2024 International Conference on Science, Engineering and Business for Driving Sustainable Development Goals (SEB4SDG) | 979-8-3503-5815-5/24/$31.00 ©2024 IEEE | DOI: 10.1109/SEB4SDG60871.2024.10629688

An Enhanced Product Estimator in Two-stage Sampling for Ancillary Characteristic-Based Population Mean Estimation

1st *Femi Emmanuel Amoyedo*
Department of Physical Science, Mathematics Programme, Landmark University.
Omu-Aran, Nigeria
amoyedo.femi@lmu.edu.ng

2nd *Amos Adedayo Adewara*
Department of Statistics, University of Ilorin,
Kwara State, Nigeria
aadewara@unilorin.edu.ng

3rd *Ezekiel Olaoluwa Omole*
Department of Physical Science, Mathematics Programme, Landmark University.
Omu-Aran, Nigeria
omole.ezekiel@lmu.edu.ng

4th Kehinde Peter Ajewole
Department of Physical Science, Mathematics Programme, Landmark University.
Omu-Aran, Nigeria
ajewale.kahinde@lmu.edu.ng

5th Peter Onu
Department of Mechanical Engineering (Mechatronics), Landmark University.
Omu-Aran, Nigeria
onu.peter@lmu.edu.ng

***Abstract* - In this research, a refined two-stage sampling product estimator for estimating the population mean is introduced, incorporating information from auxiliary attributes. The estimation is carried out in a product form within the context of two-stage sampling. The modified product estimator proposed here is a versatile family of estimators, yielding different results depending on the value of Lambda. Specifically, for the projected estimator, when lambda takes values of 0, 0.5, and 1, the product estimators align with three different estimators respectively. When the auxiliary attribute is treated as a variable, the estimator corresponds to another estimator. In the scenario where lambda is 0 and the auxiliary attribute is a variable, the result is akin to ordinary double sampling. The text further concentrates on the derivation of expressions for Bias and Mean Squared Error of the projected modified product estimator up to the first order of approximation. Through a comprehensive comparison with existing related estimators, both theoretically and empirically, it is demonstrated that the proposed modified product estimator exhibits superior efficiency at the optimal value of Lambda. This research contributes to the field by showcasing the enhanced proficiency of the proposed estimator in comparison to its counterparts.**

Keywords* - *product estimator, bias, ancillary attribute, and two-stage sampling.

I. Introduction

Sample surveys often used supplementary data to advance the accuracy of population parameters' estimators [1-3]. Double sampling was first discovered by [4]. The estimation phase using auxiliary information was discovered by [5] and [6] revealed the basic outcome of double sampling, including the modest regression estimators for this kind of sampling scheme. Double sampling represents a sampling approach that incorporates additional data obtained through a supplementary sampling process. Specifically, this method involves selecting an initial sample of entities (n') to gather supplementary information. Subsequently, a second sample (n) is chosen where the actual study variable(s) is experiential. In many instances, the second sample is a subset of the initial sample used to collect supplementary data. Double sampling is particularly useful in two common scenarios where ancillary information is utilized to enhance the estimation of study variables:

- When the study variable is challenging or expensive to measure, but there exists a more economical associated variable, a strategy is employed where a substantial number of sampling units are initially selected. The less expensive (supplementary) variable is measured on this larger sample, and the study variable is measured only on a subsample (or smaller sample).

- Non-response bias is a prevalent issue in various surveys. Double sampling, in conjunction with stratification principles, can be employed to address non-response concerns by selecting a second sample specifically from the non-respondents.

Utilizing supplementary variable becomes particularly beneficial in enhancing the accuracy of an estimator when the response variable (y) is strongly correlated with an ancillary variable (x). Additionally, circumstances arise where data is accessible in form of attributes or qualitative data (φ) that exhibits a high correlation with y. This underscores the versatility and effectiveness of double sampling in refining the accuracy of estimators in diverse survey and research contexts. For example:

- Persons' Gender and height
- Quantity of milk extracted and a specific type of the cow,

- Total produce of maize crop and a specific variety of maize etc. [7]

Given a population of size N, consider a sample of size n selected via simple random sampling without replacement (SRSWOR). Let y_i and ϕ_i signify the elements in variables y and ϕ, correspondingly, for the i[th] unit (i = 1, 2, …, N). It is presumed that the population exhibits a distinct dichotomy concerning the presence or absence of a characteristic, denoted as ϕ, it is further presumed that the attribute or quality ϕ assumed only two values, 0 and 1.

$$\phi_i = \begin{cases} 1, \text{if unit of the population has the characteristic } \phi \\ 0, \text{otherwise} \end{cases}$$

Consider $a = \sum_i^n \phi_i$ and $A = \sum_i^N \phi_i$ signify the total number of units I sample and population and correspondingly possessing characteristic ϕ. $p = \frac{a}{n}$ and $P = \frac{A}{N}$ signify the fraction of units in the sample and population correspondingly possessing characteristic ϕ [8]

Ancillary information:- in sample survey scheme, facts about the sampling unit which is supplementary to the features under study in the survey is called ancillary information. This information can be qualitative called ancillary characteristics. [9].

II. DEFINITION OF TERMS USED

n^1 = the initia stage sample size

n = the second sample size or the sub − sample size

N = Population size

Y = Response variable

$P^1 = \frac{\sum a^1}{n^1}$ = The initial stage sample proportion of the ancillary characteristic

$p = \frac{\sum a}{n}$ = the second stage sample fraction of the ancillary characteristic

Φ= ancillary characteristic

$\bar{y} = \frac{\sum_{i=1}^n y_i}{n}$ = Second stage mean of Sample of the response variable

a = $\sum_i^n \phi_i$ represent the total number of units in the second sample that possessed the attribute

$a^1 = \sum_i^n \phi_i$ represent the total number of units in the first sample that possessed the characteristic. [9]

III. MATERIAL AND METHODOLOGY

Two-stage sampling for product estimator:- Product estimators, with ancillary information, for calculating the average of a determinate population is well acknowledged. The proficiency of Product estimator is high contingent on whether the ancillary character is strongly negatively correlated with the main attribute of study. This research projected an enhanced modify family of Product estimator in two-stage sampling via auxiliary attribute.

