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COEFFICIENT PROBLEM CONCERNING SOME NEW SUBCLASSES OF P-VALENT FUNCTIONS

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Abstract. The main aim of this research is to investigate some properties of fractional p-valent function belonging to certain new subclasses of p-valent functions $S^*_{A,B,m}(\rho,\gamma,p,\phi)$ and $\mathcal{K}_{A,B,m}(\rho,\gamma,p,\phi)$ defined in the open unit disk.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{\tau=1}^{\infty} a_{\tau+p} z^{\tau+p} \tag{1}$$

where $p \in \mathbb{N} = \{1, 2, \cdots\}$ and $z \in \mathcal{D}$,
which are holomorphic in the open unit disk

$$\mathcal{D} = \{z \in \mathcal{C} : |z| < 1\}.$$

Let $\mathcal{Q}(p)$ denote the class of p-valent functions $f(z) \in \mathcal{A}(p)$ in $\mathcal{D}$. Also suppose that $S^*(p)$ denote the subclass of $\mathcal{Q}(p)$ consisting of the functions $f(z)$ which are p-valently starlike in $\mathcal{D}$. A function $f(z) \in \mathcal{K}(p)$ is said to be p-valently convex in $\mathcal{D}$ if $f(z) \in \mathcal{Q}(p)$ satisfies $zf'(z) \in S^*(p)$. see [2, 3, 4] and [7] for details. In view of the above definition, we can write that

$$\mathcal{K}(p) \subset S^*(p) \subset \mathcal{Q}(p) \subset \mathcal{A}(p)$$

and $f(z) \in S^*(p)$ if and only if

$$\int_0^z \frac{f(t)}{t} dt \in \mathcal{K}(p).$$

Example: The function $f(z)$ given by

$$f(z) = \frac{z^p}{1 - z^2} = z^p + z^{p+2} + z^{p+4} + \cdots = \sum_{\tau=0}^{\infty} z^{2\tau+p} \tag{2}$$

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is in the class $S^*(p)$ and the function $f(z)$ of the form

$$f(z) = \frac{z^p}{1-z} = z^p + z^{p+1} + z^{p+2} + \cdots = \sum_{\tau=0}^{\infty} z^{\tau+p} \tag{3}$$

is in the class $\mathcal{K}(p)$, where $p \in \mathbb{N} = \{1, 2, \cdots\}$. When $p = 1$ we have

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots = \sum_{\tau=0}^{\infty} z^{2\tau+1} \in S^*,$$

and

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots = \sum_{\tau=0}^{\infty} z^{\tau+1} \in \mathcal{K}.$$

The analytic function $f_\rho(z)$ of the from

$$f_\rho(z) = \frac{z}{1-z^\rho} = z + \sum_{\tau=1}^{\infty} z^{\rho\tau+1} \tag{4}$$

was introduced and studied Darus and Owa [2], for more details see [1, 6].
In 2020, [5] introduced and studied the generalized fractional analytic function $f_\rho(z)$ of the form

$$f_\rho(z) = \frac{A(z-\varepsilon)}{A + B(z-\varepsilon)^\rho} = (z-\varepsilon) + \sum_{\tau=1}^{\infty} (-1)^\tau \frac{B^\tau}{A^\tau} (z-\varepsilon)^{1+\tau\rho} \tag{5}$$

for some real $\rho(0 < \rho \le 2)$, $-1 \le B < A \le 1$ where ε is a fixed point in $\mathcal{D}$. Now using (5), they obtain a new class $\mathcal{A}_\rho$ of analytic functions $f_\rho(z)$ is given in $\mathcal{D}$ such that

$$f_\rho(z) = \frac{A(z-\varepsilon)}{A + B(z-\varepsilon)^\rho} = (z-\varepsilon) + \sum_{\tau=1}^{\infty} (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+1} (z-\varepsilon)^{1+\tau\rho} \tag{6}$$

for some real $\rho(0 < \rho \le 2)$, $-1 \le B < A \le 1$ where ε is a fixed point in $\mathcal{D}$.

