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Coefficients Bounds for Certain New Subclasses of Meromorphic Bi-univalent Functions Associated with Al-Oboudi Differential Operator

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Abstract In this paper, we introduce two interesting subclasses of meromorphic bi-univalent functions defined by Al-Oboudi differential operator. Estimates for the initial coefficients $|c_0|$, $|c_1|$ and $|c_2|$ are obtained for the functions in these new subclasses.

1 Introduction

Let $\mathcal{A} = \{f : \mathcal{U} \to \mathcal{C} : f$ is analytic in $\mathcal{U}, f(0) = 0 = f'(0) - 1\}$ be the class of functions of the form

$$f(z) = z + \sum_{\nu=2}^{\infty} b_\nu z^\nu \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions f univalent in $\mathcal{U} = \{z : z \in \mathcal{C}, |z| < 1\}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not to be defined on the entire unit disk $\mathcal{U}$. In fact, the Koebe one-quarter theorem [11] ensures that the image of $\mathcal{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \qquad (z \in \mathcal{U}),$$

and

$$f(f^{-1}(w)) = w, \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_3^3 - 5b_2 b_3 + b_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). For a short history and fascinating examples of functions in the class Σ, see [38] (see also [7, 8]). In fact, the aforecited work of Srivastava et al. [38] essentially revived the investigation of numerous subclasses of bi-univalent function class Σ in recent years; it was followed by such works as those by Murugusundaramoorthy et al. [21], Çaglar et al. [10], Frasin and Aouf [12], and others (for more details see; [20], [40], [6], [30], [2], [21], [22], [31], [41], [17], [37], [16], [34], [32], [25], [33]).

In this research, the concept of bi-univalency is extended to the class of meromorphic function defined on

$$\mathcal{U}^* = \{z : z \in \mathcal{C}, 1 < |z| < \infty\}.$$

Let Σ' denote the class of all meromorphic univalent functions h of the form:

$$h(z) = z + c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu}{z^\nu}, \tag{1.3}$$

defined on the domain $\mathcal{U}^*$. Since $h \in \Sigma'$ is univalent, it has an inverse denoted by $h^{-1} = l$ that satisfies the following condition:

$$h^{-1}(h(z)) = z, \qquad (z \in \mathcal{U}^*)$$

and

$$h(h^{-1}(w)) = w, \qquad (M < |w| < \infty;\ M > 0).$$

Furthermore, the inverse function $h^{-1} = l$ is of the form:

$$h^{-1}(w) = l(w) = w + \mathcal{D}_0 + \sum_{\nu=1}^{\infty} \frac{\mathcal{D}_\nu}{w^\nu}, \qquad (M < |w| < \infty). \tag{1.4}$$

A simple computation shows that

$$w = h(l(w)) = (c_0 + \mathcal{D}_0) + w + \frac{c_1 + \mathcal{D}_1}{w} + \frac{\mathcal{D}_2 - c_1\mathcal{D}_0 + c_2}{w^2} + \frac{\mathcal{D}_3 - c_1\mathcal{D}_1 + c_1\mathcal{D}_0^2 - 2c_2\mathcal{D}_0 + c_3}{w^3} + \cdots. \tag{1.5}$$

Comparing the initial coefficients in (1.5), we get

$$c_0 + \mathcal{D}_0 = 0 \quad \Longrightarrow \quad \mathcal{D}_0 = -c_0$$

$$c_1 + \mathcal{D}_1 = 0 \quad \Longrightarrow \quad \mathcal{D}_1 = -c_1$$

$$D_2 - c_1\mathcal{D}_0 + c_2 = 0 \quad \Longrightarrow \quad \mathcal{D}_2 = -(c_2 + c_0c_1)$$

$$\mathcal{D}_3 - c_1\mathcal{D}_1 + c_1\mathcal{D}_0^2 - 2c_2\mathcal{D}_0 + c_3 = 0 \quad \Longrightarrow \quad \mathcal{D}_3 = -(c_3 + 2c_0c_2 + c_0^2c_1 + c_1^2).$$

By inserting these values in (1.4), we have

$$h^{-1}(w) = l(w) = w - c_0 - \frac{c_1}{w} - \frac{c_2 + c_0c_1}{w^2} - \frac{c_3 + 2c_0c_2 + c_0^2c_1 + c_1^2}{w^3} + \cdots. \tag{1.6}$$

The coefficient problem was studied for numerous interesting subclasses of the meromorphic univalent functions (see, e.g., [1, 13, 14, 15, 9, 23, 3, 36, 24]).