The suggested enhanced family of product estimator is defined as:-

$$\bar{Y}_{adP} = \lambda\bar{y} + (1-\lambda)\,\bar{y}\left(\frac{p}{p^1}\right)$$

Existing Related Product Estimators in two-phase Sampling

Traditional estimator of population mean for (SRSWOR) .

$$\bar{Y}_a = \bar{y} = Sample\ mean \quad (1)$$

$$MSE(\bar{Y}_a) = f_n \bar{Y}^2 C_y^2 \quad (2)$$

Two-stage sampling product estimator with ancillary variable.

Product Estimator:-

$$\bar{Y}_{DSP} = \bar{y}\frac{\bar{x}}{\bar{x}^1} \quad (3)$$

$$B(\bar{Y}_{DSP}) = \bar{Y} f_n c_{yx} \quad (4)$$

$$MSE(\bar{Y}_{DSP}) = \bar{Y}^2(f_n c_y^2 + f_n\,(c_x^2 + 2\rho_{yx} c_y c_x)) \quad (5)$$

[10], proposed two-stage sampling Product estimator via ancillary attribute as:-

$$\bar{Y}_{NGP} = \bar{y}\frac{p}{p^1} \quad (6)$$

$$B(\bar{Y}_{NGP}) = \bar{Y} f_n c_{py} \quad (7)$$

$$MSE(\bar{Y}_{NGP}) = \bar{Y}^2(f_n c_y^2 + f_n\,(c_p^2 + 2\rho_{yp} c_y c_p)) \quad (8)$$

[2], recommended Product exponential estimator with twofold sampling and auxiliary attribute. The estimators were merely an advancement over the exponential estimators proposed by [10] that made use of auxiliary attributes.

$$\bar{Y}_{NSP} = \bar{y}exp^{\left(\frac{p-p^1}{p+p^1}\right)} \quad (9)$$

$$B(\bar{Y}_{NSP}) = \bar{Y}\left(\frac{1}{n} - \frac{1}{n^1}\right)\left[\frac{1}{2}\rho_{py} C_y C_p\right] \quad (10)$$

$$MSE(\bar{Y}_{NSP}) = \bar{Y}^2(f_{nn} c_y^2 + f_{nn^1}\left(\frac{1}{4} c_p^2 + \rho_{yp} c_y c_p\right) \quad (11)$$

[11], recommended employing population percentage of an ancillary character to estimate population mean more accurately.

$$\bar{Y}_{pH} = \bar{y}exp^{\left(\frac{\lambda(p-p^1)}{p+p^1}\right)} \quad (12)$$

$$B(t_{pH}) = \left(\frac{K_p C_p^2}{2}\right)(1 - K_p) \quad (13)$$

$$MSE(\bar{t}_{pH}) = \theta S_y^2(1-\rho_{pb}^2) + \theta^1 \rho_{pb}^2 S_y^2 \quad (14)$$

Where $K_p = \frac{S_{y\phi} p}{S_\phi^2 \bar{y}}$

The limitation of this estimator is that an open range between -1.25 and 2.25 is the estimator's range for alpha

In order to estimate population mean in two-stage sampling, [3] suggested the Product estimator. The exponential dual to product type estimator for finite population mean in two-stage sampling by [12] served as the model for the suggested estimator.

-

$$\bar{Y}_{\mathrm{Sub}P} = \propto \bar{y} + (1-\propto) t_{pe}^{d} \quad (15)$$

$$B(\bar{Y}_{\mathrm{Sub}P}) = \bar{Y}[\tfrac{1}{8} g^2 \gamma C_x^2 - \tfrac{1}{8} g^2 \gamma^{**} C C_x^2](1 - \tfrac{D}{4B}) \quad (16)$$

$$(\bar{Y}_{\mathrm{Sub}P}) = \bar{Y}^2 (\gamma c_y^2 + g\gamma c_x^2 (\tfrac{g}{4} + c) - \Upsilon \tfrac{D^2}{4B})] \quad (17)$$

Where $C = \rho yx \frac{Cy}{Cx}$, $\gamma^{**} = \left(\frac{1}{n} - \frac{1}{n^1}\right)$, $\gamma = \left(\frac{1}{n} - \frac{1}{N}\right)$, $\gamma^* = \left(\frac{1}{n^1} - \frac{1}{N}\right)$, $D = C_x^2(g+2c)$, $B = C_x^2$, $A = C_x^2(g-2c)$.

In this research, the optimum value of the attribute scalar α was gotten as $\alpha = \frac{A}{gB}$ and The outcomes were contrasted with those of Singh and [13] and [14]. As a result, the projected estimate was chosen above the previously stated current estimators.

.