Now we shall consider the p-valent function of the generalized form of (4),

$$f_\rho(z) = (z-\varepsilon)^p + \sum_{\tau=1}^{\infty} (-1)^\tau \frac{B^\tau}{A^\tau} (z-\varepsilon)^{p+\tau\rho} \tag{7}$$

where $z \in \mathcal{D}$, $p \in \mathbb{N} = \{1, 2, \cdots\}$, $\rho(0 < \rho \le 2)$, $-1 \le B < A \le 1$ and ε is a fixed point in $\mathcal{D}$. We have a new class $\mathcal{A}_\rho(p)$ using (7) of functions $f_\rho(z)$ is given in $\mathcal{D}$ such that

$$f_\rho(z) = (z-\varepsilon)^p + \sum_{\tau=1}^{\infty} (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p} (z-\varepsilon)^{p+\tau\rho} \tag{8}$$

where $z \in \mathcal{D}$, $p \in \mathbb{N} = \{1, 2, \cdots\}$, $\rho(0 < \rho \le 2)$, $-1 \le B < A \le 1$ and ε is a fixed point in $\mathcal{D}$.

$$D^m f_\rho(z) = p^m (z-\varepsilon)^p + \sum_{\tau=1}^{\infty} \left[p + \tau\rho + \frac{\tau\rho\gamma}{2}\right]^m (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p} (z-\varepsilon)^{p+\tau\rho} \tag{9}$$

where $-1 \le B < A \le 1$, $\gamma \ge 0$, $p \in \mathbb{N} = \{1, 2, \cdots\}$, $\rho > 0$, $m \in \mathbb{N}$ and ε is a fixed point in $\mathcal{D}$.

When $m = 1$

$$D^1 f_\rho(z) = p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[p+\tau\rho+\frac{\tau\rho\gamma}{2}\right](-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}$$

$$D^1 f_\rho(z) = p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty} p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} + \sum_{\tau=1}^{\infty}\tau\rho(-1)^\tau$$
$$\frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} + \sum_{\tau=1}^{\infty}\frac{\tau\rho\gamma}{2}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} - \frac{\gamma}{2}p(z-\varepsilon)^p$$
$$+\frac{\gamma}{2}p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\frac{\gamma}{2}p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p\tau\rho} - \sum_{\tau=1}^{\infty}\frac{\gamma}{2}p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}$$

$$D^1 f_\rho(z) = -\frac{\gamma}{2}p(z-\varepsilon)^p - \sum_{\tau=1}^{\infty}\frac{\gamma}{2}p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}$$
$$+p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty} p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} + \sum_{\tau=1}^{\infty}\tau\rho(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}$$
$$+\sum_{\tau=1}^{\infty}\frac{\tau\rho\gamma}{2}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} + \frac{\gamma}{2}p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\frac{\gamma}{2}p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p\tau\rho}$$

$$D^1 f_\rho(z) = -\frac{\gamma}{2}p\left[(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}\right] + (z-\varepsilon)$$
$$\left[p(z-\varepsilon)^{p-1} + \sum_{\tau=1}^{\infty} p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1} + \sum_{\tau=1}^{\infty}\tau\rho(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}\right.$$
$$\left.(z-\varepsilon)^{p+\tau\rho-1}\right] + \frac{\gamma}{2}(z-\varepsilon)\left[p(z-\varepsilon)^{p-1} + \sum_{\tau=1}^{\infty} p(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}\right.$$
$$\left.+\sum_{\tau=1}^{\infty}\tau\rho(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}\right]$$

$$D^1 f_\rho(z) = -\frac{\gamma}{2}p\left[(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}\right]$$
$$+\left[1+\frac{\gamma}{2}\right](z-\varepsilon)\left[p(z-\varepsilon)^{p-1} + \sum_{\tau=1}^{\infty}(p+\tau\rho)(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}\right]$$
$$D^1 f_\rho(z) = -\frac{\gamma}{2}p\left[f_\rho(z)\right] + \left[1+\frac{\gamma}{2}\right](z-\varepsilon)[f_\rho(z)]'$$

We introduce the following linear differential operator such that

$$D^0 f_\rho(z) = f_\rho(z)$$

$$D^1 f_\rho(z) + \frac{\gamma p}{2}\left[D^0 f_\rho(z)\right] = (z-\varepsilon)\left[\frac{2+\gamma}{2}\right]\left[D^0 f_\rho(z)\right]'$$

$$D^m f_\rho(z) + \frac{\gamma p}{2}\left[D^{m-1} f_\rho(z)\right] = (z-\varepsilon)\left[\frac{2+\gamma}{2}\right]\left[D^{m-1} f_\rho(z)\right]'$$

Then we have,

$$D^m f_\rho(z) = p^m (z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} \quad (10)$$

also

$$D^{m+1} f_\rho(z) = p^{m+1}(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1} (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} \quad (11)$$

where $-1 \leq B < A \leq 1$, $\gamma \geq 0$, $p \in \mathbb{N} = \{1, 2, \cdots\}$, $\rho > 0$, $m \in \mathbb{N}_0$ and ε is a fixed point in $\mathcal{D}$.

