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Research article

The Fekete-Szegö functional and the Hankel determinant for a certain class of analytic functions involving the Hohlov operator

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Abstract: In this paper, we introduce and study a new subclass of normalized functions that are analytic and univalent in the open unit disk $\mathbb{U} = \{z : z \in C \text{ and } |z| < 1\}$, which satisfies the following geometric criterion:

$$\Re\left(\frac{\mathcal{L}^{w}_{u,v} f(z)}{z}(1 - e^{-2i\phi}\mu^2 z^2)e^{i\phi}\right) > 0,$$

where $z \in \mathbb{U}$, $0 \leqq \mu \leqq 1$ and $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and which is associated with the Hohlov operator $\mathcal{L}^{w}_{u,v}$. For functions in this class, the coefficient bounds, as well as upper estimates for the Fekete-Szegö functional and the Hankel determinant, are investigated.

Keywords: analytic functions; univalent functions; coefficient bounds; Fekete-Szegö functional; Hohlov operator; Dziok-Srivastava operator; Srivastava-Wright operator; Fekete-Szegö inequality; Hankel determinant; basic q-calculus; (p, q)-variation
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1. Introduction and preliminaries

Geometric function theory is one of the most exciting areas of research in complex analysis, with applications in a wide range of mathematical fields including mathematical physics. Due to its many uses in analytical solutions to issues such as those in electrostatics, aerodynamics and fluid mechanics, researchers in the field of complex analysis have been investigating various families of analytic (or holomorphic) functions.

Analytic functions such as $\psi(z)$ can be expressed in the Taylor-Maclaurin series expansion about the origin $z = 0$ as follows:

$$\psi(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4 + \cdots \qquad (z \in \mathbb{U}),$$

which can be normalized in the following way:

$$f(z) = \frac{\psi(z) - C_0}{C_1} = z + \sum_{j=2}^{\infty} b_j z^j, \tag{1.1}$$

where

$$C_1 \neq 0, \quad b_j = \frac{C_j}{C_1}, \quad \mathbb{U} = \{z : z \in C \quad \text{and} \quad |z| < 1\},$$

and the series expansion in (1.1) is convergent in the open unit disk $\mathbb{U}$. Let $\mathcal{A}$ denote a class of functions $f(z)$ that are analytic (or holomorphic) in $\mathbb{U}$, have the form (1.1) and are normalized by the constraints $f'(0) - 1 = f(0) = 0$.

The class of functions φ that are holomorphic in $\mathbb{U}$ and have the form

$$\varphi(z) = 1 + r_1 z + r_2 z^2 + \cdots \qquad (z \in \mathbb{U}),$$

with

$$\varphi(0) = 1 \qquad \text{and} \qquad \Re(\varphi(z)) > 0 \quad (z \in \mathbb{U}),$$

is denoted by $\mathcal{P}$.

In the geometric function theory of complex analysis, studies of the concept of convolution are crucial. Various new and interesting subclasses of holomorphic and univalent functions have been introduced and investigated through the use of the Hadamard product (or convolution) in the direction of well-known ideas such as the integral mean, Hankel determinant, subordination, partial sums, superordination inequalities and so on. The Hadamard product (or convolution) of f and g, represented by $f * g$, is defined by

$$(f * g)(z) := z + \sum_{j=2}^{\infty} b_j a_j z^j =: (g * f)(z)$$

for functions f and g in $\mathcal{A}$ given by the following series:

$$f(z) = z + \sum_{j=2}^{\infty} b_j z^j \qquad g(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (z \in \mathbb{U}).$$

The Gauss hypergeometric function ${}_2F_1(u, v, w; z)$ is defined as follows:

$$ {}_2F_1(u, v, w; z) = \sum_{j=0}^{\infty} \frac{(u)_j (v)_j}{(w)_j} \frac{z^j}{(1)_j} \quad (z \in \mathbb{U}), $$

where $(\delta)_j$ signifies the Pochhammer symbol (or the shifted factorial) defined in terms of the Gamma function Γ, as follows:

$$ (\delta)_j = \frac{\Gamma(\delta + j)}{\Gamma(\delta)} = \begin{cases} 1 & (j = 0) \\ \delta(\delta + 1)(\delta + 2)(\delta + 3) \cdots (\delta + j - 1) & (j \neq 0). \end{cases} $$

Hohlov (see [19,20]) proposed and investigated a linear operator denoted by $\mathcal{L}_{u,v}^{w}$ and defined by $\mathcal{L}_{u,v}^{w} f : \mathcal{A} \longrightarrow \mathcal{A}$, with

$$ \mathcal{L}_{u,v}^{w} f(z) := z {}_2\mathcal{F}_1(u, v, w; z) * f(z) = z + \sum_{j=2}^{\infty} \frac{(u)_{j-1} (v)_{j-1}}{(w)_{j-1} (1)_{j-1}} b_j z^j \qquad (z \in \mathbb{U}). \tag{1.2} $$

The above-specified three-parameter family of operators unifies several other linear operators that have been introduced and explored previously when the parameters are appropriately chosen. The works in [7, 9, 11, 12, 40–43, 51, 53, 59, 72] provide special examples of this operator. For more details, see [17, 47, 70, 71]. It should be remarked in passing that much more general convolution operators, such as the Dziok-Srivastava operator (see [14, 15]) and the Srivastava-Wright operator (see [62]), have also been investigated rather extensively in the vast literature in geometric function theory of complex analysis.

The nth coefficient of a function belonging to the class $\mathcal{S}$ is well-known to be bounded by n, and the coefficient bounds provide information about the geometric properties of the function. For example, the nth coefficient of functions in the family $\mathcal{S}$ yields the growth and distortion properties of the function, whereas the second coefficient of functions in the family $\mathcal{S}$ yields the growth and distortion properties of the function itself. Studying a functional composed of combinations of the coefficients of the original function is a common issue in the geometric function theory of complex analysis. In most cases, the extremal value of the functional is required across a parameter. Some of our findings are related to the Fekete-Szegö functional, which is a key functional of this kind.

