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Coefficient bounds for a new family of bi-univalent functions associated with (U, V)-Lucas polynomials

Timilehin Gideon Shaba[a,*], Abbas Kareem Wanas[b]

[a] Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, Nigeria
[b] University of Al-Qadisiyah, College of science, Department of Mathematics, Al Diwaniyah, Al-Qadisiyah, Iraq

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Abstract

The aim of this paper is to use (U, V)-Lucas polynomials to introduce and study a new family of holomorphic and bi-univalent functions defined in the open unit disk which involve q-derivative operator. We investigate upper bounds for the Taylor-Maclaurin coefficients $|d_2|$ and $|d_3|$ and Fekete-Szegö problem for functions belongs to this new family. Some interesting consequences of the initial results established here are indicated.

Keywords: (U, V)-Lucas polynomials, Bi-univalent function, Coefficient bounds, Subordination
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1. Introduction and definition

Lucas, Dickson, Chebyshev, Lucas-Lehmer, Fibonacci and Lucas-Lehmer polynomials all have a lot of interest in current research. These polynomials are essential in mathematics and have a wide range of applications in combinatorics, number theory, numerical analysis and other fields. As a result, they've been thoroughly researched and many generalizations have been made (see, for more details, [16, 21, 23, 28, 51, 53, 7, 18, 25, 29]). Within the discipline of classical mathematical analysis, quantum calculus is a major topic of study. It is a broad subject of study with historical roots as well as contemporary relevance. It is important to note that quantum calculus has a long history that dates back to the work of Bernoulli and Euler. However, due to its wide range of application, it has piqued the interest of modern mathematicians in recent decades. It entails sophisticated calculations and computations, making it more challenging than the rest of the mathematics subjects.

*Corresponding author
Email addresses: shabatimilehin@gmail.com (Timilehin Gideon Shaba), abbas.kareem.w@qu.edu.iq (Abbas Kareem Wanas)

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Due to the enormous demand for mathematics that simulates quantum computing, it has recently sparked interest. Quantum calculus emerged as a link between physics and mathematics. Quantum mechanics, analytic number theory, theory of relativity, mechanical engineering, heat conduction theory, particles physics, nonlinear electric circuit theory, and physics are just some of the fields where it is used. Quantum calculus, often known as q-calculus, replaces the usual derivative with the difference operator, allowing for non-differentiable curves and multple formulas.

We let $\mathfrak{U} = \{z : z \in \mathfrak{C}, |z| < 1\}$ be a unit disk and let $\mathfrak{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{r=2}^{\infty} d_r z^r \quad (z \in \mathfrak{U}) \tag{1.1}$$

normalized by the condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S} \subset \mathfrak{A}$ be the class of holomorphic and univalent function in $\mathfrak{U}$.

Lewin [22] presented the class of bi-univalent functions as a subclass of $\mathfrak{A}$ and identified certain coefficient bounds for the class. He showed that $|n_2| \leq 1.15$. Furthermore, the Koebe 1/4 theorem (see [14]) states that the range of every function $f \in \mathcal{S}$ contains the disc $d_\omega = \{\omega : |\omega| < 0.25\}$, hence, for all $f \in \mathcal{S}$ with its inverse f^{-1}, such that

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{U})$$

and

$$f(f^{-1}(\omega)) = \omega, \quad (\omega : |\omega| < r_0(f); r_0(f) \geq 0.25)$$

where $f^{-1}(\omega)$ is expressed as

$$G(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots . \tag{1.2}$$

So, a function $f \in \mathfrak{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $f(z)$ and $G(z)$ are univalent in $\mathfrak{U}$. Let Σ denote the class of holomorhic and bi-univalent functions in $\mathfrak{U}$.

We know that some familiar functions $f \in \mathcal{S}$ such as the Koebe function $\kappa(z) = z/(1-z)^2$, its rotation function $\kappa_\varsigma(z) = z/(1-e^{i\varsigma}z)^2$, $f(z) = z - z^2/2$ and $f(z) = z/(1-z^2)$ are not members of Σ. Also some functions $f \in (\mathcal{S} \cap \Sigma)$ includes $f(z) = z$, $f(z) = 1/2\log[(1-z)/(1-z)]$, $z/(1-z)$. For more details see [1, 2, 5, 6, 12, 13, 17, 32, 38, 46, 39, 44, 48, 50, 41, 36, 27, 37, 40, 42, 43, 45, 47, 49, 54, 25, 33, 34, 35].