Analogous to the bi-univalent holomorphic functions, a function $h \in \Sigma'$ is said to be meromorphic bi-univalent if $h^{-1} \in \Sigma'$. We denote the family of all meromorphic bi-univalent functions by $\mathcal{W}_{\Sigma'}$. Estimates on the coefficients of meromorphic univalent functions were widely worked on in the literature, for example, Schiffer [28] obtained the estimates $|c_2| \leq \frac{2}{3}$ for meromorphic univalent functions $h \in \Sigma'$ with $c_0 = 0$ and Duren [11] gave an elementary proof of the inequality $|c_\nu| \leq \frac{2}{\nu+1}$ on the coefficient of meromorphic univalent functions $h \in \Sigma'$ with $c_k = 0$ for $1 \leq k < \frac{\nu}{2}$. For the coefficient of the inverse of meromorphic univalent functions $l \in \mathcal{W}_{\Sigma'}$, Springer [35] used variational methods to prove that

$$|\mathcal{D}_3 + \frac{1}{2}\mathcal{D}_1^2| \leq \frac{1}{2} \ \textit{and}\ |\mathcal{D}_3| \leq 1$$

and conjecture that

$$|\mathcal{D}_{2\nu-1}| \leq \frac{(2\nu-2)!}{\nu!(\nu-1)!}, \qquad (\nu = 1, 2, \cdots).$$

In 1977, Kubota [19] has proved that Springer [35] conjecture is true for $\nu = 3, 4, 5$ and subsequently Schober [29] obtained a sharp bounds for the coefficients $\mathcal{D}_{2\nu-1}, 1 \leq \nu \leq 7$ of the inverse of meromorphic univalent functions in $\mathcal{U}^*$. Also recently, Kapoor and Mishra [18] (also see [39]) found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order α in $\mathcal{U}^*$.

A function h in the class $\mathcal{W}_{\Sigma'}$ is said to be meromorphic bi-univalent starlike of order η where $0 \leq \eta < 1$, if it satisfies the following inequalities

$$\Re\left(\frac{zh'(z)}{h(z)}\right) > \eta \qquad \textit{and} \qquad \Re\left(\frac{wl'(w)}{l(w)}\right) > \eta \quad (z, w \in \mathcal{U}^*),$$

where l is the inverse of h given by (1.6). We denote by $\mathcal{W}^*_{\Sigma'}(\eta)$ the class of all meromorphic bi-univalent starlike functions of order η. Similarly, a function h in the class $\mathcal{W}_{\Sigma'}$ is said to be meromorphic bi-univalent strongly starlike of order ξ where $0 < \xi \leq 1$, if it satisfies the following conditions

$$\left|\arg\left(\frac{zh'(z)}{h(z)}\right)\right| < \frac{\xi\pi}{2} \quad and \quad \left|\arg\left(\frac{wl'(w)}{l(w)}\right)\right| < \frac{\xi\pi}{2} \quad (z, w \in \mathcal{U}^*),$$

where l is the inverse of h given by (1.6). We denote by $\mathcal{W}^*_{\Sigma'}(\xi)$ the class of all meromorphic bi-univalent strongly starlike functions of order ξ. The classes $\mathcal{W}^*_{\Sigma'}(\eta)$ and $\mathcal{W}^*_{\Sigma'}(\xi)$ were introduced and studied by Halim et al. [14].

For $f \in \mathcal{A}$, Al-Oboudi [4] introduced the following differential operator:

$$D^0_\zeta f(z) = f(z),$$

$$D^1_\zeta f(z) = (1-\zeta)f(z) + \zeta z f'(z) = D_\zeta f(z); \qquad (\zeta \geq 0) \tag{1.7}$$

$$D^n_\zeta f(z) = D_\zeta(D^{n-1}_\zeta f(z)); \qquad (n \in \mathfrak{N} = \{1, 2, 3, \cdots\}). \tag{1.8}$$

If f is given by (1.1), then from (1.7) and (1.8) we get,

$$D^n_\zeta f(z) = z + \sum_{\nu=2}^{\infty}[1 + (\nu-1)\zeta]^n b_\nu z^\nu; \qquad (n \in \mathfrak{N}_0 = \{0, 1, 2, 3, \cdots\}). \tag{1.9}$$

Also, when $\zeta = 0$ we have the Salagean differential operator [27].
Similarly, for $h \in \Sigma'$ as given in (1.3), Al-Oboudi differential operator can be defined as:

$$D^0_\zeta h(z) = h(z),$$

$$D^1_\zeta h(z) = (1-\zeta)h(z) + \zeta z h'(z) = D_\zeta h(z); \qquad (\zeta \geq 0) \tag{1.10}$$

$$D^n_\zeta h(z) = D_\zeta(D^{n-1}_\zeta h(z)); \qquad (n \in \mathfrak{N} = \{1, 2, 3, \cdots\}). \tag{1.11}$$

Then from (1.10) and (1.11) we get,

$$D^n_\zeta h(z) = z + (1-\zeta)^n c_0 + \sum_{\nu=1}^{\infty}[1-(\nu+1)\zeta]^n c_\nu z^{-\nu}; \qquad (n \in \mathfrak{N}_0 = \{0, 1, 2, 3, \cdots\}). \tag{1.12}$$

Babalola [5] defined the class $\mathcal{L}_\psi(\vartheta)$ of $\psi-$pseudo-starlike functions of order ϑ as follows:

Definition 1.1. [5] Let $f \in \mathcal{A}$ and if $0 \leq \vartheta < 1$ and $\psi \geq 1$. Then $f(z) \in \mathcal{L}_\psi(\vartheta)$ of ψ-pseudo-starlike functions of order ϑ in $\mathcal{U}$ if and only if

$$\Re\left(\frac{z[f'(z)]^\psi}{f(z)}\right) > \vartheta, \qquad (z \in \mathcal{U};\ 0 \leq \vartheta < 1;\ \psi \geq 1). \tag{1.13}$$

Especially, Babalola [5] proved that all ψ-pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\psi}$ and order $\vartheta^{\frac{1}{\psi}}$ and are univalent in $\mathcal{U}$.