Remark A: In particular

(1) When $\varepsilon = \gamma = 0$, $A = \rho = p = 1$ and $B = -1$ in (10), we have Salagean differential operator [8].

(2) When $p = 1$ in (10), we obtain the linear differential operator introduced by Hamzat and Raji [5].

Definition 1: Let $f_\rho(z) \in \mathcal{A}_\rho(p)$ satisfies the analytic condition

$$\Re\left[\frac{D^{m+1} f_\rho(z)}{D^m f_\rho(z)}\right] > \phi$$

for real $\phi(0 \leq \phi < 1)$ where ε is a fixed point in $\mathcal{D}$, then we say that $f_\rho(z) \in S^*_{A,B,m}(\rho, \gamma, p, \phi)$ where $S^*_{A,B,m}(\rho, \gamma, p, \phi)$ denote the class p-valently starlike functions of order ϕ.

Definition 2: Let $f_\rho(z) \in \mathcal{A}_\rho(p)$ satisfies the analytic condition

$$\Re\left[\frac{(z-\varepsilon)(D^{m+1} f_\rho(z))'}{D^{m+1} f_\rho(z)}\right] > \phi$$

for real $\phi(0 \leq \phi < 1)$ where ε is a fixed point in $\mathcal{D}$, then we say that $f_\rho(z) \in \mathcal{K}_{A,B,m}(\rho, \gamma, p, \phi)$ where $\mathcal{K}_{A,B,m}(\rho, \gamma, p, \phi)$ denote the class p-valently convex functions of order ϕ.

2. Coefficient Inequalities

Theorem 2.1 Let $f_\rho(z) \in \mathcal{A}_\rho(p)$ satisfies the inequality

$$\sum_{\tau=1}^{\infty} \frac{[2p+\tau\rho(2+\gamma)]^m}{2^{m+1}}\left[\tau\rho(2+\gamma) + (4-2\phi-2p)\right]\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| \leq (2-\phi-p)p^m(z-\varepsilon)^{p-1}, \quad (12)$$

then $f_\rho(z) \in S^*_{A,B,m}(\rho, \gamma, p, \phi)$ where $-1 \leq B < A \leq 1$, $\gamma \geq 0$, , $0 \leq \phi < 1$, $\rho > 0$, $m \in \mathbb{N}_0$ and ε is a fixed point in $\mathcal{D}$. The equation is attained for function $f_\rho(z)$ given by

$$f_\rho(z) = (z-\varepsilon)^p + \sum_{\tau=1}^{\infty} \frac{2^{m+1} A^\tau e^{i\pi}(2-\phi-p)p^m(z-\varepsilon)^{p-1}}{\tau(\tau+1)(2p+\tau\rho(2+\gamma))^m\left[\tau\rho(2+\gamma) + (4-2\phi-2p)\right]|B^\tau|}(z-\varepsilon)^{\tau\rho+p} \quad (13)$$

Proof: Suppose that $f_\rho(z) \in \mathcal{A}_\rho(p)$ is of the form (1), if $f_\rho(z)$ satisfies the iniquality (2), then

$$\left|\left(\frac{D^{m+1}f(z)}{D^m f(z)}\right) - 1\right| \leq (1-\phi)$$

$$\left|\left(\frac{D^{m+1}f(z)}{D^m f(z)}\right) - 1\right|$$

$$= \left|\frac{p^{m+1}(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^m(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}} - 1\right|$$

$$= \left|\frac{[p^m(z-\varepsilon)^p][p-1] + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} + \left[\frac{2p-2}{2} + \frac{\tau\rho(2+\gamma)}{2}\right]}{p^m(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|$$

$$= \left|(p-1) + \frac{\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^m(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|$$

$$= (p-1) + \left|\frac{\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^m(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|$$

$$\left|\frac{\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^m(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right| \leq (1-\phi)-(p-1)$$

$$\left|\frac{\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^m(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right| \leq (2-\phi-p)$$

$$\leq \frac{\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}||z-\varepsilon|^{p+\tau\rho-1}}{p^m(z-\varepsilon)^{p-1} - \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}||z-\varepsilon|^{p+\tau\rho-1}}$$

$$< \frac{\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}|}{p^m(z-\varepsilon)^{p-1} - \sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}|} \leq (2-\phi-p)$$