Bias and MSE of the Projected Enhanced Family of Product estimator

The suggested Enhanced Product Estimator $\bar{Y}_{adp}$ is given as :-

$$\bar{Y}_{adP} = \lambda \bar{y} + (1-\lambda)\, \bar{y}(\tfrac{p}{p^1}) \quad (18)$$

The Bias and mean squared error (MSE) of the suggested estimator $\bar{Y}_{adP}$ given the error terms $\bar{y} = \bar{Y}(1+\Delta\bar{y})$, $p^1 = P(1+\Delta p^1)$, $p = P(1+\Delta p)$ in a way that

$$E(\Delta p^1) = 0,\ E(\Delta p) = 0,$$
$$E(\Delta\bar{y}) = 0,\ E(\Delta_{\bar{y}}^2) = (\tfrac{1}{n} - \tfrac{1}{N}) \tfrac{S_y^2}{\bar{Y}^2},$$
$$E(\Delta_{p^1}^2) = (\tfrac{1}{n^1} - \tfrac{1}{N}) \tfrac{S_P^2}{p^2},$$
$$E(\Delta_P^2) = (\tfrac{1}{n} - \tfrac{1}{N}) \tfrac{S_P^2}{P^2},$$
$$E(\Delta\bar{y}\Delta p^1) = (\tfrac{1}{n^1} - \tfrac{1}{N}) \tfrac{S_{yp}}{\bar{Y}\bar{P}},$$
$$E(\Delta\bar{y}\Delta p) = (\tfrac{1}{n} - \tfrac{1}{N}) \tfrac{S_{yp}}{\bar{Y}\bar{P}},$$
$$E(\Delta p \Delta p^1) = (\tfrac{1}{n^1} - \tfrac{1}{N}) \tfrac{S_P^2}{p^2} \quad (19)$$

The proposed Product estimator $\bar{Y}_{adp}$ can be stated in terms of error terms $\Delta\bar{y}$, Δp^1 and Δp as follows, up to the first degree of approximation:

$$\bar{Y}_{adp} = \lambda\bar{Y}(1+\Delta\bar{y}) + (1-\lambda)\bar{Y}(1+\Delta\bar{y}) \frac{P(1+\Delta p)}{P(1+\Delta p^1)} \quad (20)$$

This might be written uniformly as

$$\bar{Y}_{adp} = \lambda\bar{Y}(1+\Delta\bar{y}) + (1-\lambda)\bar{Y}(1+\Delta\bar{y})(1+\Delta p)(1+\Delta p^1)^{-1} \quad (21)$$

$(1 - \Delta p + \Delta_P^2)$ is the expansion of $(1+\Delta p)^{-1}$ using power series up to the first order approximation $0(n^{-1})$.

As a result, equation (2.3) now yields [15]

$$\bar{Y}_{adp} = \bar{Y}[\alpha(1+\Delta\bar{y}) + (1-\alpha)\bar{Y}(1+\Delta\bar{y})(1+\Delta p)\left(1 - \Delta p^1 + \Delta_{p^1}^2\right)] \quad (22)$$

Additional simplification of equation (22)

$$\bar{Y}_{adp} = \bar{Y}[\lambda(1+\Delta\bar{y}) + (1-\lambda)\left(1 - \Delta p^1 + \Delta_{p^1}^2 + \Delta p - \Delta p\Delta p^1 + \Delta\bar{y} - \Delta\bar{y}\Delta p^1 + \Delta\bar{y}\Delta p\right) \quad (23)$$

Limiting equation (23) to the power of two or order two (2)

$$\bar{Y}_{adp} = \bar{Y}[\lambda(1+\Delta\bar{y}) + (1-\lambda)\left(1 - \Delta p^1 + \Delta_{p^1}^2 + \Delta p - \Delta p\Delta p^1 + \Delta\bar{y} - \Delta\bar{y}\Delta p^1 + \Delta\bar{y}\Delta p\right) \quad (24)$$

Bias of the Projected Enhanced Family of Product estimator

$$Bias(\bar{Y}_{adp}) = E(\bar{Y}_{adp} - \bar{Y})$$

$$(\bar{Y}_{adp} - \bar{Y}) = \left\{\bar{Y}\left[\lambda(1+\Delta\bar{y}) + (1-\lambda)\left(1 - \Delta p^1 + \Delta_{p^1}^2 + \Delta p - \Delta p\Delta p^1 + \Delta\bar{y} - \Delta\bar{y}\Delta p^1 + \Delta\bar{y}\Delta p\right)\right] - \bar{Y}\right\} \quad (25)$$

$$(\bar{Y}_{adp} - \bar{Y}) = \bar{Y}\{\left[\lambda(1+\Delta\bar{y}) + (1-\lambda)\left(1 - \Delta p^1 + \Delta_{p^1}^2 + \Delta p - \Delta p\Delta p^1 + \Delta\bar{y} - \Delta\bar{y}\Delta p^1 + \Delta\bar{y}\Delta p\right)\right] - 1\}$$

Obtain the equation expected value:

$$E(\bar{Y}_{adp} - \bar{Y}) = \bar{Y}\{\left[\lambda(E(1) + E(\Delta\bar{y})) + (1-\lambda)(E(1) - E(\Delta p^1) + E(\Delta_{p^1}^2) + E(\Delta p) - E(\Delta p\Delta p^1) + E(\Delta\bar{y}) - E(\Delta\bar{y}\Delta p^1) + E(\Delta\bar{y}\Delta p)\right] - E(1)\} \quad (26)$$

Replace the value of equation (19) in equation (26):

$$E(\bar{Y}_{adp} - \bar{Y}) = \bar{Y}\{\left[\lambda(0+0) + (1-\lambda)(0 - 0 + E(\Delta_{p^1}^2) + 0 - E(\Delta p\Delta p^1) + 0 - E(\Delta\bar{y}\Delta p^1) + E(\Delta\bar{y}\Delta p)\right] - 0\} \quad (27)$$

Adding more implications to equation (27) results in :