Recently, Srivastava et al. [36] introduced the following subclasses of the meromorphic bi-univalent function and obtained non sharp estimates on the initial coefficient $|c_0|$ and $|c_1|$ as follows.

Definition 1.2. [36] For $\psi \geq 1$ and $0 < \xi \leq 1$; a function $h(z)$ given by (1.3) is said to be in the class $\mathcal{W}_{\Sigma'}(\psi, \xi)$ if the following condition holds:

$$\left|\arg\left(\frac{z[h'(z)]^\psi}{h(z)}\right)\right| < \frac{\xi\pi}{2}, \tag{1.14}$$

and

$$\left|\arg\left(\frac{w[l'(w)]^{\psi}}{l(w)}\right)\right| < \frac{\xi\pi}{2}, \tag{1.15}$$

where $z, w \in \mathcal{U}^*$ and $h^{-1}(w) = l(w)$ is given by (1.6).

Theorem 1.3. *[36] Let $h \in \mathcal{W}_{\Sigma'}(\psi, \xi)$. Then*

$$|c_0| \leq 2\xi, \qquad |c_1| \leq \frac{2\sqrt{5}\xi^2}{1+\psi}. \tag{1.16}$$

Definition 1.4. [36] For $\psi \geq 1$ and $0 \leq \eta < 1$; a function $h(z)$ given by (1.3) is said to be in the class $\mathcal{W}_{\Sigma'}(\psi, \eta)$ if the following condition holds:

$$\Re\left(\frac{z[h'(z)]^{\psi}}{h(z)}\right) > \eta \tag{1.17}$$

and

$$\Re\left(\frac{w[l'(w)]^{\psi}}{l(w)}\right) > \eta \tag{1.18}$$

where $z, w \in \mathcal{U}^*$ and $h^{-1}(w) = l(w)$ is given by (1.6).

Theorem 1.5. *[36] Let $h(z) \in \mathcal{W}_{\Sigma'}(\psi, \eta)$. Then*

$$|c_0| \leq 2(1-\eta), \qquad |c_1| \leq \frac{2(1-\eta)\sqrt{4\eta^2 - 8\eta + 5}}{1+\psi}. \tag{1.19}$$

Motivated by the aforecited works, In our current investigation, we introduce two new subclasses of the class $\mathcal{W}_{\Sigma'}$ of meromorphic bi-univalent functions defined by Al-Oboudi differential operator and obtained the estimates for the initial coefficients $|c_0|$, $|c_1|$ and $|c_2|$ of functions in these subclasses.

In order to find out the main results, the following Lemma can be recalled here.

Lemma 1.6. *[26] If $r \in \mathcal{P}$, then $|\kappa_\tau| \leq 2$ for each τ, where $\mathcal{P}$ is the family of all functions r analytic in $\mathcal{U} = \{z : z \in \mathcal{C}, |z| < 1\}$. for which $Re(r(z)) > 0$ where*

$$r(z) = 1 + \kappa_1 z + \kappa_2 z^2 + \kappa_3 z^3 + \cdots \quad (z \in \mathfrak{D}).$$

2 Coefficient bounds for the function class $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi, \xi)$

Definition 2.1. For $\zeta \geq 0$, $n \in \mathfrak{N}$, $\psi \geq 1$ and $0 < \xi \leq 1$; a function $h(z)$ given by (1.3) is said to be in the class $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi, \xi)$ if the following condition holds:

$$\left|\arg\left(\frac{z[(D_\zeta^n h(z))']^{\psi}}{D_\zeta^n h(z)}\right)\right| < \frac{\xi\pi}{2} \tag{2.1}$$

and

$$\left|\arg\left(\frac{w[(D_\zeta^n l(w))']^{\psi}}{D_\zeta^n l(w)}\right)\right| < \frac{\xi\pi}{2} \tag{2.2}$$

where $z, w \in \mathcal{U}^*$ and $h^{-1}(w) = l(w)$ is given by (1.6).

In the ensuring theorems, the initial coefficients $|c_0|$, $|c_1|$ and $|c_2|$ for the function $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi, \xi)$ and $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi, \eta)$ are obtained.

