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Second Hankel Determinant for the Subclass of Bi-Univalent Functions Using q-Chebyshev Polynomial and Hohlov Operator

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Abstract: The q-derivative and Hohlov operators have seen much use in recent years. First, numerous well-known principles of the q-derivative operator are highlighted and explained in this research. We then build a novel subclass of analytic and bi-univalent functions using the Hohlov operator and certain q-Chebyshev polynomials. A number of coefficient bounds, as well as the Fekete–Szegö inequalities and the second Hankel determinant are provided for these newly specified function classes.

Keywords: Hankel determinant; analytic and bi-univalent functions; subordination; Hohlov operator; q-Chebyshev polynomials; coefficient bounds; Fekete–Szegö inequalities

1. Introduction and Definitions

Quantum (or q-) calculus is a vital instrument for understanding a wide range of analytic functions, and its applications in mathematics and related fields have sparked interest among scholars. Srivastava [1] was the first to use it in a univalent function context. Because of the relevance of q-analysis in mathematics and other disciplines, a large number of researchers have worked on q-calculus and studied its numerous applications. Shi et al. [2] also employed the q-differential operator to develop a novel subclass of multivalent Janowski-type q-starlike functions. A variety of adequate requirements, as well as several other intriguing aspects were investigated in both publications [2,3]. Convolution theory, moreover, allows us to examine many characteristics of analytic functions. Many scholars have studied q-calculus in depth due to its wide range of applications and the prominence of q-operators over conventional operators. Khan et al. [3], for example, created and investigated a number of subclasses of q-starlike functions with the help of certain higher-order q-derivative operators.

Furthermore, Srivastava [4] just published a survey-cum-expository review piece that may be of interest to academics and scholars working on these topics. Srivastava's recent survey-cum-expository review paper [4] further motivates the usage of the q-analysis in geometric function theory, while also commenting on the triviality of the so-called (p,q)-analysis, which involves an inconsequential and redundant parameter (p,q) (see specifically [4], p. 340).

Let the open unit disk be represented by $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Functions that are analytic and satisfy the standard normalization condition:

$$f(0) = f'(0) - 1 = 0,$$

are called analytic functions, which can be denoted by $\mathcal{W}(\mathcal{U})$. Furthermore, let $\mathcal{J}$ represent the subclass of $\mathcal{W}(\mathcal{U})$ that encloses functions of the form:

$$f(z) = z + \sum_{m=2}^{\infty} n_m z^m \qquad (z \in \mathcal{U}). \tag{1}$$

which are univalent in the unit disk $\mathcal{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by:

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U}),$$

and:

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f); \ r_0(f) \geq \frac{1}{4}\right),$$

where:

$$f^{-1}(w) = g(w) = w - n_2 w^2 + \left(2n_2^2 - n_3\right)w^3 - \left(5n_2^3 - 5n_2 n_3 + n_4\right)w^4 + \cdots. \quad (2)$$

A function is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1). The following functions are members of the class Σ,

$$z(1-z)^{-1}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left[\frac{1+z}{1-z}\right].$$

However, the functions:

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are not members of Σ.

Lewin [5] investigated a bi-univalent functions class Σ and showed that $|b_2| < 1.51$. Subsequently, Brannan and Clunie [6] conjectured that $|b_2| < \sqrt{2}$.

Netanyahu [7], on the other hand, showed that:

$$\max_{f \in \Sigma} |b_2| = 1.333333.$$

The coefficient for each of the Taylor–Maclaurin coefficients $|b_n|$ $(n \geq 3, \ n \in \mathbb{N})$ is presumably still an open problem.

Similar to the familiar subclass $\mathcal{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ of starlike and convex functions of order $\zeta (0 \leq \zeta < 1)$, respectively, Brannan and Taha [8] introduced certain subclasses of the bi-univalent function class Σ, namely $\mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of bi-starlike functions and bi-convex functions of order $\zeta (0 \leq \zeta < 1)$, respectively. For each of the function classes $\mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$, they found non-sharp bounds on the first two Taylor–Maclaurin coefficients $|b_2|$ and $|b_3|$.

Furthermore, let g_1 and g_2 be two analytic functions in the open unit disc $\mathcal{U}$. Then, the function g_1 is subordinated to g_2, symbolically denoted by:

$$g_1(z) \prec g_2(z) \quad (z \in \mathcal{U}),$$

if there is an analytic function $w(z)$ with properties that:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

with w holomorphic in $\mathcal{U}$, such that:

$$g_1(z) = g_2(w(z)).$$

If the function $g_2(z)$ is univalent in $\mathcal{U}$, using the subordination principle, the above condition is equivalent to:

$$g_1(z) \prec g_2(z) \Leftrightarrow g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathcal{U}) \subset g_2(\mathcal{U}).$$

Jackson [9] introduced and studied the q-derivative operator $\mathfrak{D}_q$ of a function as follows:

$$\mathfrak{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} = \frac{1}{z}\left\{z + \sum_{m=2}^{\infty}\left((1-q^m)(1-q)^{-1}\right)a_m z^m\right\} \tag{3}$$

and $\mathfrak{D}_q f(0) = f'(0)$. In case $f(z) = z^m$ for m is a positive integer, the q-derivative of $f(z)$ is given by:

$$\mathfrak{D}_q z^m = \frac{(zq)^m - z^m}{z(q-1)} = \frac{\left((1-q^m)(1-q)^{-1}\right)z^m}{z}, \tag{4}$$

$$\lim_{q \to 1-} [m]_q = \lim_{q \to 1-} (1-q^m)(1-q)^{-1} = m, \tag{5}$$

where $(z \neq 0,\ q \neq 0)$; for more details on the concepts of the q-derivative, see [3,10].