This shows that $f_\rho(z) \in S^*_{A,B,m}(\rho,\gamma,p,\phi)$. Now suppose that $f_\rho(z)$ is given by (6), then we have that

$$\sum_{\tau=1}^{\infty}\frac{\tau\rho(2+\gamma)}{2}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| + \sum_{\tau=1}^{\infty}(2-\phi-p)\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| \leq (2-\phi-p)p^m(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m\frac{|B^\tau|}{A^\tau}|a_{\tau+p}|\left[\frac{\tau\rho(2+\gamma)}{2}+(2-\phi-p)\right] \leq (2-\phi-p)p^m(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m\frac{|B^\tau|}{A^\tau}|a_{\tau+p}|\left[\frac{\tau\rho(2+\gamma)+(4-2\phi-2p)}{2}\right] \leq (2-\phi-p)p^m(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\frac{[2p+\tau\rho(2+\gamma)]^m}{2^{m+1}}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}|\left[\tau\rho(2+\gamma)+(4-2\phi-2p)\right] \leq (2-\phi-p)p^m(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\frac{[2p+\tau\rho(2+\gamma)]^m}{2^{m+1}}\left[\tau\rho(2+\gamma)+(4-2\phi-2p)\right]\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| = \sum_{\tau=1}^{\infty}\frac{(2-\phi-p)p^m(z-\varepsilon)^{p-1}}{\tau(\tau-1)}$$

$$= (2-\phi-p)p^m(z-\varepsilon)^{p-1}\sum_{\tau=1}^{\infty}\frac{1}{\tau(\tau-1)} = (2-\phi-p)p^m(z-\varepsilon)^{p-1}$$

If $m=0$ in Theorem 2.1, then the following corollary holds.

Corollary 2.2: Let $f_\rho(z)\in\mathcal{A}_\rho(p)$ satisfies the inequality

$$\sum_{\tau=1}^{\infty}\frac{1}{2}\left[\tau\rho(2+\gamma)+(4-2\phi-2p)\right]\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| \leq (2-\phi-p)(z-\varepsilon)^{p-1},$$

then $f_\rho(z)\in S^*_{A,B,0}(\rho,\gamma,p,\phi)$. The equation is attained for function $f_\rho(z)$ given by

$$f_\rho(z) = p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\frac{2A^\tau(2-\phi-p)(z-\varepsilon)^{p-1}}{\tau(\tau+1)\left[\tau\rho(2+\gamma)+(4-2\phi-2p)\right]|B^\tau|}(z-\varepsilon)^{\tau\rho+p}$$

Corollary 2.3: Let $f_\rho(z)\in\mathcal{A}_\rho(p)$ satisfies the inequality

$$\sum_{\tau=1}^{\infty}\frac{1}{2}\left[\tau\rho(2+\gamma)+(4-2p)\right]\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| \leq (2-p)(z-\varepsilon)^{p-1},$$

then $f_\rho(z)\in S^*_{A,B,0}(\rho,\gamma,p,0)$. The equation is attained for function $f_\rho(z)$ given by

$$f_\rho(z) = p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty}\frac{2A^\tau(2-p)(z-\varepsilon)^{p-1}}{\tau(\tau+1)\left[\tau\rho(2+\gamma)+(4-2p)\right]|B^\tau|}(z-\varepsilon)^{\tau\rho+p}$$