The famous problem solved by Fekete and Szegö [16] is to determine the greatest value of the coefficient functional $\Omega_\sigma(f) := |a_3 - \sigma a_2^2|$ over the class $\mathcal{S}$ for each $\sigma \in [0, 1]$, which was demonstrated by using the Loewner chain technique. For various subclasses of the class of $\mathcal{S}$ and associated subclasses of functions in $\mathcal{A}$, several scholars solved the Fekete-Szegö issue. For example, see [8, 13, 23, 25, 28–30, 40, 44], and so on. We refer to [68] for a thorough study on the Fekete-Szegöproblem of the traditional univalent function class $\mathcal{S}$. Srivastava et al. claimed that the inequality was sharp in [68]. However, Peng (see [54]) has demonstrated that the extremal function presented there for the situation of $\varrho \in (2/3, 1)$ is not sharp. Cho et al. [10] discovered the Fekete-Szegö inequalities for close-to-convex functions with regard to a certain convex function, which improves the bound explored in [68]. Using the Hankel or Toeplitz determinants is another approach to look at the sharp bound for the nonlinear functional. We recall that Noonan and Thomas [49] introduced and investigated the qth Hankel determinant of f for $q \geqq 1$ and $n \geqq 1$ as follows:

$$\mathcal{H}_q(j) = \begin{vmatrix} b_j & b_{j+1} & b_{j+2} & \cdots & \cdots & b_{j+q-1} \\ b_{j+1} & b_{j+2} & b_{j+3} & \cdots & \cdots & b_{j+q} \\ b_{j+2} & b_{j+3} & b_{j+4} & \cdots & \cdots & b_{j+q+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{j+q-1} & b_{j+q} & b_{j+q+1} & \cdots & \cdots & b_{j+2(q-1)} \end{vmatrix} \qquad (q, j \in \mathbb{N}). \tag{1.3}$$

Several writers, notably Noor [50], have investigated the determinant $\mathcal{H}_q(j)$, with topics ranging from the rate of development of $H_q(j)$ (as $j \longrightarrow \infty$) to the determinant of exact limits for particular subclasses of analytic functions in the unit disk $\mathbb{U}$ with specified values of j and q. When $q = 2$, $j = 1$, and $b_1 = 1$, the Hankel determinant is $H_2(1) = |b_3 - b_2^2|$. The Hankel determinant simplifies to $H_2(2) = |b_2 b_4 - b_3^2|$ when $j = q = 2$. Fekete and Szegö [12] consider the Hankel determinant $H_2(1)$ and refer to $H_2(2)$ as the second Hankel determinant. If f is univalent in $\mathbb{U}$, then the sharp upper inequality $H_2(1) = |b_3 - b_2^2| \leqq 1$ is known (see [16]). Janteng et al. [21] obtained sharp bounds for the functional $H_2(2)$ for the function f in the subclass $\mathcal{RT}$ of $\mathcal{S}$, which was introduced by MacGregor [37] and which consists of functions whose derivative has a positive real part. They demonstrated that $H_2(2) = |b_2 b_4 - b_3^2| \leqq 4/9$ for each $f \in \mathcal{RT}$. They also discovered the sharp second Hankel determinant for the classical subclasses of $\mathcal{S}$, namely, the classes S^* and $\mathcal{K}$ of starlike and convex functions, respectively (see [22]). These two classes have bounds of $|b_2 b_4 - b_3^2| \leqq 1/8$ and $|b_2 b_4 - b_3^2| \leqq 1$. The Hankel determinants for starlike and convex functions with respect to symmetric points were recently discovered by Reddy and Krishna [57]. For functions belonging to subclasses of the Ma-Minda type starlike and convex functions, Lee et al. [34] found bounds for the second Hankel determinant.

Mishra and Gochhayat [41] found the sharp bound to the nonlinear functional $|b_2 b_4 - b_3^2|$ for the subclass of analytic functions given by

$$R_\rho(\omega, t) \qquad \left(0 \leqq t < 1;\ 0 \leqq \rho < 1;\ |\omega| < \frac{\pi}{2}\right),$$

and defined as follows:

$$\Re\left(e^{i\omega} \frac{\Omega_z^\rho f(z)}{z}\right) > t \cos \omega,$$

using the Owa-Srivastava operator in [53]. Similar coefficient constraints are found for a variety of analytic function subclasses that are constructed by using other appropriate linear operators (see, for example, [1, 32, 45, 46, 74, 75]).

In the case when $q = 3$ and $j = 1$, the Hankel determinant, represented by $H_3(1)$, is given by

$$H_3(1) = b_3(b_2 b_4 - b_3^2) - b_4(b_4 - b_2 b_3) + b_5(b_3 - b_2^2).$$

Clearly, we have

$$|H_3(1)| \leqq |b_3||b_2 b_4 - b_3^2| + |b_4||b_4 - b_2 b_3| + |b_5||b_3 - b_2^2|. \tag{1.4}$$

Babalola (see [5]) recently obtained the sharp upper bound of $H_3(1)$ for functions in the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{RT}$ classes.

Krishna et al. [31] defined $\mathcal{RT}(\alpha)$ as $\Re(h'(z)) > \alpha$ and found the bound on $H_3(1)$. Ayinla and Opoola [4] introduced the class defined by using the Sălăgean derivative operator as follows:

$$\Re\left(e^{i\gamma}(1 - e^{-2i\gamma}\beta^2 z^2)\frac{D^{n+1}f(z)}{z}\right) > 0$$

and obtained inequalities for the Fekete-Szegö functional and the second Hankel determinant. Additionally, Bansal et al. [6] and Raza and Malik [56] found the bound for $H_3(1)$ for a subclass of univalent functions. Gochhayat et al. [17] recently introduced the class $\mathcal{R}^c_{a,b}$ and obtained the sharp bounds of $H_2(2)$ and $H_3(1)$ in terms of the Gauss hypergeometric function by utilizing the Hohlov operator. See also [2, 3, 26, 27, 33, 48, 55, 61, 65, 69, 73, 76] for some of the recent works on the third Hankel determinant and [66] for some developments on the fourth Hankel determinant.

Here, in this paper, we introduce a subclass of the normalized univalent function class $\mathcal{S}$ by using the Hohlov operator, as inspired by some of the above-mentioned researches.