$E(\Delta_P^2) = (\frac{1}{n} - \frac{1}{N}) \frac{S_P^2}{P^2}$ $E(\Delta\bar{y}\Delta p^1) = (\frac{1}{n^1} - \frac{1}{N}) \frac{S_{yp}}{\bar{Y}\bar{P}}$, $E(\Delta\bar{y}\Delta p) = (\frac{1}{n} - \frac{1}{N}) \frac{S_{yp}}{\bar{Y}\bar{P}}$ and $E(\Delta p\Delta p^1) = (\frac{1}{n^1} - \frac{1}{N}) \frac{S_P^2}{p^2}$

$$E(\bar{Y}_{adp} - \bar{Y}) = \bar{Y}\left[(1-\lambda)\left(\left(\frac{1}{n^1} - \frac{1}{N}\right)\frac{S_P^2}{p^2} - \left(\frac{1}{n^1} - \frac{1}{N}\right)\frac{S_P^2}{p^2} - \left(\frac{1}{n^1} - \frac{1}{N}\right)\frac{S_{yp}}{\bar{Y}\bar{P}} + \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_{yp}}{\bar{Y}\bar{P}}\right)\right] \quad (28)$$

Collecting the like terms,

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)=\bar{Y}\left[(1-\lambda)\left(\left(\frac{1}{n}-\frac{1}{N}-\frac{1}{n^1}+\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}\right)\right] \quad (29)$$

Additional simplification gives

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)=\bar{Y}\left[(1-\lambda)\left(\left(\frac{1}{n}-\frac{1}{n^1}\right)\frac{S_{yp}}{\overline{YP}}\right)\right] \quad (30)$$

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)=\bar{Y}\left[(1-\lambda)\left(\frac{1}{n}-\frac{1}{n^1}\right)\left(C_{yp}\right)\right] \quad (31)$$

The Bias connoted by $Bias(\bar{Y}_{adP})$ of the suggested Product estimator $(\bar{Y}_{adP})$ is

$$Bias\left(\bar{Y}_{adp}\right)=\bar{Y}(1-\lambda)\left(\frac{1}{n}-\frac{1}{n^1}\right)C_{yp} \quad (32)$$

MSE of the suggested family of Product estimator $(\bar{Y}_{adP})$
To get mean squared error (MSE) of the suggested Product estimator $\bar{Y}_{adP}$, replicate the stages of equations (20), (21), (22), (23), (24) and (25).

$$\left(\bar{Y}_{adp}-\bar{Y}\right)=\bar{Y}\{\left[\lambda(1+\Delta\bar{y})+(1-\lambda)\left(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p\right)\right]-1\} \quad (33)$$

$$MSE\left(\bar{Y}_{adp}\right)=E(\bar{Y}_{adp}-\bar{Y})^2$$

Square both sides of equation (33)

$$\left\{\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2(1+\Delta\bar{y})^2+2\lambda(1-\lambda)(1+\Delta\bar{y})\left(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p\right)-2\lambda(1+\Delta\bar{y})-2(1-\lambda)\left(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p\right)+(1-\lambda)^2\left(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p\right)^2-1\right\} \quad (34)$$

Expand equation (34) and limit the equation to order two

$$\left\{\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2(1+2\Delta\bar{y}+\Delta_{\bar{y}}^2)+2\lambda(1-\alpha)(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p+\Delta_{\bar{y}}^2)-2\lambda(1+\Delta\bar{y})-2(1-\lambda)\left(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p\right))+(1-\lambda)^2(1-\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p-\Delta p^1+\Delta_{p^1}^2-\Delta p\Delta p^1-\Delta\bar{y}\Delta p^1+\Delta_{p^1}^2+\Delta p-\Delta p\Delta p^1+\Delta_P^2+\Delta p\Delta\bar{y}-\Delta p\Delta p^1+\Delta\bar{y}-\Delta\bar{y}\Delta p^1+\Delta p\Delta\bar{y}+\Delta_{\bar{y}}^2-\Delta\bar{y}\Delta p^1+\Delta\bar{y}\Delta p)-1]\right\} \quad (35)$$

Take the expectation of equation (35) above

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2(E(1)+2E(\Delta\bar{y})+E(\Delta_{\bar{y}}^2))+2\lambda(1-\lambda)(E(1)-E(\Delta p^1)+E(\Delta_{p^1}^2)+E(\Delta p)-E(\Delta p\Delta p^1)-2E(\Delta\bar{y}\Delta p^1)+2E(\Delta\bar{y}\Delta p)+2E(\Delta\bar{y})+E(\Delta_{\bar{y}}^2))-2\lambda E\left(E(1)+E(\Delta\bar{y})\right)-2(1-\lambda)(E(1)-E(\Delta p^1)+E(\Delta_{p^1}^2)+E(\Delta p)-E(\Delta p\Delta p^1)+E(\Delta\bar{y})-E(\Delta\bar{y}\Delta p^1)+E(\Delta\bar{y}\Delta p)+(1-\lambda)^2(E(1)-E(\Delta p^1)+E\left(\Delta_{p^1}^2\right)+E(\Delta p)-E(\Delta p\Delta p^1)+E(\Delta\bar{y})-E(\Delta\bar{y}\Delta p^1)+E(\Delta\bar{y}\Delta p)-E(\Delta p^1)+E(\Delta_{p^1}^2)-E(\Delta p\Delta p^1)-E(\Delta\bar{y}\Delta p^1)+E(\Delta_{p^1}^2)+E(\Delta p)-E(\Delta p\Delta p^1)+E(\Delta_P^2)+E(\Delta p\Delta\bar{y})-E(\Delta p\Delta p^1)+E(\Delta\bar{y})-E(\Delta\bar{y}\Delta p^1)+E(\Delta p\Delta\bar{y})+E(\Delta_{\bar{y}}^2)-E(\Delta\bar{y}\Delta p^1)+E(\Delta\bar{y}\Delta p)) \quad (36)$$

substitute for equation (19) in equation (36)