Corollary 2.4: Let $f_\rho(z)\in\mathcal{A}_\rho(p)$ satisfies the inequality

$$\sum_{\tau=1}^{\infty}\frac{1}{2}\left[\tau\rho(2+\gamma)+(4-2p)\right]|a_{\tau+p}| \leq (2-p)(z-\varepsilon)^{p-1},$$

then $f_\rho(z) \in S^*_{1,-1,0}(\rho,\gamma,p,0)$. The equation is attained for function $f_\rho(z)$ given by

$$f_\rho(z) = p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty} \frac{2(2-p)(z-\varepsilon)^{p-1}}{\tau(\tau+1)\left[\tau\rho(2+\gamma)+(4-2p)\right]}(z-\varepsilon)^{\tau\rho+p}$$

Corollary 2.5: Let $f_\rho(z) \in \mathcal{A}_\rho(p)$ satisfies the inequality

$$\sum_{\tau=1}^{\infty} \frac{1}{2}\left[3\tau\rho+(4-2p)\right]|a_{\tau+p}| \le (2-p)(z-\varepsilon)^{p-1},$$

then $f_\rho(z) \in S^*_{1,-1,0}(\rho,1,p,0)$. The equation is attained for function $f_\rho(z)$ given by

$$f_\rho(z) = p(z-\varepsilon)^p + \sum_{\tau=1}^{\infty} \frac{2(2-p)(z-\varepsilon)^{p-1}}{\tau(\tau+1)\left[3\tau\rho+(4-2p)\right]}(z-\varepsilon)^{\tau\rho+p}$$

Remark B: In a special situation

(1) When $p=1$, then the inequality (12), which yield the result obtained by Hamzat and Raji [5].
(2) When $p=1$, $m=\gamma=0$, $A=1$ and $B=-1$, then the inequality (12), which gives the result obtain by Darus and Owa [2].

Theorem 2.6: Let the function $f_\rho(z) \in \mathcal{A}_\rho(p)$ given by (8) satisfies the inequality

$$\sum_{\tau=1}^{\infty} \frac{[2p+\tau\rho(2+\gamma)]^{m+1}}{2^{m+1}}\left[\tau\rho+(2-\phi-p)\right]\frac{|B^\tau|}{A^\tau}|a_{\tau+p}| \le (2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}, \tag{14}$$

where $-1 \le B < A \le 1$, $\gamma \ge 0$, , $0 \le \phi < 1$, $\rho > 0$, $m \in \mathbb{N}_0$ and ε is a fixed point in $\mathcal{D}$. Then

$$f_\rho(z) \in \mathcal{K}_{A,B,m}(\rho,\gamma,p,\phi),$$

equality is attained for function $f_\rho(z)$ given by

$$f_\rho(z) = (z-\varepsilon)^p + \sum_{\tau=1}^{\infty} \frac{2^{m+1}(1-\phi)}{\tau(\tau+1)(2p+\tau\rho(2+\gamma))^{m+1}\left[\tau\rho+(2-\phi-p)\right]|B^\tau|}(z-\varepsilon)^{\tau\rho+p} \tag{15}$$

Proof: Let $f_\rho(z) \in \mathcal{A}_\rho$ be given by (6). If $f_\rho(z)$ satisfies the (14), then

$$\left|\frac{(z-\varepsilon)(D^{m+1}f_\rho(z))'}{D^{m+1}f_\rho(z)} - 1\right| \le (1-\phi)$$

$$\left|\frac{(z-\varepsilon)(D^{m+1}f_\rho(z))'}{D^{m+1}f_\rho(z)}-1\right|$$

$$=\left|\frac{p^{m+2}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(p+\tau\rho)(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^{m+1}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}-1\right|$$

$$=\left|\frac{\begin{array}{c}p^{m+2}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(p+\tau\rho)(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}\\ -p^{m+1}(z-\varepsilon)^p-\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}\end{array}}{p^{m+1}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|$$

$$\left|(p-1)+\frac{\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\tau\rho(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^{m+1}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|\leq(1-\phi)$$

$$(p-1)+\left|\frac{\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\tau\rho(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^{m+1}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|\leq(1-\phi)$$

$$\left|\frac{\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\tau\rho(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^{m+1}(z-\varepsilon)^p+\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^\tau\frac{B^\tau}{A^\tau}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right|\leq(1-\phi)-(p-1))$$

$$\leq\frac{\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\tau\rho\frac{B^\tau}{A^\tau}|a_{\tau+p}||z-\varepsilon|^{p+\tau\rho-1}}{p^{m+1}(z-\varepsilon)^{p-1}-\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\frac{B^\tau}{A^\tau}|a_{\tau+p}||z-\varepsilon|^{p+\tau\rho-1}}\leq(2-\phi-p)$$