From [14], let $s(z)$, $S(z) \in \mathfrak{A}$, then $s(z) \prec S(z)$, $z \in \mathfrak{U}$, suppose ω holomorphic in $\mathfrak{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $s(z) = S(\omega(z))$. If the function $S(z)$ is univalent in $\mathfrak{U}$ then $s(z) \prec S(z) \Rightarrow s(0) = S(0)$ and $s(\mathfrak{U}) \subset S(\mathfrak{U})$. The principle of subordination is the name given to this conclusion.

Jackson [20, 19] pioneered the use of q-calculus, and investigations on quantum groups eventually recognized the geometrical meaning of q-analysis. This has inspired many q-theory scholars to extend all of the important conclusions utilizing classical analysis to their q-analogs. See [3, 4, 9, 11, 26, 30, 31] for recent work on q-calculus.

For function $f \in \mathfrak{A}$, the q-derivative of f can be defined by

$$\mathfrak{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0, \ \ 0 < q < 1) \tag{1.3}$$

where $\mathfrak{D}_q f(0) = f'(z)$ and $\mathfrak{D}_q^2 f(z) = \mathfrak{D}_q(\mathfrak{D}_q f(z))$. Applying (1.1) and (1.3) we have

$$\mathfrak{D}_q f(z) = 1 + \sum_{r=2}^{\infty} [r]_q d_r z^{r-1}$$

and

$$\mathfrak{D}_q^2 f(z) = \sum_{r=2}^{\infty} [r]_q [r-1]_q d_r z^{r-2}$$

where $[r]_q = \frac{1-q^r}{1-q}$, $[r-1]_q = \frac{1-q^{r-1}}{1-q}$, $\lim\limits_{q\uparrow 1}[r]_q = r$ and $\lim\limits_{q\uparrow 1}[r-1]_q = r-1$.

Let's get started with some definitions.

Definition 1.1. *[21] Let $U(x)$ and $V(x)$ be polynomials with real coefficients. The (U,V)-Lucas polynomials $\mathcal{L}_{U,V,t}(x)$ are defined by the recurrence relation*

$$\mathcal{L}_{U,V,t}(x) = U(x)\mathcal{L}_{U,V,t-1}(x) + V(x)\mathcal{L}_{U,V,t-2}(x) \ (t \geq 2). \tag{1.4}$$

The first few Lucas polynomials can be found in the following way:

$$\begin{aligned} \mathcal{L}_{U,V,0}(x) &= 2, \\ \mathcal{L}_{U,V,1}(x) &= U(x), \\ \mathcal{L}_{U,V,2}(x) &= U^2(x) + 2V(x), \\ \mathcal{L}_{U,V,3}(x) &= U^3(x) + 3U(x)V(x). \end{aligned} \tag{1.5}$$

Definition 1.2. *[21] Let $\mathcal{M}_{\{\mathcal{L}_t(x)\}}(z)$ be the generating function of the (U,V)-Lucas polynomial sequence $\mathcal{L}_{U,V,t}(x)$. Then*

$$\mathcal{M}_{\{\mathcal{L}_t(x)\}}(z) = \sum_{t=0}^{\infty} \mathcal{L}_{U,V,t}(x) z^t = \frac{2 - U(x)z}{1 - U(x)z - V(x)z^2}. \tag{1.6}$$

This section begins with the definition of class $\mathfrak{V}_q^{\Sigma,\delta}(s;x)$ and the estimation of coefficients $|n_2|$ and $|n_3|$ for functions in this class.