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2((0+0+E(\Delta_{\bar{y}}^2))+2\lambda(1-\lambda)(0-0+E(\Delta_{p^1}^2)+0-E(\Delta p\Delta p^1)-2E(\Delta\bar{y}\Delta p^1)+2E(\Delta\bar{y}\Delta p)+0+E(\Delta_{\bar{y}}^2)-2(1-\lambda)(0-0+E(\Delta_{p^1}^2)+0-E(\Delta p\Delta p^1)+0-E(\Delta\bar{y}\Delta p^1)+E(\Delta\bar{y}\Delta p))+(1-\alpha)^2(3E\left(\Delta_{p^1}^2\right)-4E(\Delta p\Delta p^1)-4E(\Delta\bar{y}\Delta p^1)+E(\Delta_P^2)+4E(\Delta\bar{y}\Delta p)+E(\Delta_{\bar{y}}^2)) \quad (37)$$

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2(E\left(\Delta_{\bar{y}}^2\right))+2\lambda(1-\lambda)(E(\Delta_{p^1}^2)-E(\Delta p\Delta p^1)-2E(\Delta\bar{y}\Delta p^1)+2E(\Delta\bar{y}\Delta p)+E(\Delta_{\bar{y}}^2)-2(1-\lambda)(E(\Delta_{p^1}^2)-E(\Delta p\Delta p^1)-E(\Delta\bar{y}\Delta p^1)+E(\Delta\bar{y}\Delta p))+(1-\alpha)^2(3E\left(\Delta_{p^1}^2\right)-4E(\Delta p\Delta p^1)-4E(\Delta\bar{y}\Delta p^1)+E(\Delta_P^2)+4E(\Delta\bar{y}\Delta p)+E(\Delta_{\bar{y}}^2)) \quad (38)$$

Replace equation (19) in equation (38)

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{\bar{y}}^2}{\bar{Y}^2}+2\lambda(1-\lambda)(\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_P^2}{P^2}-\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_P^2}{p^2}-2\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+2\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{\bar{y}}^2}{\bar{Y}^2})-2(1-\lambda)(\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_P^2}{P^2}-\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_P^2}{p^2}-\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+(1-\lambda)^2(3\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_P^2}{P^2}-4\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_P^2}{p^2}-4\left(\frac{1}{n^1}-\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+4\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_P^2}{p^2}+\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{\bar{y}}^2}{\bar{Y}^2})] \quad (39)$$

Collect the like terms in equation (39)

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{\bar{y}}^2}{\bar{Y}^2}+2\lambda(1-\lambda)(\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{\bar{y}}^2}{\bar{Y}^2}+2\left(\frac{1}{n}-\frac{1}{N}-\frac{1}{n^1}+\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}})-2(1-\lambda)\left(\frac{1}{n}-\frac{1}{N}-\frac{1}{n^1}+\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}})+(1-\lambda)^2(\left(\frac{1}{n}-\frac{1}{N}-\frac{1}{n^1}+\frac{1}{N}\right)\frac{S_P^2}{p^2}-4\left(\frac{1}{n}-\frac{1}{N}-\frac{1}{n^1}+\frac{1}{N}\right)\frac{S_{yp}}{\overline{YP}}+\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_{\bar{y}}^2}{\bar{Y}^2})] \quad (40)$$

Further simplify equation (40)

$$E\left(\bar{Y}_{adp}-\bar{Y}\right)^2=\bar{Y}^2[\lambda^2\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2+2\lambda(1-\lambda)(\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2+2\left(\frac{1}{n}-\frac{1}{n^1}\right)C_{yp})-2(1-\lambda)\left(\frac{1}{n}-\frac{1}{n^1}\right)C_{yp})+(1-\lambda)^2(\left(\frac{1}{n}-\right.$$

$\frac{1}{n^1}\Big) C_p^2 - 4\left(\frac{1}{n}-\frac{1}{n^1}\right) C_{yp} + (\frac{1}{n}-\frac{1}{N})C_y^2)]$ (41)

$\mathrm{MSE}(\bar{Y}_{adp}) = \mathrm{E}(\bar{\mathrm{Y}}_{\mathrm{adp}} - \bar{\mathrm{Y}})^2 = \lambda^2\left(\frac{1}{n}-\frac{1}{N}\right)\frac{S_y^2}{\bar{Y}^2} + 2\lambda(1-\lambda)((\frac{1}{n} - \frac{1}{N}\Big) C_y^2 + 2(\frac{1}{n}-\frac{1}{n^1})C_{yp}) \quad -2(1-\lambda)((\frac{1}{n} - \frac{1}{n^1}\Big) C_{yp}) + (1-\lambda)^2((\frac{1}{n}-\frac{1}{n^1}\Big) C_p^2 - 4\left(\frac{1}{n}-\frac{1}{n^1}\right) C_{yp} + (\frac{1}{n} - \frac{1}{N})C_y^2))$ (42)

Mean Squared Error (MSE) of the suggested Product estimator ($\bar{Y}_{adP}$) signified $MSE(\bar{Y}_{adP})$ was obtained as :-

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2[\left(\frac{1}{n}-\frac{1}{N}\right) C_y^2 + (1-\lambda)\left(\frac{1}{n}-\frac{1}{n^1}\right)((1-\lambda)C_p^2 + 2C_{yp})] \quad (43)$$

By applying the maxima-minima method to minimize Mean squared error $MSE(\bar{Y}_{adP})$, the most advantageous value of the described scalar λ is achieved.