$$<\frac{\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\tau\rho\frac{B^\tau}{A^\tau}|a_{\tau+p}|}{p^{m+1}(z-\varepsilon)^{p-1}-\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\frac{B^\tau}{A^\tau}|a_{\tau+p}|}\leq(2-\phi-p)$$

showing that $f_\rho(z)\in\mathcal{K}^*_{A,B,m}(\rho,\gamma,p,\phi)$.
Now suppose that $f_\rho(z)$ is given by (8), then we have that

$$\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\tau\rho\frac{B^\tau}{A^\tau}|a_{\tau+p}|+\sum_{\tau=1}^{\infty}(2-\phi-p)\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\frac{B^\tau}{A^\tau}|a_{\tau+p}|\leq(2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\frac{|B^\tau|}{A^\tau}|a_{\tau+p}|\left[\tau\rho+(2-\phi-p)\right]\leq(2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\frac{[2p+\tau\rho(2+\gamma)]^{m+1}}{2^{m+1}}\left[\tau\rho+(2-\phi-p)\right]\frac{|B^{\tau}|}{A^{\tau}}|a_{\tau+p}|\leq(2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}$$

$$\sum_{\tau=1}^{\infty}\frac{[2p+\tau\rho(2+\gamma)]^{m+1}}{2^{m+1}}\left[\tau\rho+(2-\phi-p)\right]\frac{|B^{\tau}|}{A^{\tau}}|a_{\tau+p}|=\sum_{\tau=1}^{\infty}\frac{(2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}}{\tau(\tau-1)}$$

$$=(2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}\sum_{\tau=1}^{\infty}\frac{1}{\tau(\tau-1)}=(2-\phi-p)p^{m+1}(z-\varepsilon)^{p-1}$$

.

3. Partial Sums

Theorem 3.1: Let the function $f_{\rho}(z)$ be of the form

$$f_{\rho}(z)=(z-\varepsilon)^{p}+(-1)^{\tau}\frac{B^{\tau}}{A^{\tau}}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho} \qquad (16)$$

where $\tau=p=1,2,3,\cdots$ also some real $\rho(0<\rho\leq 2)$, $-1\leq B<A\leq 1$, with $|a_{\tau+1}|\leq 1$, $\gamma\geq 0$, $0\leq\phi<1$, $\rho>0$, $m\in\mathbb{N}_0$. Then for $|z-\varepsilon|=(r+\varrho)$

$$\Re\left[\frac{D^{m+1}f_{\rho}(z)}{D^{m}f_{\rho}(z)}\right]>\frac{1-\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\frac{|B^{\tau}|}{A^{\tau}}|a_{\tau+p}|}{1-\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{B^{\tau}}{A^{\tau}}|a_{\tau+p}|} \qquad (17)$$

and

$$\Re\left[\frac{D^{m+1}f_{\rho}(z)}{D^{m}f_{\rho}(z)}\right]\geq\frac{1-\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}\frac{|B^{\tau}|}{A^{\tau}}(r+d)^{p+\tau\rho-1}}{1-\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\frac{B^{\tau}}{A^{\tau}}(r+d)^{p+\tau\rho-1}} \qquad (18)$$

Proof: Suppose that $f_{\rho}(z)$ be of the form (16), then

$$\left[\frac{D^{m+1}f_{\rho}(z)}{D^{m}f_{\rho}(z)}\right]=\Re\left[\frac{p^{m+1}(z-\varepsilon)^{p}+\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m+1}(-1)^{\tau}\frac{B^{\tau}}{A^{\tau}}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}{p^{m}(z-\varepsilon)^{p}+\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^{\tau}\frac{B^{\tau}}{A^{\tau}}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho}}\right]$$

$$\left[\frac{D^{m+1}f_{\rho}(z)}{D^{m}f_{\rho}(z)}\right]=\Re\left[\frac{p^{m+1}(z-\varepsilon)^{p-1}+\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}p(-1)^{\tau}\frac{B^{\tau}}{A^{\tau}}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}+\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}\left[\frac{\tau\rho(2+\gamma)}{2}\right](-1)^{\tau}\frac{B^{\tau}}{A^{\tau}}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}}{p^{m}(z-\varepsilon)^{p-1}+\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{m}(-1)^{\tau}\frac{B^{\tau}}{A^{\tau}}a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}}\right]$$