Theorem 2.2. *Let $h \in \mathcal{W}_{\Sigma'}^{\zeta,n}(\psi, \xi)$. Then*

$$|c_0| \leq \frac{2\xi}{(1-\zeta)^n}, \tag{2.3}$$

$$|c_1| \leq \frac{2\sqrt{5}\xi^2}{(1-2\zeta)^n(1+\psi)}, \tag{2.4}$$

$$|c_2| \leq \frac{2\xi}{(1-3\zeta)^n(1+2\psi)}\left[2\left\{\frac{(6(1-\zeta)^{3n}-1)\xi^2+3\xi-2}{3}\right\}+3-2\xi\right]. \tag{2.5}$$

Proof. Since $h(z) \in \mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\xi)$, there exist two functions κ and t such that

$$\frac{z[(D_\zeta^n h(z))']^\psi}{D_\zeta^n h(z)} = (\kappa(z))^\xi, \tag{2.6}$$

and

$$\frac{w[(D_\zeta^n l(w))']^\psi}{D_\zeta^n l(w)} = (t(w))^\xi, \tag{2.7}$$

respectively, where $\kappa(z)$ and $t(w)$ satisfy the inequality $\Re(\kappa(z)) > 0$ and $\Re(t(w)) > 0$.

Furthermore, the functions $\kappa(z)$ and $t(w)$ have the forms:

$$\kappa(z) = 1 + \frac{\kappa_1}{z} + \frac{\kappa_2}{z^2} + \frac{\kappa_3}{z^3} + \cdots \quad (z \in \mathcal{U}^*)$$

and

$$t(w) = 1 + \frac{t_1}{w} + \frac{t_2}{w^2} + \frac{t_3}{w^3} + \cdots \quad (w \in \mathcal{U}^*).$$

By definition of h and l, we get

$$\frac{z[(D_\zeta^n h(z))']^\psi}{D_\zeta^n h(z)} = 1 - \frac{(1-\zeta)^n c_0}{z} + \frac{(1-\zeta)^{2n}c_0^2 - (1-2\zeta)^n(1+\psi)c_1}{z^2} - \frac{(1-\zeta)^{3n}c_0^3 - (1-\zeta)^n(1-2\zeta)^n c_0 c_1(2+\psi) + (1-3\zeta)^n c_2(1+2\psi)}{z^3} + \cdots \tag{2.8}$$

and

$$\frac{w[(D_\zeta^n l(w))']^\psi}{D_\zeta^n l(w)} = 1 + \frac{(1-\zeta)^n c_0}{w} + \frac{(1-\zeta)^{2n}c_0^2 + (1-2\zeta)^n(1+\psi)c_1}{w^2} + \frac{(1-\zeta)^{3n}c_0^3 + (1-3\zeta)^n(1+2\psi)c_2 + \Big((1-3\zeta)^n(1+2\psi) + (1-\zeta)^n(1-2\zeta)^n(2+\psi)\Big)c_0c_1}{w^3} + \cdots. \tag{2.9}$$

A simple calculation shows

$$(\kappa(z))^\xi = 1 + \frac{\xi\kappa_1}{z} + \frac{\frac{1}{2}\xi(\xi-1)\kappa_1^2 + \xi\kappa_2}{z^2} + \frac{\frac{1}{6}\xi(\xi-1)(\xi-2)\kappa_1^3 + \xi(\xi-1)\kappa_1\kappa_2 + \xi\kappa_3}{z^3} + \cdots \tag{2.10}$$

and

$$(t(w))^\xi = 1 + \frac{\xi t_1}{w} + \frac{\frac{1}{2}\xi(\xi-1)t_1^2 + \xi t_2}{w^2} + \frac{\frac{1}{6}\xi(\xi-1)(\xi-2)t_1^3 + \xi(\xi-1)t_1t_2 + \xi t_3}{w^3} + \cdots. \tag{2.11}$$

Putting (2.8), (2.10) in (2.6) and (2.9), (2.11) in (2.7), we have

$$-(1-\zeta)^n c_0 = \xi\kappa_1, \tag{2.12}$$

$$(1-\zeta)^{2n}c_0^2-(1-2\zeta)^n(1+\psi)c_1=\frac{1}{2}\xi(\xi-1)\kappa_1^2+\xi\kappa_2, \tag{2.13}$$

$$-\left[(1-\zeta)^{3n}c_0^3-(1-\zeta)^n(1-2\zeta)^n c_0c_1(2+\psi)+(1-3\zeta)^n c_2(1+2\psi)\right] \\ =\frac{1}{6}\xi(\xi-1)(\xi-2)\kappa_1^3+\xi(\xi-1)\kappa_1\kappa_2+\xi\kappa_3, \tag{2.14}$$

$$(1-\zeta)^n c_0=\xi t_1, \tag{2.15}$$

$$(1-\zeta)^{2n}c_0^2+(1-2\zeta)^n(1+\psi)c_1=\frac{1}{2}\xi(\xi-1)t_1^2+\xi t_2, \tag{2.16}$$

$$(1-\zeta)^{3n}c_0^3+(1-3\zeta)^n(1+2\psi)c_2+\Big((1-3\zeta)^n(1+2\psi)+(1-\zeta)^n(1-2\zeta)^n \\ (2+\psi)\Big)c_0c_1=\frac{1}{6}\xi(\xi-1)(\xi-2)t_1^3+\xi(\xi-1)t_1t_2+\xi t_3. \tag{2.17}$$