Extend equation (43)

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2[\left(\frac{1}{n}-\frac{1}{N}\right) C_y^2 + (1-\lambda)^2\left(\frac{1}{n}-\frac{1}{n^1}\right) C_p^2 + 2(1-\lambda)\left(\frac{1}{n}-\frac{1}{n^1}\right) C_{yp}]$$

Where $F_{nn^1} = \left(\frac{1}{n}-\frac{1}{n^1}\right)$, $\mathrm{F_n} = \left(\frac{1}{n}-\frac{1}{N}\right)$

$$MSE(\bar{Y}_{adR}) = \bar{\mathrm{Y}}^2[\mathrm{F_n}C_y^2 + (1-\alpha)^2 F_{nn^1}C_p^2 - 2(1-\alpha)F_{nn^1}C_{yp}] \quad (44)$$

Obtain the derivative of $MSE(\bar{Y}_{adP})$ with respect to λ [16]

$$\frac{\Delta(MSE(\bar{Y}_{adp}))}{\Delta\lambda} = 2(1-\lambda)(-1)F_{nn^1}C_p^2 + 2(-1)F_{nn^1}C_{yp} \quad (45)$$

Limiting $\frac{\Delta(MSE(\bar{Y}_{adp}))}{\Delta\lambda}=0$

$$2(1-\lambda)(-1)F_{nn^1}C_p^2 + 2(-1)F_{nn^1}C_{yp}=0 \quad (46)$$

$$-2(1-\lambda)F_{nn^1}C_p^2 - 2F_{nn^1}C_{yp}=0 \quad (47)$$

$$-2(1-\lambda)F_{nn^1}C_p^2 = 2F_{nn^1}C_{yp}$$

Divide both sides by $-2F_{nn^1}C_p^2$

$$\frac{-2(1-\lambda)F_{nn^1}C_p^2}{-2F_{nn^1}C_p^2} = \frac{2F_{nn^1}C_{yp}}{-2F_{nn^1}C_p^2}$$

$$(1-\lambda) = -\frac{C_{yp}}{C_p^2} \quad (48)$$

The optimum value of the characterized scalar [9] is

$$\lambda = 1 + \frac{C_{yp}}{C_p^2} = \frac{C_p^2 + C_{yp}}{C_p^2} \quad (49)$$

Substitute for optimum lambda λ_{opt} in equation (32), where $F_{nn^1} = \left(\frac{1}{n}-\frac{1}{n^1}\right)$, $\mathrm{F_n} = \left(\frac{1}{n}-\frac{1}{N}\right)$

$$\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adp}}) = \bar{\mathrm{Y}}(1-\lambda)F_{nn^1}\mathrm{C_{yp}} \quad (50)$$

$$\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adp}}) = \bar{\mathrm{Y}}(-\frac{C_{yp}}{C_p^2})F_{nn^1}\mathrm{C_{yp}} \quad (51)$$

Simplifying equation (51)

$$\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adp}}) = \bar{\mathrm{Y}}F_{nn^1}(-\frac{\mathrm{C_{yp}}^2}{C_p^2}) \quad (52)$$

Further simplification of equation (52)

$$\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adp}}) = \bar{\mathrm{Y}}F_{nn^1(-}\rho_{\mathrm{yp}}^2\mathrm{C_y^2}) \quad (53)$$

The minimum $\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adP}})$ written as $\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adP}})_{\min}$ is obtained as

$$\mathrm{Bias}(\bar{\mathrm{Y}}_{\mathrm{adp}}) = -\bar{\mathrm{Y}}F_{nn^1}\rho_{\mathrm{yp}}^2\mathrm{C_y^2} \quad (54)$$

Substitute for optimum lambda λ_{opt} in equation (43) to obtain the minimum $\mathrm{MSE}(\bar{\mathrm{Y}}_{\mathrm{adP}})$ written as $\mathrm{MSE}(\bar{\mathrm{Y}}_{\mathrm{adP}})_{\min}$ where $F_{nn^1} = \left(\frac{1}{n}-\frac{1}{n^1}\right)$, $\mathrm{F_n} = \left(\frac{1}{n}-\frac{1}{N}\right)$

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2[\mathrm{F_n}C_y^2 + (1-\lambda)^2F_{nn^1}C_p^2 + 2(1-\lambda)F_{nn^1}C_{yp}] \quad (55)$$

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2[\mathrm{F_n}C_y^2 + \left(-\frac{C_{yp}}{C_p^2}\right)^2 F_{nn^1}C_p^2 + 2\left(-\frac{C_{yp}}{C_p^2}\right)F_{nn^1}C_{yp}] \quad (56)$$

Simplify equation (56)

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2[\mathrm{F_n}C_y^2 + \frac{{C_{yp}}^2}{C_p^2}F_{nn^1} - 2\left(\frac{{C_{yp}}^2}{C_p^2}\right)F_{nn^1}] \quad (57)$$

Further simplification of equation (57)

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2[\mathrm{F_n}C_y^2 - \left(\frac{{C_{yp}}^2}{C_p^2}\right)F_{nn^1}] \quad (58)$$

The minimum $\mathrm{MSE}(\bar{\mathrm{Y}}_{\mathrm{adP}})$ written as $\mathrm{MSE}(\bar{\mathrm{Y}}_{\mathrm{adP}})_{\min}$ is obtained as

$$MSE(\bar{Y}_{adp}) = \bar{\mathrm{Y}}^2 C_y^2[\mathrm{F_n} - \rho_{\mathrm{yp}}^2F_{nn^1}] \quad (59)$$

where $\rho_{\mathrm{yp}}^2 = \frac{{C_{yp}}^2}{C_y^2\,C_p^2}$, $C_y^2\rho_{\mathrm{yp}}^2 = \frac{{C_{yp}}^2}{C_p^2}$ [9].

IV. EFFECTIVENESS AND EXPERIMENTAL EVALUATION

We used the following two populations to determine the effectiveness of the proposed Product estimator with various current population mean estimators in two-stage sampling.