In geometric function theory, studies of convolution are crucial. Various new and interesting subclasses of holomorphic and univalent functions have been introduced and investigated through the use of the Hadamard product (or convolution) in the direction of well-known ideas such as integral mean, Hankel determinant, subordination, partial sums, superordination inequalities, and so on. The Hadamard product (or convolution) of f and g, represented by $f * g$, is defined by:

$$(f * g)(z) = z + \sum_{j=2}^{\infty} b_j a_j z^j = (g * f)(z) \quad z \in \mathcal{U}.$$

If f and g are functions in $\mathcal{A}$ and are given by the power series:

$$f(z) = z + \sum_{j=2}^{\infty} b_j z^j \qquad g(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad z \in \mathcal{U}.$$

The Gauss hypergeometric function ${}_2\mathcal{F}_1(u,v,w;z)$ is defined as:

$${}_2\mathcal{F}_1(u,v,w;z) = \sum_{j=0}^{\infty} \frac{(u)_j(v)_j}{(w)_j} \frac{z^j}{(\delta)_j} \quad z \in \mathcal{U},$$

where $(\delta)_j$ signifies the Pochhammer symbol (or shifted factorial) provided in terms of the Gamma function Γ, by:

$$(\delta)_j = \frac{\Gamma(\delta + j)}{\Gamma(\delta)} = \begin{cases} 1; & \text{if } j = 0, \\ \delta(\delta+1)(\delta+2)(\delta+3)\cdots(\delta+j-1); & \text{if } j \neq 0. \end{cases}$$

Hohlov (cf. [11,12]) proposed and investigated a linear operator denoted by $\mathcal{L}_{u,v}^{w}$ and defined by $\mathcal{L}_{u,v}^{w} f : \mathcal{A} \longrightarrow \mathcal{A}$, with:

$$\mathcal{L}_{u,v}^{w} f(z) := z\,{}_2\mathcal{F}_1(u,v,w;z) * f(z) = z + \sum_{j=2}^{\infty} \frac{(u)_{j-1}(v)_{j-1}}{(w)_{j-1}(1)_{j-1}} b_j z^j, \quad z \in \mathcal{U}. \tag{6}$$

The above-specified three-parameter family of operators unifies several other linear operators that have been introduced and explored previously when the parameters are appropriately chosen. The following citations [13–20] provide special examples of this operator. For more details, see [21,22].

In 2017, Altinkaya and Yalcin [23] studied the Chebyshev polynomial expansions to provide estimates for the initial coefficients of some subclasses of bi-univalent functions defined by the symmetric q-derivative operator. They also established Fekete–Szegö inequalities for the class $\widetilde{H}_{\Sigma}^{q}(t)$. After some time, other researchers started introducing dif-

ferent subclasses of bi-univalent functions linked with Chebyshev polynomials. Ayinla and Opoola [24] introduced the class defined by the Sălăgean differential operator as follows:

$$\Re\left(e^{i\gamma}\left(1 - e^{-2i\gamma}\beta^2 z^2 \frac{D^{n+1}f(z)}{z}\right)\right) > 0$$

and obtained the Fekete–Szegö inequality and the second Hankel determinant. Furthermore, in 2018, Orhan et al. [25] obtained an upper bound estimate for the second Hankel determinant of a subclass $\mathcal{N}_\sigma^\mu(\lambda, t)$ of analytic bi-univalent function class Σ, which is associated with Chebyshev polynomials in the open unit disk.

Al Salam and Ismail [26] found a set of polynomials known as q-analogues of second-order bivariate Chebyshev polynomials. Johann Cigler first introduced and studied the q-Chebyshev polynomials in 2012, as shown below.

Definition 1 ([27]). *The polynomials:*

$$G_n(t, x, q) = P_{n+1}(t, -1, x, q)(-q; q)_n$$

$$= \sum_{m=0}^{\left[\frac{k}{2}\right]} q^{m^2} \begin{bmatrix} n-m \\ m \end{bmatrix} (1 + q^{m+1}) \cdots (1 + q^{n-m}) x^m t^{n-2m} \tag{7}$$

are called q-Chebyshev polynomials of the second kind.

Theorem 1 ([27]). *The q-Chebyshev polynomials of the second kind satisfy:*

$$G_n(t, x, q) = (1 + q^n)t G_{n-1}(t, x, q) + q^{n-1} x G_{n-2}(t, x, q) \tag{8}$$

with initial values

$$G_0(t, x, q) = 1 \quad and \quad G_1(t, x, q) = (1 + q)t.$$

Remark 1. *It is obvious that:*

$$G_n(t, -1, 1) = G_n(t),$$

where $G_n(t)$ is the classical Chebyshev polynomial of the second kind.