Definition 1.3. *For $\delta \geq 0$, $|s| \leq 1$ but $s \neq 1$, a function $f \in \Sigma$ is called in the class $\mathfrak{V}_q^{\Sigma,\delta}(s;x)$ if the following subordination conditions are satisfied:*

$$\left[\frac{(1-s)z}{f(z)-f(sz)}\right]^{\delta} (\mathfrak{D}_q f)(z) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(z) - 1 \tag{1.7}$$

and

$$\left[\frac{(1-s)\omega}{G(\omega)-G(s\omega)}\right]^{\delta} (\mathfrak{D}_q G)(\omega) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{1.8}$$

By choosing special values for s and δ the class $\mathfrak{V}_q^{\Sigma,\delta}(s;x)$ reduces some interesting new classes:

Remark 1.4. *For $s = 0$, we have the new class*

$$\mathfrak{V}_q^{\Sigma,\delta}(0;x) = \mathfrak{V}_q^{\Sigma,\delta}(x).$$

The class $\mathfrak{V}_q^{\Sigma,\delta}(x)$ consists of the functions of $f \in \Sigma$ satisfying

$$\left[\frac{z}{f(z)}\right]^{\delta} (\mathfrak{D}_q f)(z) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(z) - 1 \tag{1.9}$$

and

$$\left[\frac{\omega}{G(\omega)}\right]^{\delta} (\mathfrak{D}_q G)(\omega) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{1.10}$$

Remark 1.5. *For $\delta = 0$, we have the new class*

$$\mathfrak{V}_q^{\Sigma,0}(s;x) = \mathfrak{V}_q^{\Sigma}(x).$$

The class $\mathfrak{V}_q^{\Sigma}(x)$ consists of the functions of $f \in \Sigma$ satisfying

$$(\mathfrak{D}_q f)(z) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(z) - 1 \tag{1.11}$$

and

$$(\mathfrak{D}_q G)(\omega) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{1.12}$$

Remark 1.6. *For $\delta = 1$, we have the new class*

$$\mathfrak{V}_q^{\Sigma,1}(s;x) = \mathfrak{V}_q^{\Sigma}(s;x).$$

The class $\mathfrak{V}_q^{\Sigma}(s;x)$ consists of the functions of $f \in \Sigma$ satisfying

$$\left[\frac{(1-s)z}{f(z)-f(sz)}\right](\mathfrak{D}_q f)(z) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(z) - 1 \tag{1.13}$$

and

$$\left[\frac{(1-s)\omega}{G(\omega)-G(s\omega)}\right](\mathfrak{D}_q G)(\omega) \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{1.14}$$

Remark 1.7. *For $\delta = 1$ and $s = 0$, we have the new class*

$$\mathfrak{V}_q^{\Sigma,1}(0;x) = \mathfrak{V}_q^{\Sigma}(x).$$

The class $\mathfrak{V}_q^{\Sigma}(x)$ consists of the functions of $f \in \Sigma$ satisfying

$$\left[\frac{z(\mathfrak{D}_q f)(z)}{f(z)}\right] \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(z) - 1 \tag{1.15}$$

and

$$\left[\frac{\omega(\mathfrak{D}_q G)(\omega)}{G(\omega)}\right] \prec \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{1.16}$$

Our first main result is given by Theorem 1.8 below.

Theorem 1.8. *Let $f(z) \in \mathfrak{V}_q^{\Sigma,\delta}(s;x)$, then*

$$|d_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| U^2(x)\left[\delta\left[\frac{1+\delta}{2}(1+s)^2 - [2]_q(1+s)\right] + ([3]_q - \delta(1+s+s^2)) - ([2]_q - \delta(1+s))^2\right] - 2V(x)([2]_q - \delta(1+s))^2\right|}}, \tag{1.17}$$

and

$$|d_3| \leq \frac{U^2(x)}{[[2]_q - \delta(1+s)]^2} + \frac{|U(x)|}{|[3]_q - \delta(1+s+s^2)|}. \tag{1.18}$$

Proof . Let $f(z) \in \mathfrak{V}_q^{\Sigma,\delta}(s;x)$. Then, from Definition (1.2), for some holomorphic functions Υ, Φ such that $\Upsilon(0) = \Phi(0) = 0$ and $|\Upsilon(z)| < 1$, $|\Phi(z)| < 1$, for all $z, \omega \in \mathfrak{U}$, we can have

$$\left[\frac{(1-s)z}{f(z)-f(sz)}\right]^{\delta} (\mathfrak{D}_q f)(z) = \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\Phi(z)) - 1$$

and

$$\left[\frac{(1-s)\omega}{G(\omega)-G(s\omega)}\right]^{\delta} (\mathfrak{D}_q G)(\omega) = \mathcal{M}_{\{\mathcal{L}_{U,V,t}(x)\}}(\Upsilon(\omega)) - 1,$$