Definition 1.1. *A function $f(z)$ of the form* (1.1) *that is holomorphic and univalent in $\mathbb{U}$ is said to belong to the class $\mathcal{J}^{\phi}_{\mu}(u, v, w)$ if it satisfies the following geometric criterion:*

$$\Re\left(\frac{\mathcal{L}^w_{u,v}f(z)}{z}(1 - e^{-2i\phi}\mu^2 z^2)e^{i\phi}\right) > 0, \tag{1.5}$$

where $z \in \mathbb{U}$, $0 \leqq \mu \leqq 1$ and $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Remark 1.1. *Choosing $u = 2$, $v = w = 1$ and $\mu = 0$ gives we get the class*

$$\mathcal{J}^{\phi}_0(2, 1, 1) := \mathcal{J}^{\phi}.$$

This class $\mathcal{J}^{\phi}$ was introduced and studied by Noshiro [52].

Remark 1.2. *Choosing $u = 2$, $v = w = 1$ and $\mu = \phi = 0$ gives we get the class*

$$\mathcal{J}^{0}_0(2, 1, 1) =: \mathcal{J}.$$

This class $\mathcal{J}$ was introduced and studied by MacGregor [37].

Remark 1.3. *Choosing $u = 2$, $v = w = 1$, $\phi = 0$ and $\mu = 1$ gives we get the class*

$$\mathcal{J}^{0}_1(2, 1, 1) =: \mathcal{J}_1.$$

This class $\mathcal{J}_1$ was introduced and studied by Hengartner and Schober [18].

Remark 1.4. *Choosing $u = 2$, $v = w = 1$ and $\mu = 1$ gives we get the class*

$$\mathcal{J}^{\phi}_1(2, 1, 1) =: \mathcal{J}^{\phi}_1.$$

This class $\mathcal{J}^{\phi}_1$ was introduced and studied by Royster and Ziegl [58].

Remark 1.5. *Choosing $u = 2$, $v = w = 1$ and $\phi = 0$ gives we get the class*

$$\mathcal{J}^{0}_{\mu}(2, 1, 1) =: \mathcal{J}_{\mu}.$$

This class $\mathcal{J}_{\mu}$ was introduced and studied by Kanas and Lecko [24].

Remark 1.6. *Choosing $u = 2$ and $v = w = 1$ gives we get the class*

$$\mathcal{J}_{\mu}^{\phi}(2, 1, 1) =: \mathcal{J}_{\mu}^{\phi}.$$

This class $\mathcal{J}_{\mu}^{\phi}$ was introduced and studied by Lecko [35].

In this article, we establish the coefficient estimates, Fekete-Szegö type inequality, and the bounds for the second and the third Hankel determinants for functions belonging to the class $\mathcal{J}_{\mu}^{\phi}(u, v, w)$.

Lemma 1.1. (see [12]) *Let $\varphi(z) \in \mathcal{P}$. Then*

$$|r_j| \leqq 2 \qquad (j \in \mathbb{N}).$$

Lemma 1.2. (see [38]) *Let $\varphi(z) \in \mathcal{P}$. Then*

$$\left| r_2 - \upsilon \frac{r_1^2}{2} \right| \leqq \begin{cases} 2(1 - \upsilon), & (\upsilon \leqq 0) \\ 2 & (0 \leqq \upsilon \leqq 2) \\ 2(\upsilon - 1) & (\upsilon \geqq 2) \end{cases}$$

for $\upsilon \in \mathbb{R}$.

Lemma 1.3. (see [36]) *Let $\varphi(z) \in \mathcal{P}$. Then*

$$2r_2 = r_1^2 + x\left(4 - r_1^2\right),$$
$$4r_3 = r_1^3 + 2r_1(4 - r_1^2)x - r_1(4 - r_1^2)x^2 + 2(4 - r_1^2)(1 - |x|^2)z,$$

for some complex numbers x and z such that $|x| \leqq 1$ and $|z| \leqq 1$.

Theorem 1.1. *Let $f(z) \in \mathcal{J}_{\mu}^{\phi}(u, v, w)$. Then*

$$|b_2| \leqq \frac{2(w)_1 r_1 \cos\phi}{(u)_1 (v)_1}, \tag{1.6}$$

$$|b_3| \leqq \frac{2(w)_2}{(u)_2 (v)_2}(2\cos\phi + \mu^2), \tag{1.7}$$

$$|b_4| \leqq \frac{12\cos\phi (w)_3}{(u)_3 (v)_3}(1 + \mu^2), \tag{1.8}$$

$$|b_5| \leqq \frac{24(w)_4}{(u)_4 (v)_4}(2\cos\phi + 2\mu^2\cos\phi + \mu^4) \tag{1.9}$$

and

$$|b_6| \leqq \frac{240\cos\phi (w)_5}{(u)_5 (v)_5}(1 + \mu^2 + \mu^4), \tag{1.10}$$

where $\mu \in [0, 1]$ and $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. Consider the function $\vartheta(z)$ given by

$$\vartheta(z) = \cos\phi + i\sin\phi + \sum_{j=1}^{\infty} \kappa_j z^j \Longrightarrow \varphi(z) = \frac{\vartheta(z) - i\sin\phi}{\cos\phi}. \tag{1.11}$$

Then, by (1.5), we can have

$$\frac{\mathcal{L}^{w}_{u,v}f(z)}{z}(1-e^{-2i\phi}\mu^2z^2)e^{i\phi}=\vartheta(z). \tag{1.12}$$

Also, from (1.11) and (1.12), we get

$$\frac{\mathcal{L}^{w}_{u,v}f(z)}{z}(1-e^{-2i\phi}\mu^2z^2)e^{i\phi}=\varphi(z)\cos\phi+i\sin\phi. \tag{1.13}$$