Furthermore, from (8), we have the following:

$$G_1(t, x, q) = t + tq$$
$$G_2(t, x, q) = qx + (1 + q)(1 + q^2)t^2$$
$$G_3(t, x, q) = qxt(1 + q)(1 + q^2) + t^3(1 + q)(1 + q^2)(1 + q^3)$$
$$G_4(t, x, q) = q^4 x + (1 + q)(1 + q^2)(1 + q^3)(1 + q^4)t^4$$
$$+ qx^2 t(1 + q)(1 + q^2)(1 + q^4 + q^2).$$

We shall discuss the following intriguing points in light of these recurrence relations:

1. The Chebyshev polynomials of the second kind denoted by $G_n(t)$ are obtained when $x = -1$ and $q = 1$;
2. The Fibonacci polynomials denoted by $F_{n+1}(t)$ are obtained when $t = \frac{1}{2}$, $x = 1$, and $q = 1$;
3. The Fibonacci numbers denoted by F_{n+1} are obtained when $t = \frac{1}{2}$, $x = 1$, and $q = 1$;
4. The Pell polynomials denoted by $P_{n+1}(t)$ are obtained when $x = 1$ and $q = 1$;
5. The Pell numbers denoted by P_{n+1} are obtained when $t = 1$, $x = 1$, and $q = 1$;
6. The Jacobsthal polynomials denoted by $J_{n+1}(y)$ are obtained when $t = \frac{1}{2}$, $x = 2y$, and $q = 1$;
7. The Jacobsthal numbers denoted by J_{n+1} are obtained when $t = \frac{1}{2}$, $x = 2$, and $q = 1$.

Let D be the q-differentiation operator defined by:

$$Df(t) = \frac{f(t) - f(qt)}{t - qt}.$$

The q-Chebyshev polynomials satisfy the q-differential equation.

$$(t^2 + qx)D^2 T_n(t, q^2 x, q) + q^{n-1} t D T_n(t, x, q) = [n]^2 T_n(t, x, q) \tag{9}$$

and:

$$(t^2 + qx)D^2 G_n(t, q^2 x, q) + q^{n-1}[3]t D G_n(t, x, q) = [n][n+2]G_n(t, x, q). \tag{10}$$

Equations (9) and (10) are applications of q-Chebyshev polynomials in the field of differential equations. For more details, see [27].

Now making, use q-Chebyshev polynomials, we define the following.

Definition 2. *Let $\Lambda_{z,t}^{x,q}$ be defined as follows:*

$$\Lambda_{z,t}^{x,q} = \sum_{n=0}^{\infty} G_n(t, x, q) z^n. \tag{11}$$

By using the principle of subordination and the Hohlov operator, we define the following subclasses of analytic and bi-univalent functions.

Definition 3. *A function $f \in \Sigma$ given by (1) is said to be in the class $Y_{\Sigma, u, v, w}^{q, x, \phi}(t, \mu)$, if the following conditions are satisfied:*

$$\frac{\mathcal{L}_{u,v}^w f(z)}{z}\left(1 - e^{-2i\phi}\mu^2 z^2\right) \prec \Lambda_{z,t}^{x,q} \tag{12}$$

and:

$$\frac{\mathcal{L}_{u,v}^w f^{-1}(w)}{z}\left(1 - e^{-2i\phi}\mu^2 w^2\right) \prec \Lambda_{w,t}^{x,q}, \tag{13}$$

where $-1 \le x \le 2, \frac{1}{2} < t < 1, 0 < q < 1, z, w \in \mathcal{U}, 0 \le \mu \le 1$, and $\phi \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

From (11), we have:

$$\Lambda_{z,t}^{x,q} = 1 + G_1(t, x, q)z + G_2(t, x, q)z^2 + G_3(t, x, q)z^3 + \cdots$$

where $z \in \mathcal{U}$ and $t \in (-1, 1)$.

As far as we know, there are no studies in the literature of q-Chebyshev polynomials that are linked to bi-univalent functions. The main goal of this research was to start looking at the properties of bi-univalent functions that are linked to q-Chebyshev polynomials and are defined by the Hohlov operator. In this paper, the initial coefficient estimates, the Fekete–Szegö problem, and the $|H_2(2)|$ Hankel determinant for a subclass $Y_{\Sigma, u, v, w}^{q, x, \phi}(t, \mu)$ of analytic and bi-univalent functions are determined using the q-Chebyshev polynomial expansion. With this idea, the authors focused on the bound of coefficient functionals for a new subclass of analytic and bi-univalent functions using the Hohlov operator and certain q-Chebyshev polynomials.

Lemma 1 ([28]). *Let $\varphi(z) \in \mathcal{P}$, then:*

$$2h_2 = p_1^2 + b\left(4 - h_1^2\right)$$

$$4h_3 = h_1^3 + 2h_1(4 - h_1^2)b - h_1(4 - h_1^2)b^2 + 2(4 - h_1^2)(1 - |b|^2)z$$

for some complex number satisfying $b, z, |b| \le 1$, and $|z| \le 1$.