by equivalence

$$\left[\frac{(1-s)z}{f(z)-f(sz)}\right]^{\delta} (\mathfrak{D}_q f)(z) = -1 + \mathcal{L}_{U,V,0}(x) + \mathcal{L}_{U,V,1}(x)\Phi(z) + \mathcal{L}_{U,V,2}(x)\Phi^2(z) + \cdots \tag{1.19}$$

and

$$\left[\frac{(1-s)\omega}{G(\omega)-G(s\omega)}\right]^{\delta} (\mathfrak{D}_q G)(\omega) = -1 + \mathcal{L}_{U,V,0}(x) + \mathcal{L}_{U,V,1}(x)\Upsilon(\omega) + \mathcal{L}_{U,V,2}(x)\Upsilon^2(\omega) + \cdots . \tag{1.20}$$

From (1.19) and (1.20), yields

$$\left[\frac{(1-s)z}{f(z)-f(sz)}\right]^{\delta} (\mathfrak{D}_q f)(z) = 1 + \mathcal{L}_{U,V,1}(x)y_1 z + \left[\mathcal{L}_{U,V,1}(x)y_2 + \mathcal{L}_{U,V,2}(x)y_1^2\right] z^2 + \cdots \tag{1.21}$$

and

$$\left[\frac{(1-s)\omega}{G(\omega)-G(s\omega)}\right]^{\delta} (\mathfrak{D}_q G)(\omega) = 1 + \mathcal{L}_{U,V,1}(x)\mu_1 \omega + \left[\mathcal{L}_{U,V,1}(x)\mu_2 + \mathcal{L}_{U,V,2}(x)\mu_1^2\right] \omega^2 + \cdots . \tag{1.22}$$

If for $z, \omega \in \mathfrak{U}$, it is already known that

$$|\Phi(z)| = \left|\sum_{j=1}^{\infty} y_j z^j\right| < 1$$

and

$$|\Upsilon(\omega)| = \left|\sum_{j=1}^{\infty} \mu_j \omega^j\right| < 1,$$

then

$$|y_j| < 1 \tag{1.23}$$

and

$$|\mu_j| < 1 \tag{1.24}$$

where $j \in \mathfrak{N}$. When the corresponding coefficients in (1.21) and (1.22) are compared, we get

$$[[2]_q - \delta(1+s)]d_2 = \mathcal{L}_{U,V,1}(x)y_1, \tag{1.25}$$

$$\delta\left[\frac{1+\delta}{2}(1+s)^2 - [2]_q(1+s)\right] d_2^2 + ([3]_q - \delta(1+s+s^2))d_3 = \mathcal{L}_{U,V,1}(x)y_2 + \mathcal{L}_{U,V,2}(x)y_1^2 \tag{1.26}$$

$$[\delta(1+s)-[2]_q]d_2=\mathcal{L}_{U,V,1}(x)\mu_1, \tag{1.27}$$

$$\left(\delta\left[\frac{1+\delta}{2}(1+s)^2-[2]_q(1+s)\right]+2([3]_q-\delta(1+s+s^2))\right)d_2^2 \\ -([3]_q-\delta(1+s+s^2))d_3=\mathcal{L}_{U,V,1}(x)\mu_2+\mathcal{L}_{U,V,2}(x)\mu_1^2. \tag{1.28}$$

From (1.25) and (1.27)

$$y_1=-\mu_1, \tag{1.29}$$

$$2[[2]_q-\delta(1+s)]^2d_2^2=\mathcal{L}_{U,V,1}^2(x)(y_1^2+\mu_1^2). \tag{1.30}$$

Summation of (1.26) and (1.28) gives

$$2\delta\left[\frac{1+\delta}{2}(1+s)^2-[2]_q(1+s)\right]d_2^2+2([3]_q-\delta(1+s+s^2))d_2^2 \\ =\mathcal{L}_{U,V,1}(x)(y_2+\mu_2)+\mathcal{L}_{U,V,2}(x)(y_1^2+\mu_1^2)=\mathcal{L}_{U,V,1}(x)(y_2+\mu_2) \\ +\mathcal{L}_{U,V,2}(x)\left[\frac{2[[2]_q-\delta(1+s)]^2d_2^2}{\mathcal{L}_{U,V,1}^2(x)}\right]. \tag{1.31}$$