As a result, the right-hand side of (1.13) is given by

$$\cos\phi+r_1z\cos\phi+r_2z^2\cos\phi+\cdots=\cos\phi+\kappa_1z+\kappa_2z^2+\cdots,$$

which implies that

$$\kappa_j=r_j\cos\phi \qquad (j\in\mathbb{N}). \tag{1.14}$$

Furthermore, from the left-hand side of (1.13), we have

$$\begin{aligned}\frac{\mathcal{L}^{w}_{u,v}f(z)}{z}(1-e^{-2i\phi}\mu^2z^2)e^{i\phi}&=e^{i\phi}\left[(1-e^{-2i\phi}\mu^2z^2)\left(1+\sum_{j=2}^{\infty}\frac{(u)_{j-1}(v)_{j-1}}{(w)_{j-1}(1)_{j-1}}b_jz^{j-1}\right)\right]\\&=e^{i\phi}+\frac{(u)(v)}{(w)}e^{i\phi}b_2z+\left(\frac{(u)_2(v)_2}{2(w)_2}e^{i\phi}b_3-e^{-i\phi}\mu^2\right)z^2+\left(\frac{(u)_3(v)_3}{6(w)_3}e^{i\phi}b_4-e^{-i\phi}\frac{(u)_1(v)_1}{(w)_1}\mu^2b_2\right)z^3\\&+\left(\frac{(u)_4(v)_4}{24(w)_4}e^{i\phi}b_5-e^{-i\phi}\frac{(u)_2(v)_2}{2(w)_2}\mu^2b_3\right)z^4+\left(\frac{(u)_5(v)_5}{120(w)_5}e^{i\phi}b_6-e^{-i\phi}\frac{(u)_3(v)_3}{6(w)_3}\mu^2b_4\right)z^5+\cdots.\end{aligned} \tag{1.15}$$

Now, upon comparing the coefficients of z, z^2, z^3, z^4 and z^5 in (1.14) and (1.15), we get

$$b_2=\frac{(w)_1r_1\cos\phi e^{-i\phi}}{(u)_1(v)_1}, \tag{1.16}$$

$$b_3=\frac{2(w)_2(r_2\cos\phi e^{-i\phi}+e^{-2i\phi}\mu^2)}{(u)_2(v)_2}. \tag{1.17}$$

$$b_4=\frac{6(w)_3(r_3\cos\phi e^{-i\phi}+e^{-3i\phi}r_1\cos\phi\mu^2)}{(u)_3(v)_3}, \tag{1.18}$$

$$b_5=\frac{24(w)_4(r_4\cos\phi e^{-i\phi}+r_2\mu^2e^{-3i\phi}\cos\phi+e^{-4i\phi}\mu^4)}{(u)_4(v)_4} \tag{1.19}$$

and

$$b_6=\frac{120(w)_5(r_5\cos\phi e^{-i\phi}+r_3\mu^2e^{-3i\phi}\cos\phi+e^{-5i\phi}r_1\cos\phi\mu^4)}{(u)_5(v)_5}. \tag{1.20}$$

The desired estimate is obtained by first applying the triangle inequality to (1.16) to (1.20) and then using Lemma 1.1. The proof of Theorem 1.1 is thus completed. □

Theorem 1.2. *Let $f(z) \in \mathcal{J}_{\mu}^{\phi}(u, v, w)$, $0 \leqq \mu \leqq 1$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Then*

$$\left|b_3 - \sigma b_2^2\right| \leqq \begin{cases} \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi) - e^{-i\phi}\sigma\left(\frac{2(w)_1 \cos\phi}{(u)_1(v)_1}\right)^2, & (\sigma \leqq 0), \\ \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi), & (0 \leqq \sigma \leqq A_1), \\ \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 - 2\cos\phi) + e^{-i\phi}\sigma\left(\frac{2(w)_1 \cos\phi}{(u)_1(v)_1}\right)^2, & (\sigma \geqq A_1), \end{cases}$$

where

$$A_1 = \frac{2(w)_2 e^{i\phi}}{(u)_2(v)_2 \cos\phi}\left(\frac{(u)_1(v)_1}{(w)_1}\right)^2$$

for any real number σ.

Proof. By applying (1.16) and (1.17), we have

$$\begin{aligned} |b_3 - \sigma b_2^2| &= \left|\frac{2r_2 \cos\phi e^{-i\phi}(w)_2}{(u)_2(v)_2} + \frac{2\mu^2 e^{-2i\phi}(w)_2}{(u)_2(v)_2} - \frac{\sigma r_1^2 \cos^2\phi e^{-2i\phi}(w)_1^2}{(u)_1^2(v)_1^2}\right| \\ &\leqq \frac{2\mu^2(w)_2}{(u)_2(v)_2} + \frac{2(w)_2\cos\phi}{(u)_2(v)_2}\left|r_2 - r_1^2 \frac{\sigma(w)_1^2(u)_2(v)_2\cos\phi e^{-i\phi}}{2(u)_1^2(v)_1^2(w)_2}\right| \\ &\leqq \frac{2\mu^2(w)_2}{(u)_2(v)_2} + \frac{2(w)_2\cos\phi}{(u)_2(v)_2}\left|r_2 - \upsilon\frac{r_1^2}{2}\right|, \end{aligned}$$

where

$$\upsilon = \frac{\sigma(w)_1^2(u)_2(v)_2\cos\phi e^{-i\phi}}{(u)_1^2(v)_1^2(w)_2}. \tag{1.21}$$

By Lemma 1.2 and for $\upsilon \leqq 0$, we get

$$|b_3 - \sigma b_2^2| \leqq \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi) - e^{-i\phi}\sigma\left(\frac{2(w)_1\cos\phi}{(u)_1(v)_1}\right)^2 \tag{1.22}$$

and, for $\upsilon \leqq 0$ in (1.21), we have

$$\frac{\sigma(w)_1^2(u)_2(v)_2\cos\phi e^{-i\phi}}{(u)_1^2(v)_1^2(w)_2} \leqq 0. \tag{1.23}$$

Also, by applying Lemma 1.2, and for $0 \leqq \upsilon \leqq 2$, we obtain

$$|b_3 - \sigma b_2^2| \leqq \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi) \tag{1.24}$$

and for $0 \leqq \upsilon \leqq 2$ in (1.21), we get

$$0 \leqq \sigma \leqq \frac{2(w)_2 e^{i\phi}}{(u)_2(v)_2\cos\phi}\left(\frac{(u)_1(v)_1}{(w)_1}\right)^2. \tag{1.25}$$