Population 1 [17]

Y=Area under non-Agriculture, X= permanent Pasture, ϕ =Area of Barren and un-Cultivated Land greater than 2

$\bar{Y}$ =2.725, $\bar{X}$ =3.37, P=0.75, C_x =0.6751, C_y =0.09456, C_p=1.2405, ρ_{yx}= -0.2246, ρ_{yp}= -0.3305, ρ_{xp}= -0.1858, n=15, n^1=25, N=40

Population 2 [18]

Y=Birth Weight, X= Placenta Weight, ϕ =Gender

$\bar{Y}$ =2.8828, $\bar{X}$ =0.6017, P=0.5, C_x =0.2893, C_y =0.1935, C_p=0.4989, ρ_{yx}= 0.1866, ρ_{yp}= -0.21501, ρ_{xp}= -0.41696, n=30, n^1=40, N=60

Table 1: Theoretical and Empirical Conditions for Efficiency of $\bar{Y}_{adP}$ on
$\bar{Y}_{srs}$, $\bar{Y}_{DSp}$, $\bar{Y}_{NG}$, $\bar{Y}_{NS}$, $\bar{Y}_{SS}$, and $\bar{Y}_{Sub}$ using population 1, n=15, n^1=25, N=40

S/No	Author	Estimator	Theoretical or Mathematical Condition	Empirical Condition
1	Conventional Mean	$\bar{Y}_{srs}$	$\rho_{yp}^2 \geq 0$	$0.1092 > 0$
2	Conventional Two-Stage estimator	$\bar{Y}_{DSP}$	$\rho_{yp}^2 \geq \frac{-(2C_{yx} + C_x^2)}{C_y^2}$	$0.1092 > -47.7633$
3	Naik, Gupta (1996)	$\bar{Y}_{NG}$	$\rho_{yp} \geq -\frac{C_p}{C_y}$	$-0.3305 > -25.3179$
4	Nirmala Sawan (2010)	$\bar{Y}_{NS}$	$\rho_{yp} \geq -\frac{C_p}{2C_y}$	$-0.3305 > -12.6589$
5	Subhash et al (2016)	$\bar{Y}_{Sub}$	$\rho_{yp}^2 \geq \frac{D^2 - gC_x^2B(g-4c)}{4BC_y^2}$	$0.1092 > 0.05044$

From Table 1, all the stated conditions are met.:

Table 2: **Bias, Mean Squared Error and Percentage Relative Efficiency of $\bar{Y}_{adP}$, $\bar{Y}_{srs}$, $\bar{Y}_{DSP}$, $\bar{Y}_{NG}$, $\bar{Y}_{NS}$, and $\bar{Y}_{Sub}$ for Population 1 where ρ < 0**

S/No	Author	Estimator	Bias	Mean Square Error (MSE)	Percentage Relative Efficiency (PRE)
1	Conventional Mean	$\bar{Y}_{srs}$		0.002767	100
2	Conventional Two-Stage estimator	$\bar{Y}_{DSP}$	-0.00163	0.0874	3.2
3	Naik, Gupta (1996)	$\bar{Y}_{NG}$	-0.9358	190.722	0.25
4	Nirmala Sawan (2010)	$\bar{Y}_{NS}$	-0.4679	46.7627	1
5	Subhash *et al* (2016)	$\bar{Y}_{Sub}$	0.03728	0.4607	103
6	Modified Product Estimator	$\bar{Y}_{adP}$	-7.097E-05	0.002573	108

From Table 2:

$$MSE(\bar{Y}_{adP}) << MSE(\bar{Y}_{DSP}) < MSE(\bar{Y}_{Sub}) < MSE(\bar{Y}_{srs}) < MSE(\bar{Y}_{NS}) < MSE(\bar{Y}_{NG})$$
$$PRE(\bar{Y}_{adP}) > PRE(\bar{Y}_{DSP}) > PRE(\bar{Y}_{Sub}) > PRE(\bar{Y}_{srs}) > PRE(\bar{Y}_{NS}) > PRE(\bar{Y}_{NG})$$
$$Bias(\bar{Y}_{adP}) < Bias(\bar{Y}_{DSP}) < Bias(\bar{Y}_{Sub}) < Bias(\bar{Y}_{NS}) < Bias(\bar{Y}_{NG})$$

Table 3: Theoretical and Empirical Conditions for Efficiency of $\bar{Y}_{adP}$ on $\bar{Y}_{srs}$, $\bar{Y}_{DSR}$, $\bar{Y}_{NG}$, $\bar{Y}_{NS}$, and $\bar{Y}_{Sub}$ using population 2, n=30, n^1=40, N=60

S/No	Author	Estimator	Theoretical or Mathematical Condition	Empirical Condition
1	Conventional Mean	$\bar{Y}_{srs}$	$\rho_{yp}^2 \geq 0$	$0.04623 > 0$
2	Conventional Two-Stage estimator	$\bar{Y}_{DSP}$	$\rho_{yp}^2 \geq \frac{-(2C_{yx} + C_x^2)}{C_y^2}$	$0.0462 > -2.7930$
3	Naik, Gupta (1996)	$\bar{Y}_{NG}$	$\rho_{yp} \geq -\frac{C_p}{C_y}$	$-0.21501 > -2.5783$
4	Nirmala Sawan (2010)	$\bar{Y}_{NS}$	$\rho_{yp} \geq -\frac{C_p}{2C_y}$	$-0.21501 > -1.2891$
5	Subhash et al (2016)	$\bar{Y}_{Sub}$	$\rho_{yp}^2 \geq \frac{D^2 - gC_x^2 B(g-4c)}{4BC_y^2}$	$0.04623 > 0.034697$

From Table 3, all the stated conditions are met.

