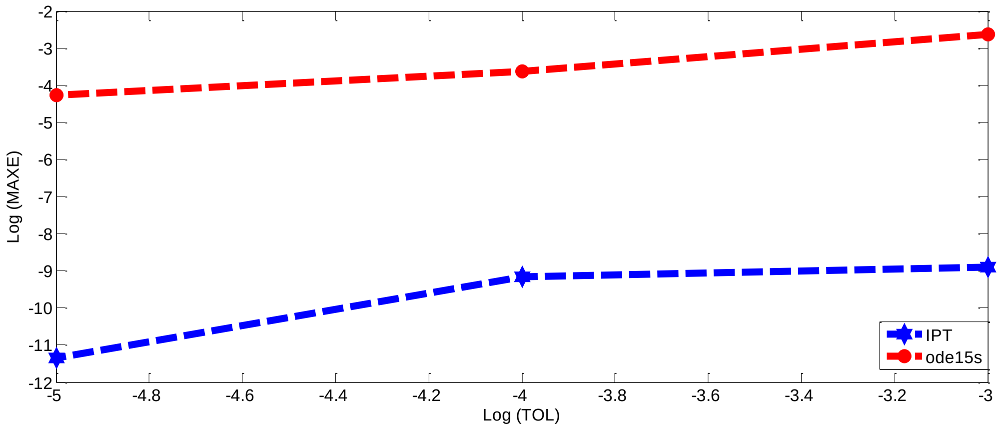

Fig. 3. Accuracy curves for Problem 5.1

Table 4. Numerical results for Problem 5.2

TOL	Method	TS	MAXE	TIME
10^{-2}	ode15s	37	$7.92983e-03$	0.009659
	IBP	26	$2.15015e-01$	0.000224
	IPT	20	$1.23172e-05$	0.000101
10^{-4}	ode15s	86	$1.16049e-04$	0.018109
	IBP	36	$8.24059e-03$	0.000344
	IPT	28	$3.77241e-07$	0.000171
10^{-6}	ode15s	162	$1.77877e-06$	0.028360
	IBP	58	$4.71831e-05$	0.000919
	IPT	40	$1.62001e-10$	0.000544
10^{-8}	ode15s	305	$3.83010e-08$	0.073445
	IBP	139	$1.45542e-09$	0.001936
	IPT	108	$7.26503e-12$	0.001032

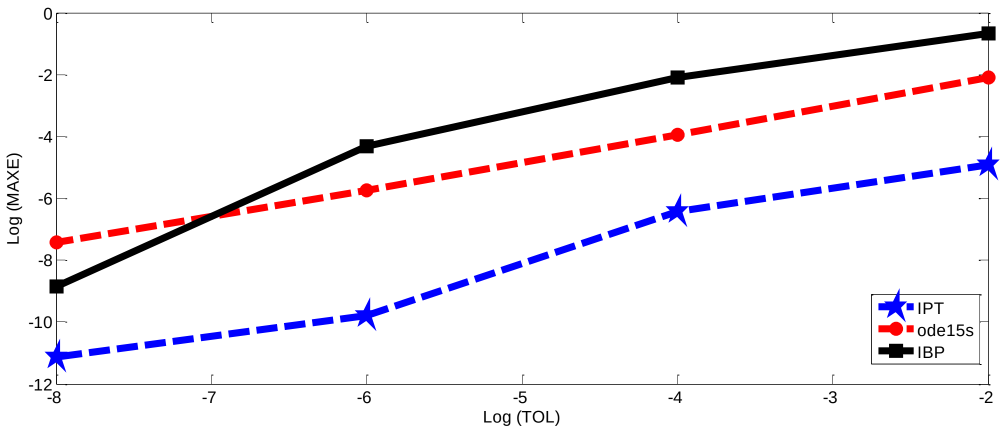

Fig. 3. Accuracy curves for Problem 5.2

The newly formulated IPT was applied in solving two differential systems of the form (1). The results obtained in Table 3 showed that the method is efficient as well as accurate. This is because the maximum absolute errors of the IPT are smaller than those of ode15s. The IPT also had smaller execution time than the ode15s implying that it is more efficient. The accuracy curves obtained in Fig. 3 clearly showed that the IPT is more computationally reliable. It was also observed in Table 3 that the IPT took fewer numbers of steps to achieve accuracy in comparison with ode15s. Similarly the numerical and graphical results obtained in Table 4 and Fig. 4 respectively showed that the IPT is accurate and efficient than the ode15s and IBP developed by [1]. Further study could explore the performance of the IPT on problems discussed in [20], [21] and [22-28].

VI. CONCLUSSIONS

The new partitioning strategy, which is the IPT proposed in this research has been proven to be more efficient than the other methods we compared our results with. This is in addition to the improved accuracy exhibited by the new technique. Analysis of basic properties of the IPT showed that the technique is consistent, zero-stable and convergent. Thus, the IPT has proven to be computationally reliable in solving differential systems of the form (1).

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