Next, for $\upsilon \geqq 2$ in Lemma 1.2, we have

$$|b_3 - \sigma b_2^2| \leqq \frac{2\mu^2(w)_2}{(u)_2(v)_2} + \frac{2(w)_2\cos\phi}{(u)_2(v)_2}\left[2\left(\frac{\sigma(w)_1^2(u)_2(v)_2\cos\phi e^{-i\phi}}{(u)_1^2(v)_1^2(w)_2} - 1\right)\right], \tag{1.26}$$

which gives

$$|b_3 - \sigma b_2^2| \leqq \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 - 2\cos\phi) + e^{-i\phi}\sigma\left(\frac{2(w)_1\cos\phi}{(u)_1(v)_1}\right)^2 \tag{1.27}$$

and, for $\upsilon \geqq 2$ in (1.21), we get

$$\sigma \geqq \frac{2(w)_2 e^{i\phi}}{(u)_2(v)_2\cos\phi}\left(\frac{(u)_1(v)_1}{(w)_1}\right)^2.$$

This completes the proof of Theorem 1.2. □

Theorem 1.3. *Let $f(z) \in \mathcal{J}_\mu^\phi(u, v, w)$, $0 \leqq \mu \leqq 1$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Then*

$$H_2(1) = |b_3 - b_2^2| \leqq \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi). \tag{1.28}$$

Proof. By applying (1.16) and (1.17), we have

$$\begin{aligned}|b_3 - b_2^2| &= \left|\frac{2r_2\cos\phi e^{-i\phi}(w)_2}{(u)_2(v)_2} + \frac{2\mu^2 e^{-2i\phi}(w)_2}{(u)_2(v)_2} - \frac{r_1^2\cos^2\phi e^{-2i\phi}(w)_1^2}{(u)_1^2(v)_1^2}\right| \\ &\leqq \frac{2\mu^2(w)_2}{(u)_2(v)_2} + \frac{2(w)_2\cos\phi}{(u)_2(v)_2}\left|r_1 - \frac{(w)_1^2(u)_2(v)_2\cos\phi e^{-i\phi}}{(u)_1^2(v)_1^2(w)_2}\frac{r_1^2}{2}\right|.\end{aligned}$$

Thus, by applying Lemma 1.2, we find that

$$|b_3 - b_2^2| \leqq \frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi).$$

□

Theorem 1.4. *Let $f(z) \in \mathcal{J}_\mu^\phi(u, v, w)$, $0 \leqq \mu \leqq 1$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Then*

$$\begin{aligned}H_2(2) = |b_2b_4 - b_3^2| &\leqq \left(\frac{2(w)_2}{(u)_2(v)_2}\right)^2(\mu^4 + 4\mu^2\cos\phi + 4\cos^2\phi) \\ &\quad + \frac{3\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}(\mu^4 + 6\mu^2 + 9).\end{aligned} \tag{1.29}$$

Proof. From the equations (1.16) to (1.18), we have

$$\begin{aligned}|b_2b_4 - b_3^2| \leqq \Bigg|&\frac{6r_1r_2(w)_1(w)_3\cos^2\phi e^{-2i\phi}}{(u)_1(v)_1(u)_3(v)_3} + \frac{6r_1^2(w)_1(w)_3\cos^2\phi e^{-4i\phi}\mu^2}{(u)_1(v)_1(u)_3(v)_3} - \frac{4r_2^2 e^{-2i\phi}\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2} \\ &- \frac{8r_2\mu^2 e^{-3i\phi}\cos\phi(w)_2^2}{(u)_2^2(v)_2^2} - \frac{4\mu^4 e^{-4i\phi}(w)_2^2}{(u)_2^2(v)_2^2}\Bigg|.\end{aligned}$$

Applying Lemma 1.3, and after some simplification, we find that

$$
\begin{aligned}
X|b_2b_4-b_3^2| =& \Bigg| \frac{3r_1^4 e^{-2i\phi}\cos^2\phi (w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} + \frac{3r_1^2(4-r_1^2)e^{-2i\phi}x\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} \\
& - \frac{3r_1^2(4-r_1^2)e^{-2i\phi}x^2\cos^2\phi (w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} + \frac{3r_1(4-r_1^2)(1-|x|^2)e^{-2i\phi}\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} \\
& + \frac{6r_1^2\mu^2 e^{-4i\phi}\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} - \frac{r_1^4 e^{-2i\phi}\cos^2\phi (w)_2^2}{(u)_2^2(v)_2^2} - \frac{2r_1^2 x e^{-2i\phi}\cos^2\phi (4-r_1^2)(w)_2^2}{(u)_2^2(v)_2^2} \\
& - \frac{x^2 e^{-2i\phi}\cos^2\phi (4-r_1^2)^2(w)_2^2}{(u)_2^2(v)_2^2} - \frac{4r_1^2\mu^2 e^{-3i\phi}\cos\phi (w)_2^2}{(u)_2^2(v)_2^2} - \frac{4\mu^2 x(4-r_1^2)e^{-3i\phi}\cos\phi (w)_2^2}{(u)_2^2(v)_2^2} \\
& - \frac{4\mu^4 e^{-4i\phi}(w)_2^2}{(u)_2^2(v)_2^2} \Bigg|.
\end{aligned}
$$

Let $r_1 = r$ and recall that $|r_1| \leqq 2$. We may assume without restriction that $r \in [0,2]$. Then, by using the triangle inequality, we get

$$
\begin{aligned}
|b_2b_4-b_3^2| \leqq & \frac{3r^4\cos^2\phi (w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} + \frac{3r^2(4-r_1^2)|x|\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} \\
& + \frac{3r^2(4-r^2)|x|^2\cos^2\phi (w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} + \frac{3r(4-r^2)(1-|x|^2)\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} \\
& + \frac{6r^2\mu^2\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} + \frac{r^4\cos^2\phi (w)_2^2}{(u)_2^2(v)_2^2} + \frac{2r^2|x|\cos^2\phi (4-r^2)(w)_2^2}{(u)_2^2(v)_2^2} \\
& + \frac{|x|^2\cos^2\phi (4-r^2)^2(w)_2^2}{(u)_2^2(v)_2^2} + \frac{4r^2\mu^2\cos\phi (w)_2^2}{(u)_2^2(v)_2^2} + \frac{4\mu^2|x|(4-r^2)\cos\phi (w)_2^2}{(u)_2^2(v)_2^2} + \frac{4\mu^4(w)_2^2}{(u)_2^2(v)_2^2}.
\end{aligned}
$$