Lemma 2 (see [29]). *Let the function p given by:*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$ of functions with positive real part. Then:

$$|p_n| \le 2 \qquad (n \in \mathbb{N}).$$

This last inequality is sharp.

2. Coefficients' Bounds for the Functions Class $Y^{q,x,\phi}_{\Sigma,u,v,w}(t,\mu)$

Theorem 2. *Let $f \in Y^{q,x,\phi}_{\Sigma,u,v,w}(t,\mu)$. Then:*

$$|n_2| \le \sqrt{\Phi_1(u,v,w,t,q,\mu)},$$

$$|n_3| \le \frac{(w)_1^2(1+q)^2 t^2}{(u)_1^2(v)_1^2} + \frac{2(c)_2(1+q)t}{(u)_2(v)_2}$$

and:

$$\begin{aligned}
|n_4| &\le \frac{5(w)_2(w)_1(1+q)^2 t^2}{(u)_2(v)_2(u)_1(v)_1} + \frac{6(w)_3(1+q)t}{(u)_3(v)_3} + \frac{12(w)_3[t(1+q)[t(1+q^2)-1]-qx]}{(u)_3(v)_3} \\
&\quad + \frac{6(w)_3[(1+q)t[1+q(1+q^2)x - 2t(1+q^2) + (1+q^2)(1+q^3)t^2] - 2qx]}{(u)_3(v)_3} \\
&\quad + \frac{6\mu^2(1+q)t(w)_3}{(u)_3(v)_3}.
\end{aligned}$$

where:

$$\Phi_1(u,v,w,t,q,\mu) = \frac{2(w)_2(w)_1^2(1+q)^3 t^3 + 2(w)_2(w)_1^2\mu^2(1+q)^2 t^2}{\left| \begin{array}{l} (1+q)t^2[(u)_2(v)_2(w)_1^2(1+q) - 2(w)_2(u)_1^2(v)_1^2(1+q^2)] \\ \qquad\qquad - 2(w)_2(u)_1^2(v)_1^2(qx-(1+q)t) \end{array} \right|}$$

Proof. Let $f \in \Sigma$ given by (1) be in the class $Y^{q,x,\phi}_{\Sigma,u,v,w}(t,\mu)$. Then:

$$\frac{\mathcal{L}^w_{u,v} f(z)}{z}\left(1 - e^{-2i\phi}\mu^2 z^2\right) = \Lambda^{x,q}_{\omega(z),t} \tag{14}$$

and:

$$\frac{\mathcal{L}^w_{u,v} f^{-1}(w)}{z}\left(1 - e^{-2i\phi}\mu^2 w^2\right) = \Lambda^{x,q}_{\varpi(w),t}. \tag{15}$$

let $h, l \in \mathcal{P}$ be defined as:

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots$$

$$\Rightarrow \omega(z) = \frac{h(z)-1}{h(z)+1} \qquad (z \in \mathfrak{D}) \tag{16}$$

and:

$$l(w) = \frac{1 + \varpi(w)}{1 - \varpi(w)} = 1 + l_1 w + l_2 w^2 + l_3 w^3 + \cdots$$

$$\Rightarrow \varpi(w) = \frac{l(w) - 1}{l(w) + 1} \qquad (w \in \mathfrak{D}). \tag{17}$$

It follows from (16) and (17) that:

$$\omega(z) = \frac{1}{2}\left[h_1 z + \left(h_2 - \frac{h_1^2}{2}\right)z^2 + \left(h_3 - h_1 h_2 + \frac{h_1^3}{4}\right)z^3 + \cdots \right] \tag{18}$$

and:

$$\varpi(w) = \frac{1}{2}\left[l_1 w + \left(l_2 - \frac{l_1^2}{2}\right)w^2 + \left(l_3 - l_1 l_2 + \frac{l_1^3}{4}\right)w^3 + \cdots \right] \tag{19}$$

From (18) and (19), applying $\Lambda_{z,t}^{x,q}$ as given in (11), we see that:

$$\begin{aligned}
\Lambda_{\omega(z),t}^{x,q} = 1 &+ \frac{G_1(t,x,q)}{2} h_1 z \\
&+ \left[\frac{G_1(t,x,q)}{2}\left(h_2 - \frac{h_1^2}{2}\right) + \frac{G_2(t,x,q)}{4} h_1^2 \right] z^2 \\
&+ \left[\frac{G_1(t,x,q)}{2} \cdot \left(h_3 - h_1 h_2 + \frac{h_1^3}{4}\right) \right. \\
&\left. + \frac{G_2(t,x,q)}{2} h_1\left(h_2 - \frac{h_1^2}{2}\right) + \frac{G_3(t,x,q)}{8} h_1^3 \right] z^3 + \cdots
\end{aligned} \tag{20}$$

and:

$$\begin{aligned}
\Lambda_{\varpi(w),t}^{x,q} = 1 &+ \frac{G_1(t,x,q)}{2} l_1 w + \\
&\left[\frac{G_1(t,x,q)}{2}\left(l_2 - \frac{l_1^2}{2}\right) + \frac{G_2(t,x,q)}{4} l_1^2 \right] w^2 \\
&+ \left[\frac{G_1(t,x,q)}{2}\left(l_3 - l_1 l_2 + \frac{l_1^3}{4}\right) \right. \\
&\left. + \frac{G_2(t,x,q)}{2} l_1\left(l_2 - \frac{l_1^2}{2}\right) + \frac{G_3(t,x,q)}{8} l_1^3 \right] w^3 + \cdots
\end{aligned} \tag{21}$$