From (2.12) and (2.15) , it follows that

$$c_0=-\xi\kappa_1=\xi t_1 \qquad (\kappa_1=-t_1) \tag{2.18}$$

and

$$c_0^2=\frac{\xi^2(\kappa_1^2+t_1^2)}{2(1-\zeta)^{2n}}. \tag{2.19}$$

As $\Re(\kappa(z))>0$ in $\mathcal{U}^*$, the function $\kappa(\frac{1}{z})\in\mathcal{P}$. Similarly $t(\frac{1}{w})\in\mathcal{P}$. So, the coefficients of $\kappa(z)$ and $t(w)$ satisfy the inequality of Lemma 1.6. Applications of triangle inequality and followed by Lemma 1.6 in (2.19) we get,

$$|c_0|\le\frac{2\xi}{(1-\zeta)^n}.$$

Furthermore, in order to find the bound on $|c_1|$, by applying (2.13) and (2.16), we have

$$\left[(1-\zeta)^{2n}c_0^2-(1-2\zeta)^n(1+\psi)c_1\right]\cdot\left[(1-\zeta)^{2n}c_0^2+(1-2\zeta)^n(1+\psi)c_1\right] \\ =\left(\frac{1}{2}\xi(\xi-1)\kappa_1^2+\xi\kappa_2\right)\cdot\left(\frac{1}{2}\xi(\xi-1)t_1^2+\xi t_2\right)$$

$$(1-2\zeta)^{2n}(1+\psi)^2c_1^2=(1-\zeta)^{4n}(c_0^2)^2-\frac{1}{4}\xi^2(\xi-1)^2\kappa_1^2t_1^2 \\ -\frac{1}{2}\xi^2(\xi-1)(\kappa_2t_1^2+\kappa_1^2t_2)-\xi^2\kappa_2t_2$$

and

$$(1-2\zeta)^{2n}(1+\psi)^2c_1^2=(1-\zeta)^{4n}\left(\frac{\xi^2(\kappa_1^2+t_1^2)}{2(1-\zeta)^{2n}}\right)^2-\frac{1}{4}\xi^2(\xi-1)^2\kappa_1^2t_1^2 \\ -\frac{1}{2}\xi^2(\xi-1)(\kappa_2t_1^2+\kappa_1^2t_2)-\xi^2\kappa_2t_2.$$

Applying Lemma 1.6, we have

$$(1-2\zeta)^{2n}(1+\psi)^2|c_1^2|\le 16\xi^4+4\xi^2(\xi-1)^2+8\xi^2(\xi-1)+4\xi^2$$

that is,

$$|c_1| \leq \frac{2\sqrt{5}\xi^2}{(1-2\zeta)^n(1+\psi)},$$

Finally, to obtain the bounds on c_2, consider the sum of (2.14) and (2.17) with $\kappa_1 = -t_1$, we get

$$c_0c_1 = \frac{\xi(\xi-1)\kappa_1(\kappa_2-t_2)+\xi(\kappa_3+t_3)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}. \tag{2.20}$$

Subtracting (2.17) from (2.14) with $\kappa_1 = -t_1$, we have

$$\begin{aligned} -2(1-3\zeta)^n(1+2\psi)c_2 &= 2(1-\zeta)^{3n}c_0^3 + (1-3\zeta)^n(1+2\psi)c_0c_1 \\ &\quad + \frac{1}{3}\xi(\xi-1)(\xi-2)\kappa_1^3 + \xi(\xi-1)\kappa_1(\kappa_2+t_2) + \xi(\kappa_3-t_3). \end{aligned} \tag{2.21}$$

Putting (2.18) and (2.20) in (2.21) gives

$$\begin{aligned} \frac{2(1-3\zeta)^n(1+2\psi)c_2}{\xi} &= \frac{(6(1-\zeta)^{3n}-1)\xi^2+3\xi-2}{3}\kappa_1^3 \\ &\quad + \frac{2(1-3\zeta)^n(1+2\psi)(1-\xi)+2(1-\xi)(1-\zeta)^n(1-2\zeta)^n(2+\psi)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}\kappa_1\kappa_2 \\ &\quad + \frac{2(1-\xi)(1-\zeta)^n(1-2\zeta)^n(2+\psi)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}\kappa_1 t_2 \\ &\quad + \frac{2(1-3\zeta)^n(1+2\psi)+2(1-\zeta)^n(1-2\zeta)^n(2+\psi)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}\kappa_3 \\ &\quad + \frac{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}t_3. \end{aligned}$$

By applying Lemma 1.6 for the above equation we have

$$|c_2| \leq \frac{2\xi}{(1-3\zeta)^n(1+2\psi)}\left[2\left\{\frac{(6(1-\zeta)^{3n}-1)\xi^2+3\xi-2}{3}\right\}+3-2\xi\right].$$

which is the desired estimates on c_2 given by (2.5).

Taking $n = 1$ in Theorem 2.2, we get the following results.