Now, putting $\lambda = |x| \leqq 1$, we have

$$
\begin{aligned}
|b_2b_4-b_3^2| \leqq & \frac{3r^4\cos^2\phi (w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} + \frac{3r^2(4-r_1^2)\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\lambda \\
& + \frac{3r^2(4-r^2)\cos^2\phi (w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3}\lambda^2 + \frac{3r(4-r^2)\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} \\
& - \frac{3r(4-r^2)\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\lambda^2 + \frac{6r^2\mu^2\cos^2\phi (w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} + \frac{r^4\cos^2\phi (w)_2^2}{(u)_2^2(v)_2^2} \\
& + \frac{2r^2\cos^2\phi (4-r^2)(w)_2^2}{(u)_2^2(v)_2^2}\lambda + \frac{(4-r^2)^2\cos^2\phi (w)_2^2}{(u)_2^2(v)_2^2}\lambda^2 + \frac{4r^2\mu^2\cos\phi (w)_2^2}{(u)_2^2(v)_2^2} \\
& + \frac{4\mu^2(4-r^2)\cos\phi (w)_2^2}{(u)_2^2(v)_2^2}\lambda + \frac{4\mu^4(w)_2^2(1)_2^2}{(u)_2^2(v)_2^2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|b_2b_4-b_3^2| \leqq & \left\{\frac{3r^4\cos^2\phi(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3}+\frac{3r(4-r^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}+\frac{6r^2\mu^2\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\right.\\
&\left.+\frac{r^4\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}+\frac{4r^2\mu^2\cos\phi(w)_2^2}{(u)_2^2(v)_2^2}+\frac{4\mu^4(w)_2^2}{(u)_2^2(v)_2^2}\right\}+\left\{\frac{3r^2(4-r_1^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\right.\\
&\left.+\frac{2r^2(4-r^2)\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}+\frac{4\mu^2(4-r^2)\cos\phi(w)_2^2}{(u)_2^2(v)_2^2}\right\}\lambda+\left\{\frac{3r^2(4-r^2)\cos^2\phi(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3}\right.\\
&\left.-\frac{3r(4-r^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}+\frac{(4-r^2)^2\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}\right\}\lambda^2\\
=&G_1(r,\lambda).
\end{aligned}
$$

Now, maximizing the function $G_1(r,\lambda)$ in the closed interval $0 \leqq \lambda \leqq 1$, we obtain

$$
\begin{aligned}
\frac{\partial G_1(\lambda,r)}{\partial\lambda} = & \left\{\frac{3r^2(4-r_1^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}+\frac{2r^2(4-r^2)\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}\right.\\
&\left.+\frac{4\mu^2(4-r^2)\cos\phi(w)_2^2}{(u)_2^2(v)_2^2}\right\}+2\left\{3\frac{r^2(4-r^2)\cos^2\phi(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3}\right.\\
&\left.-\frac{3r(4-r^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}+\frac{(4-r^2)^2\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}\right\}\lambda\\
>&\,0
\end{aligned}
$$

for $0 \leqq r \leqq 1$. Thus, clearly, $G_1(\lambda,r)$ is an increasing function. Hence, it has the maximum point at $\lambda = 1$ and we have

$$
\begin{aligned}
\max_{0\leqq\lambda\leqq1} G_1(\lambda,r) = G_1(1,r) \leqq & \frac{3r^4\cos^2\phi(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3}+\frac{3r(4-r^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\\
&+\frac{6r^2\mu^2\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}+\frac{r^4\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}+\frac{4r^2\mu^2\cos\phi(w)_2^2}{(u)_2^2(v)_2^2}\\
&+\frac{4\mu^4(w)_2^2}{(u)_2^2(v)_2^2}+\frac{3r^2(4-r_1^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\\
&+\frac{2r^2(4-r^2)\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2}+\frac{4\mu^2(4-r^2)\cos\phi(w)_2^2}{(u)_2^2(v)_2^2}\\
&+\frac{3r^2(4-r^2)\cos^2\phi(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3}+\frac{3r(4-r^2)\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\\
&+\frac{(4-r^2)^2\cos^2\phi(w)_2^2}{(u)_2^2(v)_2^2} = G(r).
\end{aligned} \tag{1.30}
$$

After simplifying and differentiating with respect to r, we have

$$
G'(r) = [\mu^2+3]\frac{12\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}r-\frac{12\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}r^3.
$$

By equating $G'(r)$ to zero and doing some simple calculations, we have the critical points at

$$r_0 = 0, \quad r_1 = \sqrt{\mu^2+3} \quad \text{and} \quad r_2 = -\sqrt{\mu^2+3}.$$

The maximum point occurs at $r_1 = \sqrt{\mu^2+3}$, so by using (1.30), we get

$$\begin{aligned} G(r) =& \frac{4(w)_2^2}{(u)_2^2(v)_2^2}\{\mu^4 + 4\mu^2\cos\phi + 4\cos^2\phi\} + \frac{6\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\{\mu^4+6\mu^2+9\} \\ & - \frac{3\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\{\mu^4+6\mu^2+9\}. \end{aligned}$$

Hence, we have

$$|b_2b_4 - b_3^2| \leqq \left(\frac{2(w)_2}{(u)_2(v)_2}\right)^2\{\mu^4+4\mu^2\cos\phi+4\cos^2\phi\} + \frac{3\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\{\mu^4+6\mu^2+9\}.$$

□

Theorem 1.5. *Let $f(z) \in \mathcal{J}_\mu^\phi(u,v,w)$, $0 \leqq \mu \leqq 1$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Then*

$$\begin{aligned} |b_2b_3 - b_4| \leqq & -\frac{3\cos\phi(w)_3}{(u)_3(v)_3}\times \\ & \left[\left(\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}\right)^{\frac{3}{2}}\right] \\ & + \sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}} \\ & \times\left(\frac{2\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2}(\mu^2+2\cos\phi)\right)+\left(\frac{6\cos\phi(w)_3}{(u)_3(v)_3}(\mu^3+3)\right) \\ & \times\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}. \end{aligned} \tag{1.31}$$

Proof. Applying the equations (1.16) to (1.18), we have

$$|b_2b_3 - b_4| \leqq \left|\frac{2r_1r_2e^{-2i\phi}\cos^2\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{2r_1\mu^2e^{-3i\phi}\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} - \frac{6r_3e^{-i\phi}\cos\phi(r)_3}{(u)_3(v)_3} - \frac{6\mu^2r_1e^{-3i\phi}\cos\phi(r)_3}{(u)_3(v)_3}\right|.$$