It follows from (14), (15), (20), and (21) that we have:

$$\frac{(u)_1(v)_1}{(w)_1} n_2 = \frac{G_1(t,x,q)}{2} h_1 \tag{22}$$

$$\frac{(u)_2(v)_2}{2(w)_2} n_3 - e^{-2i\phi} \mu^2 = \frac{G_1(t,x,q)}{2}\left(h_2 - \frac{h_1^2}{2}\right) + \frac{G_2(t,x,q)}{4} h_1^2 \tag{23}$$

$$\begin{aligned}
\frac{(u)_3(v)_3}{6(w)_3} n_4 - e^{-2i\phi}\frac{(u)_1(v)_1}{(w)_1}\mu^2 n_2 = {}& \frac{G_1(t,x,q)}{2}\left(h_3 - h_1 h_2 + \frac{h_1^3}{4}\right) \\
&+ \frac{G_2(t,x,q)}{2} h_1\left(h_2 - \frac{h_1^2}{2}\right) + \frac{G_3(t,x,q)}{8} h_1^3 \tag{24}
\end{aligned}$$

$$-\frac{(u)_1(v)_1}{(w)_1}n_2 = \frac{G_1(t,x,q)}{2}l_1 \tag{25}$$

$$\frac{(u)_2(v)_2}{2(w)_2}(2n_2^2 - n_3) - e^{-2i\phi}\mu^2 = \frac{G_1(t,x,q)}{2}\left(l_2 - \frac{l_1^2}{2}\right) + \frac{G_2(t,x,q)}{4}l_1^2 \tag{26}$$

$$-\frac{(u)_3(v)_3}{6(w)_3}(5n_2^3 - 5n_2n_3 + n_4) + \frac{(u)_1(v)_1}{(w)_1}e^{-2i\phi}\mu^2 n_2 = \frac{G_1(t,x,q)}{2}\left(l_3 - l_1l_2 + \frac{l_1^3}{4}\right)$$
$$+ \frac{G_2(t,x,q)}{2}l_1\left(l_2 - \frac{l_1^2}{2}\right)$$
$$+ \frac{G_3(t,x,q)}{8}l_1^3. \tag{27}$$

Adding (22) and (25), we have:

$$h_1 = -l_1, \quad h_1^2 = l_1^2 \quad \text{and} \quad h_1^3 = -l_1^3 \tag{28}$$

and:

$$n_2^2 = \frac{(w)_1^2 G_1^2(t,x,q)(h_1^2 + l_1^2)}{8(u)_1^2(v)_1^2} \tag{29}$$

Furthermore, adding (23) and (26) and applying (28) yield:

$$\frac{2(u)_2(v)_2}{(w)_2}n_2^2 - 4e^{-2i\phi}\mu^2 = G_1(t,x,q)(h_2 + l_2) - l_1^2(G_1(t,x,q) - G_2(t,x,q)) \tag{30}$$

Applying (28) in (29) gives:

$$l_1^2 = \frac{4(u)_1^2(v)_1^2 n_2^2}{(w)_1^2 G_1^2(t,x,q)} \tag{31}$$

Putting (31) into (30) and with some calculations, we have:

$$|n_2|^2 = \left| \frac{(w)_2(w)_1^2 G_1^3(t,x,q)(h_2 + l_2) + 4(w)_2(w)_1^2 e^{-2i\phi}\mu^2 G_1^2(t,x,q)}{2\left[(u)_2(v)_2(w)_1^2 G_1^2(t,x,q) - 2(w)_2(u)_1^2(v)_1^2(G_2(t,x,q) - G_1(t,x,q))\right]} \right|$$

Applying the triangular inequality and Lemma 2, we have:

$$|n_2| \leq \sqrt{\Phi_1(u,v,w,t,q,\mu)}.$$

Subtracting (26) from (23) and with some calculations, we have:

$$n_3 = n_2^2 + \frac{(w)_2 G_1(t,x,q)[h_2 - l_2]}{2(u)_2(v)_2} \tag{32}$$

$$n_3 = \frac{(w)_1^2 G_1^2(t,x,q)p_1^2}{4(u)_1^2(v)_1^2} + \frac{(w)_2 G_1(t,x,q)[h_2 - l_2]}{2(u)_2(v)_2}. \tag{33}$$

Applying the triangular inequality and Lemma 2, we have:

$$|n_3| \leq \frac{(w)_1^2(1+q)^2 t^2}{(u)_1^2(v)_1^2} + \frac{2(c)_2(1+q)t}{(u)_2(v)_2}. \tag{34}$$