Corollary 2.3. *Let $h \in \mathcal{W}_{\Sigma'}(\psi, \xi)$. Then*

$$|c_0| \leq 2\xi,$$

$$|c_1| \leq \frac{2\sqrt{5}\xi^2}{1+\psi},$$

$$|c_2| \leq \frac{2\xi}{1+2\psi}\left[2\left\{\frac{5\xi^2+3\xi-2}{3}\right\}+3-2\xi\right].$$

3 Coefficient bounds for the function class $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\eta)$

Definition 3.1. For $\zeta \geq 0$, $n \in \mathfrak{N}$, $\psi \geq 1$ and $0 \leq \eta < 1$; a function $h(z)$ given by (1.3) is said to be in the class $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\eta)$ if the following condition holds:

$$\Re\left(\frac{z[(D_\zeta^n h(z))']^\psi}{D_\zeta^n h(z)}\right) > \eta \tag{3.1}$$

and

$$\Re\left(\frac{w[(D_\zeta^n l(w))']^\psi}{D_\zeta^n l(w)}\right) > \eta \tag{3.2}$$

where $z, w \in \mathcal{U}^*$ and $h^{-1}(w) = l(w)$ is given by (1.6).

Theorem 3.2. *Let* $h(z) \in \mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\eta)$. *Then*

$$|c_0| \leq \frac{2(1-\eta)}{(1-\zeta)^n}, \tag{3.3}$$

$$|c_1| \leq \frac{2(1-\eta)\sqrt{4\eta^2-8\eta+5}}{(1-2\zeta)^n(1+\psi)} \tag{3.4}$$

and

$$|c_2| \leq \frac{2(1-\eta)}{(1-3\zeta)^n(1+2\psi)}\left[1+4(1-\eta)^2\right]. \tag{3.5}$$

Proof. Let $h \in \mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\eta)$. Then, by definition of the class $\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\eta)$,

$$\frac{z[(D_\zeta^n h(z))']^\psi}{D_\zeta^n h(z)} = \eta + (1-\eta)\kappa(z) \tag{3.6}$$

and

$$\frac{w[(D_\zeta^n l(w))']^\psi}{D_\zeta^n l(w)} = \eta + (1-\eta)t(w) \tag{3.7}$$

where κ and t are as in Theorem 2.2.

Equating coefficients in (3.6) and (3.7) yields

$$-(1-\zeta)^n c_0 = (1-\eta)\kappa_1, \tag{3.8}$$

$$(1-\zeta)^{2n}c_0^2 - (1-2\zeta)^n(1+\psi)c_1 = (1-\eta)\kappa_2, \tag{3.9}$$

$$-[(1-\zeta)^{3n}c_0^3 - (1-\zeta)^n(1-2\zeta)^n c_0c_1(2+\psi) + (1-3\zeta)^n c_2(1+2\psi)] = (1-\eta)\kappa_3, \tag{3.10}$$

$$(1-\zeta)^n c_0 = (1-\eta)t_1, \tag{3.11}$$

$$(1-\zeta)^{2n}c_0^2 + (1-2\zeta)^n(1+\psi)c_1 = (1-\eta)t_2, \tag{3.12}$$

$$(1-\zeta)^{3n}c_0^3 + (1-3\zeta)^n(1+2\psi)c_2 + ((1-3\zeta)^n(1+2\psi) + (1-\zeta)^n (1-2\zeta)^n(2+\psi))c_0c_1 = (1-\eta)t_3. \tag{3.13}$$

From (3.8) and (3.11), we have

$$\kappa_1 = -t_1$$

and

$$c_0^2 = \frac{(1-\eta)^2(\kappa_1^2+t_1^2)}{2(1-\zeta)^{2n}}. \tag{3.14}$$

An application of triangle inequality and lemma 1.6 in (3.14) we have

$$|c_0| \leq \frac{2(1-\eta)}{(1-\zeta)^n},$$

Furthermore, in order to find the bound on $|c_1|$, by applying (3.9) and (3.12), we have

$$\left[(1-\zeta)^{2n}c_0^2 - (1-2\zeta)^n(1+\psi)c_1\right]\cdot\left[(1-\zeta)^{2n}c_0^2 + (1-2\zeta)^n(1+\psi)c_1\right] = ((1-\eta)\kappa_2)\cdot((1-\eta)t_2)$$

$$(1-2\zeta)^{2n}(1+\psi)^2 c_1^2 = (1-\zeta)^{4n}(c_0^2)^2 - (1-\eta)^2\kappa_2 t_2$$

and

$$(1-2\zeta)^{2n}(1+\psi)^2 c_1^2 = (1-\zeta)^{4n}\left(\frac{(1-\eta)^2(\kappa_1^2+t_1^2)}{2(1-\zeta)^{2n}}\right)^2 - (1-\eta)^2\kappa_2 t_2.$$

Applying Lemma 1.6, we have

$$(1-2\zeta)^{2n}(1+\psi)^2|c_1^2| \leq 4(1-\eta)^2(4\eta^2-8\eta+5)$$

that is,

$$|c_1| \leq \frac{2(1-\eta)\sqrt{4\eta^2-8\eta+5}}{(1-2\zeta)^n(1+\psi)}.$$

Finally, in order to obtain the bound on c_2, adding (3.10) and (3.13) yields

$$c_0 c_1 = \frac{(1-\eta)(\kappa_3+t_3)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}. \tag{3.15}$$