Applying (1.30) in (1.31), yields

$$\left[2\mathcal{L}_{U,V,1}^2(x)\left[\delta\left[\frac{1+\delta}{2}(1+s)^2-[2]_q(1+s)\right]+([3]_q-\delta(1+s+s^2))\right] \right. \\ \left. -2\mathcal{L}_{U,V,2}(x)[[2]_q-\delta(1+s)]^2\right]d_2^2=\mathcal{L}_{U,V,1}^3(x)(y_2+\mu_2) \tag{1.32}$$

$$\left[U^2(x)\left[2\left[\delta\left[\frac{1+\delta}{2}(1+s)^2-[2]_q(1+s)\right]+([3]_q-\delta(1+s+s^2))\right]\right.\right. \\ \left.\left.-2[[2]_q-\delta(1+s)]^2\right]-4V(x)[[2]_q-\delta(1+s)]^2\right]d_2^2=U^3(x)(y_2+\mu_2)$$

which gives

$$|d_2|\le\frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x)\left[\delta\left[\frac{1+\delta}{2}(1+s)^2-[2]_q(1+s)\right]+([3]_q-\delta(1+s+s^2)) -([2]_q-\delta(1+s))^2\right]-2V(x)([2]_q-\delta(1+s))^2\right|}}.$$

Hence, (1.26) minus (1.28) gives us

$$2[[3]_q-\delta(1+s+s^2)]d_3-2[[3]_q-\delta(1+s+s^2)]d_2^2=\mathcal{L}_{U,V,1}(x)(y_2-\mu_2). \tag{1.33}$$

Then, by using (1.29) and (1.30) in (1.33), we get

$$d_3 = d_2^2 + \frac{\mathcal{L}_{U,V,1}(x)(y_2-\mu_2)}{2[[3]_q - \delta(1+s+s^2)]} \tag{1.34}$$

$$= \frac{\mathcal{L}^2_{U,V,1}(x)(y_1^2+\mu_1^2)}{2[[2]_q - \delta(1+s)]^2} + \frac{\mathcal{L}_{U,V,1}(x)(y_2-\mu_2)}{2[[3]_q - \delta(1+s+s^2)]}. \tag{1.35}$$

Applying (1.5), we have

$$|d_3| \leq \frac{U^2(x)}{[[2]_q - \delta(1+s)]^2} + \frac{|U(x)|}{|[3]_q - \delta(1+s+s^2)|}.$$

As a result, the proof of our primary theorem is complete. □

2. Corollaries

We get the following by specializing the parameters s, δ, in Theorem 1.8:

Corollary 2.1. *Let $f(z) \in \mathfrak{V}_q^{\Sigma,1}(s;x) = \mathfrak{V}_q^{\Sigma}(s;x)$, then*

$$|d_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| \begin{array}{r} U^2(x)\left[s(1-[2]_q) - [2]_q + [3]_q - ([2]_q - (1+s))^2\right] \\ - V(x)[[2]_q - (1+s)]^2 \end{array} \right|}}, \tag{2.1}$$

$$|d_3| \leq \frac{U^2(x)}{[[2]_q - (1+s)]^2} + \frac{|U(x)|}{|[3]_q - (1+s+s^2)|}. \tag{2.2}$$

Corollary 2.2. *Let $f(z) \in \mathfrak{V}_q^{\Sigma,\delta}(0;x) = \mathfrak{V}_q^{\Sigma,\delta}(x)$, then*

$$|d_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| \begin{array}{r} U^2(x)\left[\delta\left[\frac{1+\delta}{2} - [2]_q\right] + ([3]_q - \delta) - ([2]_q - \delta)^2\right] \\ - 2V(x)([2]_q - \delta)^2 \end{array} \right|}}, \tag{2.3}$$

$$|d_3| \leq \frac{U^2(x)}{[[2]_q - \delta]^2} + \frac{|U(x)|}{|[3]_q - \delta|}. \tag{2.4}$$