Subtracting (27) from (24), we have:

$$\frac{(u)_3(v)_3}{3(w)_3} n_4 = \frac{5(u)_3(v)_3(w)_2(w)_1 G_1^2(t,t,q) h_1(h_2 - l_2)}{24(w)_3(u)_2(v)_2(u)_1(v)_1} + \frac{G_1(t,x,q)(h_3 - l_3)}{2}$$
$$+ \frac{[G_2(t,x,q) - G_1(t,x,q)] h_1(h_2 + l_2)}{2}$$
$$+ \frac{(G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)) h_1^3}{4} + e^{-2i\phi} \mu^2 G_1(t,x,q) h_1. \quad (35)$$

Applying the triangular inequality and Lemma 2, we have:

$$|n_4| \leq \frac{5(w)_2(w)_1(1+q)^2 t^2}{(u)_2(v)_2(u)_1(v)_1} + \frac{6(w)_3(1+q)t}{(u)_3(v)_3} + \frac{12(w)_3[t(1+q)[t(1+q^2) - 1] - qx]}{(u)_3(v)_3}$$
$$+ \frac{6(w)_3[(1+q)t[1 + q(1+q^2)x - 2t(1+q^2) + (1+q^2)(1+q^3)t^2] - 2qx]}{(u)_3(v)_3}$$
$$+ \frac{6\mu^2(1+q)t(w)_3}{(u)_3(v)_3}.$$

□

3. Fekete–Szegö Inequalities for The Function Class $Y_{\Sigma,u,v,w}^{q,x,\phi}(t,\mu)$

The *nth* coefficient of a function class $\mathcal{S}$ is well known to be constrained by n, and the coefficient limits provide information about the geometric properties of the functions. The famous problem solved by Fekete–Szegö [30] is to determine the highest value of the coefficient functional $\Omega_\delta(f) := |n_3 - \delta n_2^2|$ over the class $\mathcal{S}$ for any $\delta \in [0,1]$, which was shown using the Loewner technique.

The upper bonds of the coefficient functional $|n_3 - \delta n_2^2|$ for the function class $Y_{\Sigma,u,v,w}^{q,x,\phi}(t,\mu)$ are determined in this section.

Theorem 3. *Let* $f \in Y_{\Sigma,u,v,w}^{q,x,\phi}(t,\mu)$. *Then, for some* $\delta \in \mathbb{R}$,

$$\left| n_3 - \delta n_2^2 \right| \leq \begin{cases} 2|1 - \delta| \Phi_1(u,v,w,t,q,\mu) & \left(|1 - \delta| \geq \frac{2(w)_2(1+q)t}{(u)_2(v)_2 \Phi_1(u,v,w,t,q,\mu)} \right) \\ \\ \frac{2(w)_2(1+q)t}{(u)_2(v)_2} & \left(|1 - \delta| \leq \frac{2(w)_2(1+q)t}{(u)_2(v)_2 \Phi_1(u,v,w,t,q,\mu)} \right), \end{cases}$$

where $\Phi_1(u,v,w,t,q,\mu)$ *is given in* (2).

Proof. From (32), we have:

$$n_3 - \delta n_2^2 = n_2^2 + \frac{G_1(t,x,q)[h_2 - l_2](w)_2}{2(u)_2(v)_2} - \delta n_2^2$$

By the triangular inequality, we have:

$$|n_3 - \delta n_2^2| \leq \frac{2(w)_2(1+q)t}{(u)_2(v)_2} + |1 - \delta| \Phi_1(u,v,w,t,q,\mu).$$

Suppose:

$$|1 - \delta| \Phi_1(u,v,w,t,q,\mu) \geq \frac{2(w)_2(1+q)t}{(u)_2(v)_2}$$

then we have:

$$|n_3 - \delta n_2^2| \leq 2|1 - \delta| \Phi_1(u,v,w,t,q,\mu)$$

where:

$$|1 - \delta| \geq \frac{2(w)_2(1+q)t}{(u)_2(v)_2\Phi_1(u,v,w,t,q,\mu)}$$

and suppose:

$$|1 - \delta|\Phi_1(u,v,w,t,q,\mu) \leq \frac{2(w)_2(1+q)t}{(u)_2(v)_2},$$

then, we have:

$$|a_3 - \delta a_2^2| \leq \frac{4(w)_2(1+q)t}{(u)_2(v)_2}$$

where:

$$|1 - \delta| \leq \frac{2(w)_2(1+q)t}{(u)_2(v)_2\Phi_1(u,v,w,t,q,\mu)}$$

and $\Phi_1(u,v,w,t,q,\mu)$ is given in (2). $\square$

4. Second Hankel Determinant for the Class $Y_{\Sigma,u,v,w}^{q,x,\phi}(t,\mu)$

Noonan and Thomas [31] introduced and investigated the *mth* Hankel determinant of f for $m \geq 1$ and $n \geq 1$ as:

$$\mathcal{H}_m(j) = \begin{vmatrix} b_j & b_{j+1} & b_{j+2} & \cdots & \cdots & b_{j+m-1} \\ b_{j+1} & b_{j+2} & b_{j+3} & \cdots & \cdots & b_{j+m} \\ b_{j+2} & b_{j+3} & b_{j+4} & \cdots & \cdots & b_{j+m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{j+m-1} & b_{j+m} & b_{j+m+1} & \cdots & \cdots & b_{j+2(m-1)} \end{vmatrix} \quad (m,j \in \mathcal{N}).$$