Subtracting (3.13) from (3.10), we have

$$-2(1-3\zeta)^n(1+2\psi)c_2 = 2(1-\zeta)^{3n}c_0^3 + (1-3\zeta)^n(1+2\psi)c_0 c_1 + (1-\eta)(\kappa_3-t_3). \tag{3.16}$$

Putting (3.8) and (3.15) in (3.16) gives

$$c_2 = \frac{(1-\eta)}{(1-3\zeta)^n(1+2\psi)} \left[(1-\eta)^2\kappa_1^3 - \frac{(1-3\zeta)^n(1+2\psi)+(1-\zeta)^n(1-2\zeta)^n(2+\psi)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}\kappa_3 + \frac{(1-\zeta)^n(1-2\zeta)^n(2+\psi)}{2(1-\zeta)^n(1-2\zeta)^n(2+\psi)+(1-3\zeta)^n(1+2\psi)}t_3\right].$$

By applying Lemma 1.6 for the above equation we have

$$|c_2| \leq \frac{2(1-\eta)}{(1-3\zeta)^n(1+2\psi)}\left[1+4(1-\eta)^2\right].$$

Choosing $n=1$ in Theorem 3.2, yields:

Corollary 3.3. *Let $h \in \mathcal{W}_{\Sigma'}(\psi,\eta)$. Then*

$$|c_0| \leq 2(1-\eta),$$

$$|c_1| \leq \frac{2(1-\eta)\sqrt{4\eta^2-8\eta+5}}{(1+\psi)}$$

and

$$|c_2| \leq \frac{2(1-\eta)}{(1+2\psi)}\left[1+4(1-\eta)^2\right].$$

4 Conclusion

Here, in our present investigation, we have introduced and studied coefficient problems associated with each of the following two new subclasses:

$$\mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\xi) \quad and \quad \mathcal{W}_{\Sigma'}^{\zeta,n}(\psi,\eta)$$

of the class $\mathcal{W}_{\Sigma'}$ of meromorphic bi-univalent functions associated with Al-Oboudi differential operator defined on $\mathcal{U}^* = \{z : z \in \mathcal{C}, 1 < |z| < \infty\}$. These class $\mathcal{W}_{\Sigma'}$ of meromorphic bi-univalent functions associated with Al-Oboudi differential operator are given by Definition 2.1

and 3.1, respectively. For function in each of these two meromorphic bi-univalent functions classes, we have obtained the estimates for the coefficients $|c_0|$, $|c_1|$ and $|c_2|$. The results presented in this research have been shown to considerably improve the earlier results of Srivastava et al. [36] in terms of the bounds.
Using the Feber polynomial expansion for the two classes $\mathcal{W}_{\Sigma'}^{\zeta;n}(\psi,\xi)$ and $\mathcal{W}_{\Sigma'}^{\zeta;n}(\psi,\eta)$ is still an interesting open problem, as well as for $|c_n|$ where $n \geq 3$. Another investigation to consider, Amol B. Patil and Uday H. Naik [24] obtained initial coefficient for certain subclass of meromorphic bi-univalent function class Σ' of complex order $\gamma \in \mathcal{C}\backslash\{0\}$, using Al-Oboudi differential operator. Obtaining complex order $\gamma \in \mathcal{C}\backslash\{0\}$ for the two classes $\mathcal{W}_{\Sigma'}^{\zeta;n}(\psi,\xi)$ and $\mathcal{W}_{\Sigma'}^{\zeta;n}(\psi,\eta)$ are issues to be investigated.

References

[1] E. A. Adegani, A. Motamednezhad and S. Bulut, Coefficient estimates for a subclass of meromorphic bi-univalent functions defined by subordination, *Stud. Univ. Babeş-Bolyai Math.* **65**(1), 57-66 (2020).

[2] A. Aldawish, T. Al-Hawary and B.A. Frasin, Subclasses of bi-univalent function defined by Frasin differential operator, *E. Mathematics.* **8**, 1-11 (2020).

[3] A. G. Alamoush, A Subclass of pseudo-type meromorphic bi-univalent functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **69**(2), 31-38 (2020).

[4] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, *Int. J. Math. Sci.* **27**, 1429-1436 (2004).

[5] K.O. Babalola, On λ-pseudo-starlike function, *J. Class. Anal.* **3**, 137-147 (2013).

[6] D.A. Brannan, J. Clunie and W.E. Kirwan, Coefficient estimate for a class of starlike functions, *Canad. J. Math.* **22**, 476-485 (2009).

[7] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, In Mathematical Analysis and its Applications; Pergamon Press: Pergamon, Turkey. 53-60 (2009).

[8] D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, *Stud. Univ. Babeş-Bolyai Math.*, *Canad. J. Math.* **31**(2), 70–77 (1986).

[9] S. Bulut, Coefficient estimate for new subclasses of meromorphic bi-univalent functions, *International Sch. Research Notices.* Article ID 376076, 476-485.

[10] M. Çağlar, H. Orhan, and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat.* **27**(7), 1165–1171 (2013).

[11] P.L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, (2004).

[12] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* **24**(9), 1569-1573 (2011).