Corollary 2.3. *Let $f(z) \in \mathfrak{V}_q^{\Sigma,0}(s;x) = \mathfrak{V}_q^{\Sigma}(s;x)$, then*

$$|d_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| U^2(x)\{[3]_q - [2]_q^2\} + 2[2]_q^2 V(x) \right|}},$$

$$|d_3| \leq \frac{U^2(x)}{[2]_q^2} + \frac{|U(x)|}{[3]_q}.$$

Corollary 2.4. *[54] Let $f(z) \in \mathfrak{V}_q^{\Sigma,0}(s;x) = \mathfrak{V}_q^{\Sigma}(s;x)$, then as $q \uparrow 1$*

$$|d_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x) + 8V(x)\right|}},$$

$$|d_3| \leq \frac{U^2(x)}{4} + \frac{|U(x)|}{3}.$$

Corollary 2.5. *Let $f(z) \in \mathfrak{V}_q^{\Sigma,1}(0;x) = \mathfrak{V}_q^{\Sigma}(x)$, then*

$$|d_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x)\left[[3]_q + [2]_q - [2]_q^2 - 1\right] - 2V(x)([2]_q - 1)^2\right|}}, \tag{2.5}$$

$$|d_3| \leq \frac{U^2(x)}{[[2]_q - 1]^2} + \frac{|U(x)|}{|[3]_q - 1|}. \tag{2.6}$$

Corollary 2.6. *[24] Let $f(z) \in \mathfrak{V}_q^{\Sigma,1}(0;x) = \mathfrak{V}_q^{\Sigma}(x)$, then as $q \uparrow 1$*

$$|d_2| \leq U(x)\frac{\sqrt{|U(x)|}}{\sqrt{2\,|V(x)|}}, \tag{2.7}$$

$$|d_3| \leq U^2(x) + \frac{|U(x)|}{2}. \tag{2.8}$$

The Fekete-Szegö functional upper bound is given by the following theorem:

Theorem 2.7. *For $\delta \geq 0$, $|s| \leq 1$ but $s \neq 1$, let $f \in \mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,\delta}(s;x)$. Then*

$$\left|d_3 - \chi d_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|[3]_q - \delta(1+s+s^2)|}, & |\chi - 1| \leqq H \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Lambda - 2V(x)[[2]_q - \delta(1+s)]^2|}, & |\chi - 1| \geqq H. \end{cases}$$

Where

$$H = \frac{1}{|[3]_q - \delta[1+s+s^2]|}\left|\Lambda - 2([2]_q - \delta(1+s))^2\frac{V(x)}{U^2(x)}\right|$$

$$\Lambda = \delta\left[\frac{1+\delta}{2}(1+s)^2 - [2]_q(1+s)\right] + ([3]_q - \delta(1+s+s^2)) - ([2]_q - \delta(1+s))^2.$$

Proof . From (1.32) and (1.33), we get

$$d_3 - \chi d_2^2 = \mathcal{L}_{U,V,1}(x)\left[\left(G(\chi;x) + \frac{1}{2[[3]_q - \delta(1+s+s^2)]}\right)y_2 \right.$$
$$\left. + \left(G(\chi;x) - \frac{1}{2[[3]_q - \delta(1+s+s^2)]}\right)\mu_2\right]$$

where

$$G(\chi;x)=\frac{\mathcal{L}_{U,V,1}^{2}(x)(1-\chi)}{2\mathcal{L}_{U,V,1}^{2}(x)\left[\delta\left[\frac{1+\delta}{2}(1+s)^2-[2]_q(1+s)\right]+([3]_q-\delta(1+s+s^2))\right]-2\mathcal{L}_{U,V,2}(x)[[2]_q-\delta(1+s)]^2}.$$

Thus, according to (1.5), we have

$$\left|d_3-\chi d_2^2\right|\leq\begin{cases}\frac{U(x)}{|[3]_q-\delta(1+s+s^2)|}, & 0\leqq|G(\chi;x)|\leqq\frac{1}{2[[3]_q-\delta(1+s+s^2)]}\\[2ex] 2|G(\chi;x)|\cdot|U(x)|, & |G(\chi;x)|\geqq\frac{1}{2[[3]_q-\delta(1+s+s^2)]}\end{cases}$$

hence, after some calculations, gives

$$\left|d_3-\chi d_2^2\right|\leq\begin{cases}\frac{|U(x)|}{|[3]_q-\delta(1+s+s^2)|}, & |\chi-1|\leqq H\\[2ex] \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Lambda-2V(x)[[2]_q-\delta(1+s)]^2|}, & |\chi-1|\geqq H.\end{cases}$$