Several writers, notably Noor [32], have investigated this determinant, with topics ranging from the rate of development of $H_m(j)$ (as $j \longrightarrow \infty$) to the determinant of exact limits for particular subclasses of analytic functions on the unit disk $\mathcal{U}$ with specified values of j and m. When $m = 2$, $j = 1$, and $b_1 = 1$, the Hankel determinant is $H_2(1) = |b_3 - b_2^2|$. The Hankel determinant simplifies to $H_2(2) = |b_2b_4 - b_3^2|$ when $j = m = 2$. Fekete and Szegö [29] considered the Hankel determinant $H_2(1)$ and referred to $H_2(2)$ as the second Hankel determinant. If f is univalent in $\mathcal{U}$, then the sharp upper inequality $H_2(1) = |b_3 - b_2^2| \leq 1$ is known (see [30]). Janteng et al. [33] obtained sharp bounds for the functional $H_2(2)$ for the function f in the subclass $\mathcal{RT}$ of $\mathcal{S}$, which was introduced by MacGregor [34] and consists of functions whose derivative has a positive real part. They demonstrated that $H_2(2) = |b_2b_4 - b_3^2| \leq 4/9$ for each $f \in \mathcal{RT}$. They also discovered the sharp second Hankel determinant for the classical subclass of $\mathcal{S}$, namely S^* and $\mathcal{K}$, which are the classes of starlike and convex functions (see [33]). These two classes have bounds of $|b_2b_4 - b_3^2| \leq 1/8$ and $|b_2b_4 - b_3^2| \leq 1$. The Hankel determinants for starlike and convex functions with respect to symmetric points were recently discovered by Ready and Krishna [35]. For functions belonging to subclasses of M-a-Minda starlike and convex functions, Lee et al. [36] found the second Hankel determinant. Mishra and Gochhayat [14] found the sharp bound to the nonlinear functional $|b_2b_4 - b_3^2|$ for the subclass of analytic functions.

Deniz et al. [37] discussed the upper bounds of $H_2(2)$ for the classes S^* and $\mathcal{K}$ recently. Later, Altinkaya and Yalcin [38], Caglar et al. [39], Kanas et al. [40], and Orhan et al. [41] determined the upper bounds of $H_2(2)$ for several subclasses of Σ.

Theorem 4. *Let the function $f(z)$ given by (1) be in the class $Y^{q,x,\phi}_{\Sigma,u,v,w}(t,\mu)$. Then:*

$$
H_2(2) = \left| n_2 n_4 - n_3^2 \right| \leq
\begin{cases}
T(2,t) & (D_1 \geq 0 \ \ and \ \ D_2 \geq 0) \\[2mm]
\max\left\{ \dfrac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2 (v)_2^2}, T(2,t) \right\} & (D_1 > 0 \ \ and \ \ D_2 < 0) \\[2mm]
\dfrac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2 (v)_2^2} & (D_1 \leq 0 \ \ and \ \ D_2 \leq 0) \\[2mm]
\max\{T(m_0,t), T(2,t)\} & (D_1 < 0 \ \ and \ \ D_2 > 0).
\end{cases}
$$

where:

$$
\begin{aligned}
T(2,t) = {} & \frac{6(w)_1 (w)_3 G_1^2(t,x,q)[G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)]}{(u)_1 (v)_1 (u)_3 (v)_3} \\
& + \frac{12(w)_3 (w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)]}{(u)_1 (v)_1 (u)_3 (v)_3} \\
& - \frac{15(w)_1 (w)_3 G_1^2(t,x,q)}{4(u)_1 (v)_1 (u)_3 (v)_3} + \frac{(w)_1^4 G_1^4(t,x,q)}{(u)_1^4 (v)_1^4} \\
& + \frac{6\mu^2 G_1^2(t,x,q)(w)_1 (w)_3}{(u)_1 (v)_1 (u)_3 (v)_3} - \frac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2 (v)_2^2},
\end{aligned}
$$

$$
T(m_0,t) = \frac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2 (v)_2^2} + \frac{(u)_1^2 (v)_1^2 D_2^4}{(u)_2^2 (v)_2^2 (u)_3 (v)_3 D_1^3} + \frac{(u)_1^2 (v)_1^2 D_2^3}{(u)_2^2 (v)_2^2 (u)_3 (v)_3 D_1^2},
$$

and D_1, D_2 are given by (52) and (53).