$$\left[\frac{D^{m+1}f_\rho(z)}{D^m f_\rho(z)}\right] = \Re\left[p + \frac{\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \left[\frac{\tau\rho(2+\gamma)}{2}\right] (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}}{p^m(z-\varepsilon)^{p-1} + \left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m (-1)^\tau \frac{B^\tau}{A^\tau} a_{\tau+p}(z-\varepsilon)^{p+\tau\rho-1}}\right]$$

$$\left[\frac{D^{m+1}f_\rho(z)}{D^m f_\rho(z)}\right] = p + \Re\left[\frac{\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \left[\frac{\tau\rho(2+\gamma)}{2}\right] \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}e^{i\theta(p+\tau\rho-1)}}{p^m(z-\varepsilon)^{p-1} + \left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}e^{i\theta(p+\tau\rho-1)}}\right]$$

$$\Re\left[\frac{D^{m+1}f_\rho(z)}{D^m f_\rho(z)}\right] = p + \frac{\left[\frac{\tau\rho(2+\gamma)}{2}\right]\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \left[p^m(z-\varepsilon)^{p-1}\right] \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}\cos((p+\tau\rho-1)\theta+\vartheta) + \frac{B^{2\tau}}{A^{2\tau}}|a_{\tau+p}|^2(r+d)^{2(p+\tau\rho-1)}\left[\frac{\tau\rho(2+\gamma)}{2}\right]\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{2m}}{p^{2m}(z-\varepsilon)^{2p-2} + 2p^m(z-\varepsilon)^{p-1}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}\cos((p+\tau\rho-1)\theta+\vartheta) + \left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{2m}\frac{B^{2\tau}}{A^{2\tau}}|a_{\tau+p}|^2(r+d)^{2(p+\tau\rho-1)}}$$

$$\Re\left[\frac{D^{m+1}f_\rho(z)}{D^m f_\rho(z)}\right] = p + \frac{\left[\frac{\tau\rho(2+\gamma)}{2}\right]\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}\left[\left[p^m(z-\varepsilon)^{p-1}\right]\cos((p+\tau\rho-1)\theta+\vartheta) + \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{(p+\tau\rho-1)}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m\right]}{p^{2m}(z-\varepsilon)^{2p-2} + 2p^m(z-\varepsilon)^{p-1}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}\cos((p+\tau\rho-1)\theta+\vartheta) + \left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{2m}\frac{B^{2\tau}}{A^{2\tau}}|a_{\tau+p}|^2(r+d)^{2(p+\tau\rho-1)}}$$

Suppose $l(t)$ is define by

$$\text{l(t)}= \frac{\frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{(p+\tau\rho-1)}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m + t}{p^{2m}(z-\varepsilon)^{2p-2} + 2p^m(z-\varepsilon)^{p-1}\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}t + \left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^{2m}\frac{B^{2\tau}}{A^{2\tau}}|a_{\tau+p}|^2(r+d)^{2(p+\tau\rho-1)}}$$

where

$$t = \cos((p+\tau\rho-1)\theta+\vartheta)$$

Then, we have $l'(t) > 0$ and $|a_{\tau+p}| \leq 1.$ we obtain

$$\Re\left[\frac{D^{m+1}f_\rho(z)}{D^m f_\rho(z)}\right] > 1 - \frac{\left[\frac{\tau\rho(2+\gamma)}{2}\right]\left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}}{1 - \left[\frac{2p+\tau\rho(2+\gamma)}{2}\right]^m \frac{B^\tau}{A^\tau}|a_{\tau+p}|(r+d)^{p+\tau\rho-1}} \tag{19}$$

Putting $r \longrightarrow 0$ and $d = 0$ in (19), we obtain the desired result in (17) while we have the inequality in (18) by putting $|a_{\tau+p}| = 1$ in (19).

Remark C: In the special cases:

(1) When $m = 0$, $\gamma = 0$ and $p = 1$ in (17) and (18), then we obtain result of Darus and Owa [2].

(2) When $p = 1$ in (17) and (18), then we obtain result of Hamzat and Raji [5].

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