□

Corollary 2.8. *For $s=0$, let $f\in\mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,\delta}(0;x)=\mathfrak{V}_q^{\Sigma,\delta}(x)$. Then*

$$\left|d_3-\chi d_2^2\right|\leq\begin{cases}\frac{|U(x)|}{|[3]_q-\delta|}, & |\chi-1|\leqq H_1\\[2ex] \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Lambda_1-2V(x)[[2]_q-\delta]^2|}, & |\chi-1|\geqq H_1.\end{cases}$$

Where

$$H_1=\frac{1}{|[3]_q-\delta|}\left|\Lambda_1-2([2]_q-\delta)^2\frac{V(x)}{U^2(x)}\right|$$

$$\Lambda_1=\delta\left[\frac{1+\delta}{2}-[2]_q\right]+([3]_q-\delta)-([2]_q-\delta)^2.$$

Corollary 2.9. *For $\delta=0$, let $f\in\mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,0}(s;x)=\mathfrak{V}_q^{\Sigma}(x)$. Then*

$$\left|d_3-\chi d_2^2\right|\leq\begin{cases}\frac{|U(x)|}{[3]_q}, & |\chi-1|\leqq H_2\\[2ex] \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Lambda_2-2V(x)[2]_q^2|}, & |\chi-1|\geqq H_2.\end{cases}$$

Where

$$H_2=\frac{1}{[3]_q}\left|\Lambda_2-2[2]_q^2\frac{V(x)}{U^2(x)}\right|$$

$$\Lambda_2=[3]_q-[2]_q^2.$$

Corollary 2.10. *[8] For $\delta=0$, let $f\in\mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,0}(s;x)=\mathfrak{V}_q^{\Sigma}(x)$. Then as $q\uparrow 1$*

$$\left|d_3-\chi d_2^2\right|\leq\begin{cases}\frac{|U(x)|}{3}, & |\chi-1|\leqq\frac{1}{3}\left|1+8\frac{V(x)}{U^2(x)}\right|\\[2ex] \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)+8V(x)|}, & |\chi-1|\geqq\frac{1}{3}\left|1+8\frac{V(x)}{U^2(x)}\right|.\end{cases}$$

Corollary 2.11. *For $\delta = 1$, let $f \in \mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,1}(s;x) = \mathfrak{V}_q^{\Sigma}(s;x)$. Then*

$$\left|d_3 - \chi d_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|[3]_q-(1+s+s^2)|}, & |\chi - 1| \leqq H_3 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Lambda_3-2V(x)[[2]_q-(1+s)]^2|}, & |\chi - 1| \geqq H_3. \end{cases}$$

Where

$$H_3 = \frac{1}{|[3]_q - [1+s+s^2]|}\left|\Lambda_3 - 2([2]_q - (1+s))^2\frac{V(x)}{U^2(x)}\right|$$

$$\Lambda_3 = s(1-[2]_q) - [2]_q + [3]_q - ([2]_q - (1+s))^2.$$

Corollary 2.12. *For $\delta = 1$, $s = 0$, let $f \in \mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,1}(0;x) = \mathfrak{V}_q^{\Sigma}(x)$. Then*

$$\left|d_3 - \chi d_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|[3]_q-1|}, & |\chi - 1| \leqq H_4 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Lambda_4-2V(x)[[2]_q-1]^2|}, & |\chi - 1| \geqq H_4. \end{cases}$$

Where

$$H_4 = \frac{1}{|[3]_q - 1|}\left|\Lambda_4 - 2([2]_q - 1)^2\frac{V(x)}{U^2(x)}\right|$$

$$\Lambda_4 = [3]_q + [2]_q - [2]_q^2 - 1.$$

Corollary 2.13. *[52] For $\delta = 1$, $s = 0$, let $f \in \mathfrak{A}$ be in the class $\mathfrak{V}_q^{\Sigma,1}(0;x) = \mathfrak{V}_q^{\Sigma}(x)$. Then as $q \uparrow 1$*

$$\left|d_3 - \chi d_2^2\right| \leq \begin{cases} \frac{|U(x)|}{2}, & |\chi - 1| \leqq \frac{|V(x)|}{U^2(x)} \\ \frac{|1-\chi|\cdot|U^3(x)|}{2|V(x)|}, & |\chi - 1| \geqq \frac{|V(x)|}{U^2(x)}. \end{cases}$$

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