Applying Lemma 1.3, we obtain

$$\begin{aligned} |b_2b_3 - b_4| = & \left|\frac{r_1^3e^{-2i\phi}\cos^2\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{r_1(4-r_1^2)e^{-2i\phi}x\cos^2\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2}\right. \\ & + \frac{2r_1\mu^2e^{-3i\phi}\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} - \frac{3r_1^3e^{-i\phi}\cos\phi(w)_3}{2(u)_3(v)_3} - \frac{3r_1(4-r_1^2)e^{-i\phi}x\cos\phi(w)_3}{(u)_3(v)_3} \\ & \left. + \frac{3r_1(4-r_1^2)e^{-i\phi}x^2\cos\phi(w)_3}{2(u)_3(v)_3} - \frac{3(4-r_1^2)(1-|x|^2)e^{-i\phi}\cos\phi z(w)_3}{(u)_3(v)_3} - \frac{6r_1\mu^2e^{-3i\phi}\cos\phi(w)_3}{(u)_3(v)_3}\right|. \end{aligned}$$

Let $r_1 = r$, assuming that $|r| = |r_1| \leqq 2$, so that without restriction, $r \in [0, 2]$, and by applying triangle inequality with $|x| = \lambda \leqq 1$ and $|z| \leqq 1$, we find that

$$\begin{aligned} |b_2b_3 - b_4| \leqq & \left\{ \frac{r^3 \cos^2 \phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{2r\mu^2 \cos\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{3r^3 \cos\phi (w)_3}{2(u)_3(v)_3} \right. \\ & \left. + \frac{3(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} + \frac{6r\mu^2 \cos\phi (w)_3}{(u)_3(v)_3} \right\} + \left\{ \frac{r(4-r^2)\cos^2\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} \right. \\ & \left. + \frac{3r(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} \right\}\lambda + \left\{ \frac{3r(4-r^2)\cos\phi (w)_3}{2(u)_3(v)_3} - \frac{3(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} \right\}\lambda^2 \\ = & \, G_2(\lambda, r). \end{aligned} \tag{1.32}$$

By differentiating with respect to λ, we have

$$\begin{aligned} \frac{\partial G_2(\lambda, r)}{\partial \lambda} = & \left\{ \frac{r(4-r^2)\cos^2\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{3r(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} \right\} \\ & + \left\{ \frac{3r(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} - \frac{6(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} \right\}\lambda \\ > & \, 0 \end{aligned}$$

for $0 \leqq \lambda \leqq 1$. Since $G_2'(\lambda, r) > 0$ for $0 \leqq \lambda \leqq 1$, it means that $G_2(\lambda, r)$ is an increasing function with its maximum point at $\lambda = 1$. Hence, from (1.32), we have

$$\begin{aligned} \max_{0 \leqq \lambda \leqq 1} G_2(\lambda, r) = G_2(1, r) \leqq & \left\{ \frac{r^3 \cos^2 \phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{2r\mu^2 \cos\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} \right\} \\ & + \left\{ \frac{3r^3 \cos\phi (w)_3}{2(u)_3(v)_3} + \frac{3(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} + \frac{6r\mu^2 \cos\phi (w)_3}{(u)_3(v)_3} \right\} \\ & + \left\{ \frac{r(4-r^2)\cos^2\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{3r(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} \right\} \\ & + \left\{ \frac{3r(4-r^2)\cos\phi (w)_3}{2(u)_3(v)_3} - \frac{3(4-r^2)\cos\phi (w)_3}{(u)_3(v)_3} \right\} = G(r). \end{aligned} \tag{1.33}$$

After some simple calculations and simplification, we get

$$\begin{aligned} G(r) = & \frac{2r\mu^2 \cos\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{6r\mu^2 \cos\phi (w)_3}{(u)_3(v)_3} + \frac{4r\cos^2\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} \\ & + \frac{12r\cos\phi (w)_3}{(u)_3(v)_3} - \frac{3r^3\cos\phi (w)_3}{(u)_3(v)_3} + \frac{12\cos\phi (w)_3}{(u)_3(v)_3}. \end{aligned} \tag{1.34}$$

By differentiating $G(r)$ with with respect to r and equating it to zero, the critical point will be seen to occur at

$$\frac{2\mu^2 \cos\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{4\cos^2\phi (w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2} + \frac{18\cos\phi (w)_3}{(u)_3(v)_3} + \frac{6\mu^2\cos\phi (w)_3}{(u)_3(v)_3} = \frac{9\cos\phi (w)_3}{(u)_3(v)_3} r^2.$$

Hence, we have

$$r=\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}},$$

$$r=-\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}.$$

Also, we have

$$\begin{aligned}G'(r)&=\frac{-18\cos\phi(w)_2}{(u)_3(v)_3}r\\&=-\frac{18\cos\phi(w)_2}{(u)_3(v)_3}\times\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}.\end{aligned}$$

From (1.34), we get

$$\begin{aligned}G(r)=-\frac{3\cos\phi(w)_3}{(u)_3(v)_3}\times&\\ \Bigg[\left(\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}\right)^{\frac{3}{2}}\Bigg]&\\ +\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}&\\ \times\left(\frac{2\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2}(\mu^2+2\cos\phi)\right)+\left(\frac{6\cos\phi(w)_3}{(u)_3(v)_3}(\mu^3+3)\right)&\\ \times\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}&.\end{aligned}$$

Hence, we have

$$\begin{aligned}|b_2b_3-b_4|\leqq-\frac{3\cos\phi(w)_3}{(u)_3(v)_3}\times&\\ \Bigg[\left(\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}\right)^{\frac{3}{2}}\Bigg]&\\ +\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}&\\ \times\left(\frac{2\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2}(\mu^2+2\cos\phi)\right)+\left(\frac{6\cos\phi(w)_3}{(u)_3(v)_3}(\mu^3+3)\right)&\\ \times\sqrt{\frac{2(w)_1(w)_2(u)_3(v)_3(\mu^2+2\cos\phi)+6(w)_3(u)_1(v)_1(u)_2(v)_2(3+\mu^2)}{9(w)_3(u)_1(v)_1(u)_2(v)_2}}&.\end{aligned}$$