Table 4: Bias, Mean Squared Error and Percentage Relative Efficiency of $\bar{Y}_{adP}$, $\bar{Y}_{srs}$, $\bar{Y}_{DSR}$, $\bar{Y}_{NG}$, $\bar{Y}_{NS}$, and $\bar{Y}_{Sub}$ for Population 2 where ρ < 0

S/No	Author	Estimator	Bias	Mean Square Error (MSE)	Percentage Relative Efficiency (PRE)
1	Conventional Mean	$\bar{Y}_{srs}$		0.005184	100
2	Conventional Two-Stage estimator	$\bar{Y}_{DSP}$	0.0005026	0.01241	42
3	Naik, Gupta (1996)	$\bar{Y}_{NG}$	-0.0004983	0.01954	27
4	Nirmala Sawan (2010)	$\bar{Y}_{NS}$	-0.00024922	0.008055	64
5	Subhash *et al* (2016)	$\bar{Y}_{Sub}$	-0.002102	0.007263	71
6	Modified Product Estimator	$\bar{Y}_{adP}$	-0.00004154	0.005064	102

From Table 4,

$$MSE(\bar{Y}_{adP}) < MSE(\bar{Y}_{srs}) < \text{MSE}(\bar{Y}_{Sub}) < MSE(\bar{Y}_{NS}) < MSE(\bar{Y}_{DSP}) < MSE(\bar{Y}_{NG})$$
$$\text{PRE}(\bar{Y}_{adP}) > PRE(\bar{Y}_{srs}) > \text{PRE}(\bar{Y}_{Sub}) > PRE(\bar{Y}_{NS}) > PRE(\bar{Y}_{DSP}) > PRE(\bar{Y}_{NG})$$
$$Bias(\bar{Y}_{adP}) < Bias(\bar{Y}_{NS}) < Bias(\bar{Y}_{NG}) < Bias(\bar{Y}_{DSP}) < \text{Bias}(\bar{Y}_{Sub})$$

V. CONCLUSION

In conclusion, the use of auxiliary attributes in a suggested product estimator in double sampling proves to be a superior approach, outperforming existing estimators [19], [20]. This innovative method not only enhances precision but also demonstrates a clear advantage in accuracy, making it a valuable and efficient tool for product estimation. The combination of double sampling and auxiliary attributes stands as a robust solution, paving the way for more reliable and advanced estimations in comparison to conventional methods.

REFERENCES

[1] V. D Naik, and P. C. Gupta. A note on estimation of mean with known population proportion of an auxiliary character.Jour. Ind. Soc. Agr. Stat., 48(2), 151-158. 1996.

[2]. S. Nirmala. Suggested ratio-product type exponential estimator for estimating finite population mean using information on auxiliary attribute. *Pak. Sta. Oper. Res.* 5.47 53. 2010.

[3]. K. Y. Subhash, G. Sat G., S. S. Mishra and Shukla A. K. Modified ratio and Product Estimators for Estimating Population Mean in Two-Phase Sampling. *American Journal Of OpeProductnal Research.* 6(3): 61-68, 2016.

[4] J. Nayman,. Contribution to the theory of sampling human populations. *Jour. Amer. Stat. assoc.* 33, 101-116. *1938.*

[5] W. G. Cochran. - The estimation of the yields of cereal experiments by sampling for the Product of grain to total produce, *J. Agric. Sci.,* 30, 262–275, 1940.

[6] W. G. Cochran W. G. Sampling techniques, 3rd Edition. *John Wiley, New York*. 1977.

[7] H. S. Jhajj., M. K. Sharma, and L. K. Grover. A family of estimators of population means using information on auxiliary attribues, *Pakistan Journal of Statistics,* 22(1), 43 50. 2006

[8] N. Singh, P. Chauhan, and N. Sawan. On linear combination of ratio and product type exponential estimator for estimating the finite population mean. *Statistics in Transition New Series, 9*(1), 105-115. 2008

[9]. F. E. Amoyedo, and A. A. Adewara., Improved modifiedcratio estimator for estimating population mean in double sampling using information on auxiliary attribute. *Annals of Computer Science*. 17(2). 121-128. 2019.

[10]. S. Bahl ., and R. K. Tuteja product type exponential estimator*, Inform. and Optimize. Science*, 12(1), 159-163. 1991.

[11]. R. S. Solanki and H. P. .Singh.. Improved estimation of population mean using population proportion of an auxiliary character, Chilean *Journal of Statistics*, 1-16. 2013

[12].B. K. Singh, S. Choudhury and D. Kalita. A class of exponential chain ratio-product type estimator with two auxiliary variables under double sampling scheme. *Electronic Journal of Applied Statistical Analysis*, 2013.

[13]. H. P Singh., and G. .Vishwakarma G. (2007) Modified exponential Ratio and product estimatrs of finite population mean in Double sampling, *Aust. J. of Statist.,* 36(3), 217-225. 2007.

[14]. S. Choudhury and B. K Singh**.** A Class of Product-Cum-Dual to Product Estimators of the Population Mean In Survey Sampling using Auxiliary Information*, Asian J Math Stat,* 12(3). 9. 2012.

[15]. T. G. Shaba, B. Aladeitan, and S. E. Fadugba. Fekete-Szegö problem and second Hankel determinant for a class of τ-pseudo bi-univalent functions involving Euler polynomials. *Asia Pac J Math*. 2023 *10*,-16.

[16]. S. E. Fadugba, V. Shaalini, J. T. Okunlola, S. Olukayode, and F. A. Olanrewaju. Development and Analysis of a New Third Order Method for Solving Initial Value Problems in Ordinary Differential Equations. Development, 1(2), 2013. 130-135. [17]. Census of India (2011) Series 22, Part Xii-A Page 80-209

[18]. University of Ilorin Teaching Hospital, Ilorin Kwara State. (2018) Life data on child delivery

[19]. P. Onu, A. Pradhan, and C. Mbohwa, "Potential to use metaverse for future teaching and learning," *Educ Inf Technol (Dordr)*, 2023, doi: 10.1007/s10639-023-12167-9.

[20]. T. G. Shaba, S. Araci, B. O. Adebesin, F. Tchier, S. Zainab, and B. Khan. Sharp bounds of the Fekete–Szegö problem and second hankel determinant for certain bi-Univalent Functions Defined by a novel q-differential Operator associated with q-limaçon domain. *Fractal and Fractional*, 2023. *7*(7), 506