[13] S. G. Hamidi, S. A. Halim, and J. M. Jahangiri, Coefficent estimates for bi-univalent strongly starlike and Bazilevic functions, *International Journal of Mathematics Research,* **5**(1), 87–96 (2013).

[14] S. A. Halim, S. G. Hamidi and V. Ravichandran, Coefficients estimates for meromorphic bi-univalent functions, arXiv: 1108.4089vl, 1-9 (2011).

[15] T. Janani and G. Murugusundaramoorthy, Coefficients estimates of meromorphic bi-starlike functions of complex order, *Internat. J. Anal. Appl.* **4**(1), 68-77 (2014).

[16] S.B. Joshi and P.P. Yadav, Coefficient bounds for new subclasses of bi-univalent function associated with pseudo-starlike functions, *Ganita J*. **69**, 67-74 (2019).

[17] J. Jothibasu, Certain subclasses of bi-univalent functions defined by salagean operator, *Elec. J. Math. Anal. Appl.* **3**, 150-157 (2015).

[18] G. P. Kapoor and A. K. Mishra, Coefficient estimates for inverses of starlike functions of positive order, *Journal of Mathematical Analysis and Applications* **329** (2), 922–934 (2017).

[19] Y. Kubota, Coefficients of meromorphic univalent functions, *Kōdai Math. Sem. Rep.* **28** (2-3), 253–261 (1976/77).

[20] M. Lewin, On a coefficients problem of bi-univalent functions, *Proc. Amer. Math*. Soc. **18**, 63-68 (1967).

[21] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent functions, *Abs. Appl. Anal.*, 1-3 (2013).

[22] S.0. Olatunji and P.T. Ajayi, On subclasses of bi-univalent functions of Bazelevic type involving linear salagean operator, *Internat. J. Pure. Appl. Math.* **92**, 645-656 (2014).

[23] T. Panigrahi, Coefficient bounds for certain subclasses of meromorphic and bi-univalent functions, *Bull. Korean. Math. Soc.,* **50**(5), 1531-1538 (2013).

[24] A. B. Patil and U. H. Naik, Coefficient estimates for a new subclass of meromorphic bi-univalent function defined by Al-Oboudi differential operator, *Global Journal of Pure and Applied Mathematics,* **13**(9), 4405-4414 (2017).

[25] A. B. Patil, T. G. Shaba, On sharp Chebyshev polynomial bounds for general subclassof bi-univalent functions. *Applied Sciences*, **23**, 109-117 (2021).

[26] C.H. Pommerenke, Univalent Functions, Vandendoeck and Rupercht, Gottingen, (1975).

[27] G.S. Salagean, *Subclasses of Univalent functions*, Lecture Notes in Math., Spinger Verlag, Berlin. **1013**, 362–372 (1983).

[28] M. Schiffer, Sur un probléme dextrémum de la représentation conforme, *Bull. Soc. Math. France,* **66**, 48–55 (1938).

[29] G. Schober, Coefficients of inverses of meromorphic univalent functions, *Proc. Amer. Math. Soc.,* **67**, 111–116 (1977).

[30] T. G. Shaba, On some new subclass of bi-univalent functions associated with Opoola differential operator, *Open J. Math. Anal.,* **4**(2), 74–79 (2020).

[31] T. G. Shaba, Certain new subclasses of analytic and bi-univalent functions using salagean operator, *Asia Pac. J. Math.,* **7**(29), 1–11 (2020).

[32] T. G. Shaba and A. K. Wanas, Coefficients bounds for a new family of bi-univalent functions associated with (U, V)-Lucas polynomial, *Int. J. Nonlinear Anal. Appl.* **13**(1), 615- 626 (2022).

[33] T. G. Shaba, On some subclasses of bi-pseudo-starlike functions defined by Salagean differential operator, *Asia Pac. J. Math.* **8**(6), 1–11 (2021) .

[34] T. G. Shaba, A. A. Ibrahim and A. A. Jimoh, On a new subclass of bi-pseudo-starlike functions defined by frasin differential operator, *Adv. Math., Sci. J.* **9**(7), 4829–4841 (2020).

[35] G. Springer, The coefficient problem for schlicht mappings of the exterior of the unit circle, *Trans. Amer. Math. Soc.* **70**, 421–450 (1951).

[36] H.M. Srivastava, S.B. Joshi, S.S. Joshi, and H. Pawar, Coefficients estimates for certain subclasses of meromorphically bi-univalent function, *Palestine J. Math.* **5**(1), 250-258 (2016).

[37] H. M. Srivastava, S. S. Eker, and R. M. Ali, Coefficients estimates for a certain class of analytic and bi-univalent functions, *Filomat*, **29**, 1839-1845 (2015).

[38] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23**(10), 1188–1192 (2010).

[39] H. M. Srivastava, A. K. Mishra, and S. N. Kund, Coefficient estimates for the inverses of starlike functions represented by symmetric gap series, *Panamerican Mathematical Journal* **21**(4), 105–123 (2011).

[40] T.S. Taha, Topics in univalent functions theory, Ph.D. Thesis, University of London, London, UK, (1998).

[41] Q.H. Xu, Y.C. Gui and H.M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* 11461-11465 (2012).

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