□

Theorem 1.6. *Let $f(z) \in \mathcal{J}_{\mu}^{\phi}(u, v, w)$, $0 \leqq \mu \leqq 1$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Then*

$$\begin{aligned}
\mathcal{H}_3(1) \leqq & \left[\frac{2(w)_2}{(u)_2(v)_2}(2\cos\phi + \mu^2)\right]\left[\left(\frac{2(w)_2}{(u)_2(v)_2}\right)^2 \{\mu^4 + 4\mu^2\cos\phi + 4\cos^2\phi\}\right. \\
& \left. + \frac{3\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\{\mu^4 + 6\mu^2 + 9\}\right] + \left[\frac{12\cos\phi(w)_3}{(u)_3(v)_3}(1 + \mu^2)\right]\left[-\frac{3\cos\phi(w)_3}{(u)_3(v)_3}\right. \\
& \left. \cdot\left(\frac{2A(\mu^2 + 2\cos\phi) + 2B(3 + \mu^2)}{3B}\right)^{\frac{3}{2}}\right] + \left[\frac{12\cos\phi(w)_3}{(u)_3(v)_3}(1 + \mu^2)\right] \\
& \left[\sqrt{\frac{2A(\mu^2 + 2\cos\phi) + 2B(3 + \mu^2)}{3B}}\frac{2\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2}(\mu^2 + 2\cos\phi)\right] \\
& + \left[\frac{12\cos\phi(w)_3}{(u)_3(v)_3}(1 + \mu^2)\right]\left[\sqrt{\frac{2A(\mu^2 + 2\cos\phi) + 2B(3 + \mu^2)}{3B}}\right. \\
& \left. \cdot\frac{6\cos\phi(w)_3}{(u)_3(v)_3}(\mu^3 + 3)\right] + \frac{24(w)_4}{(u)_4(v)_4}(2\cos\phi + 2\mu^2\cos\phi + \mu^4)\left[\frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi)\right], \quad (1.35)
\end{aligned}$$

where

$$A = 2(w)_1(w)_2(u)_3(v)_3 \qquad \textit{and} \qquad B = 6(w)_3(u)_1(v)_1(u)_2(v)_2.$$

Proof. Taking it from (1.4), we have

$$\mathcal{H}_3(1) = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{vmatrix} \quad (b_1 = 1) \tag{1.36}$$

$$= b_3(b_2 b_4 - b_3^2) - b_4(b_4 - b_2 b_3) + b_5(b_3 - b_2^2). \tag{1.37}$$

Applying Theorems 1.1 as well as 1.3 to 1.5, and by using the triangle inequality, we have

$$\begin{aligned}
\mathcal{H}_3(1) \leqq & \left[\frac{2(w)_2}{(u)_2(v)_2}(2\cos\phi + \mu^2)\right]\left[\left(\frac{2(w)_2}{(u)_2(v)_2}\right)^2 \{\mu^4 + 4\mu^2\cos\phi + 4\cos^2\phi\}\right. \\
& \left. + \frac{3\cos^2\phi(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3}\{\mu^4 + 6\mu^2 + 9\}\right] + \left[\frac{12\cos\phi(w)_3}{(u)_3(v)_3}(1 + \mu^2)\right]\left[-\frac{3\cos\phi(w)_3}{(u)_3(v)_3}\right. \\
& \left. \left(\frac{2A(\mu^2 + 2\cos\phi) + 2B(3 + \mu^2)}{3B}\right)^{\frac{3}{2}}\right] + \left[\frac{12\cos\phi(w)_3}{(u)_3(v)_3}(1 + \mu^2)\right] \\
& \left[\sqrt{\frac{2A(\mu^2 + 2\cos\phi) + 2B(3 + \mu^2)}{3B}}\frac{2\cos\phi(w)_1(w)_2}{(u)_1(v)_1(u)_2(v)_2}(\mu^2 + 2\cos\phi)\right] \\
& + \left[\frac{12\cos\phi(w)_3}{(u)_3(v)_3}(1 + \mu^2)\right]\left[\sqrt{\frac{2A(\mu^2 + 2\cos\phi) + 2B(3 + \mu^2)}{3B}}\right. \\
& \left. \cdot\frac{6\cos\phi(w)_3}{(u)_3(v)_3}(\mu^3 + 3)\right] + \frac{24(w)_4}{(u)_4(v)_4}(2\cos\phi + 2\mu^2\cos\phi + \mu^4)\left[\frac{2(w)_2}{(u)_2(v)_2}(\mu^2 + 2\cos\phi)\right].
\end{aligned}$$

□

2. Concluding remarks and observations

Our present investigation was motivated by a number of recent developments on the Fekete-Szegö functional, the Hankel determinants of the third and the fourth kinds, and the associated Taylor-Maclaurin coefficient estimates and coefficient inequalities. Here, in this paper, we have introduced and systematically studied a new subclass of normalized analytic and univalent functions in the open unit disk $\mathbb{U}$, which satisfies the following geometric criterion:

$$\Re\left(\frac{\mathcal{L}_{u,v}^{w}f(z)}{z}(1-e^{-2i\phi}\mu^2z^2)e^{i\phi}\right)>0,$$

where $z\in\mathbb{U}$, $0\leqq\mu\leqq1$ and $\phi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, and which is associated with the Hohlov operator $\mathcal{L}_{u,v}^{w}$. For functions in this class, we have investigated several coefficient bounds, as well as upper estimates for the Fekete-Szegö functional and the Hankel determinant.

It should be remarked that, in many recent investigations dealing with some of the topics of our presentation in this paper, the basic or quantum (or q-) calculus was extensively used (see [39, 60, 67]).

We conclude this paper by recalling a recently-published survey-cum-expository review article in which Srivastava [63] explored the mathematical applications of the q-calculus, the fractional q-calculus and the fractional q-derivative operators in geometric function theory of complex analysis, especially in the study of Fekete-Szegö functional. Srivastava [63] also exposed the not-yet-widely-understood fact that the so-called (p,q)-variation of the classical q-calculus is, in fact, a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [63, p. 340]; see also [64, pp. 1511–1512]).

Conflicts of interest

The authors declare that they have no conflicts of interest.

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