Proof. From (22), (35), and (33) and with some calculations, we have:

$$
\begin{aligned}
n_2 n_4 - n_3^2 = {} & \frac{5(w)_1^2 (w)_2 G_1^3(t,x,q)(h_2 - l_2)}{16(u)_2 (v)_2 (u)_1^2 (v)_1^2} h_1^2 + \frac{3(w)_1 (w)_3 G_1^2(t,x,q)(h_3 - l_3)}{4(u)_1 (u)_3 (v)_1 (v)_3} h_1 \\
& + \frac{3(w)_3 (w)_2 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)] h_1 (h_2 + l_2)}{4(u)_1 (v)_1 (u)_3 (v)_3} \\
& + \frac{3(w)_3 (w)_2 G_1(t,x,q)[G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)]}{8(u)_1 (v)_1 (u)_3 (v)_3} h_1^4 \\
& - \frac{(w)_1^4 G_1^4(t,x,q)}{16(u)_1^4 (v)_1^4} h_1^4 - \frac{(w)_1^2 (w)_2 G_1^3(t,x,q)(h_2 - l_2)}{4(u)_1^2 (v)_1^2 (u)_2 (v)_2} h_1^2 \\
& - \frac{(w)_2^2 G_1^2(t,x,q)(h_2 - l_2)^2}{4(u)_2^2 (v)_2^2} + \frac{3e^{-2i\phi} \mu^2 G_1^2(t,x,q)(w)_1 (w)_3}{2(u)_1 (v)_1 (u)_3 (v)_3} h_1^2.
\end{aligned}
\tag{36}
$$

By using Lemma 1,

$$
h_2 - l_2 = \frac{4 - h_1^2}{2}(b - c)
\tag{37}
$$

$$
h_2 + l_2 = h_1^2 + \frac{4 - h_1^2}{2}(b + c)
\tag{38}
$$

and:

$$
\begin{aligned}
h_3 - l_3 = {} & \frac{h_1^3}{2} + \frac{4 - h_1^2}{2} h_1 (b + c) - \frac{4 - h_1^2}{4} h_1 (b^2 + c^2) \\
& + \frac{4 - h_1^2}{2}\left[(1 - |b|^2 z) - (1 - |c|^2) w \right]
\end{aligned}
\tag{39}
$$

for some b, c, z, w with $|b| \leq 1$, $|c| \leq 1$, $|z| \leq 1$, $|w| \leq 1$, $|p_1| \in [0, 2]$, substituting $(h_2 + l_2)$, $(h_2 - l_2)$ and $(h_3 - l_3)$, and after some straightforward simplifications, we have:

$$n_2 n_4 - n_3^2 = \frac{(w)_1^2 (w)_2 G_1^3(t, x, q)(4 - h_1^2)(b - c)}{32(u)_2(v)_2(u)_1^2(v)_1^2} h_1^2$$

$$+ \frac{3(w)_3(w)_1 G_1(t, x, q)[G_2(t, x, q) + G_1(t, x, q)]}{4(u)_1(v)_1(u)_3(v)_3} h_1^4 + \frac{3(w)_1(w)_3 G_1^2(t, x, q)}{16(u)_1(v)_1(u)_3(v)_3} h_1^4$$

$$+ \frac{3(w)_3(w)_1 G_1(t, x, q)[G_2(t, x, q) + G_1(t, x, q)](4 - h_1^2)(b + c)}{8(u)_1(v)_1(u)_3(v)_3} h_1^2$$

$$+ \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - h_1^2)(b + c)}{16(u)_1(v)_1(u)_3(v)_3} h_1^2$$

$$- \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - h_1^2)(b^2 + c^2)}{32(u)_1(v)_1(u)_3(v)_3} h_1^2 - \frac{(w)_1^4 G_1^4(t, x, q)}{16(u)_1^4(v)_1^4} h_1^4$$

$$+ \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - h_1^2)[(1 - |b|^2)z - (1 - |c|^2)w]}{16(u)_1(v)_1(u)_3(v)_3} h_1$$

$$- \frac{(w)_2^2 G_1^2(t, x, q)(4 - h_1^2)^2(b - c)^2}{16(u)_2^2(v)_2^2} + \frac{3e^{-2i\phi} \mu^2 G_1^2(t, x, q)(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} h_1^2$$

$$+ \frac{3(w)_1(w)_3 G_1^2(t, x, q)[G_1(t, x, q) - 2G_2(t, x, q) + G_3(t, x, q)]}{8(u)_1(v)_1(u)_3(v)_3} h_1^4.$$

Let $m = h_1$, and assume without any restriction that $m \in [0, 2]$, $\lambda_1 = |b| \leq 1$ and $\lambda_2 = |c| \leq 1$; by applying the triangular inequality, we have:

$$|n_2 n_4 - n_3^2| \leq \left\{ \frac{3(w)_1(w)_3 G_1^2(t, x, q)[G_1(t, x, q) - 2G_2(t, x, q) + G_3(t, x, q)]}{8(u)_1(v)_1(u)_3(v)_3} m^4 \right.$$

$$+ \frac{3(w)_3(w)_1 G_1(t, x, q)[G_2(t, x, q) + G_1(t, x, q)]}{4(u)_1(v)_1(u)_3(v)_3} m^4 + \frac{3(w)_1(w)_3 G_1^2(t, x, q)}{16(u)_1(v)_1(u)_3(v)_3} m^4$$

$$+ \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - m^2)}{8(u)_1(v)_1(u)_3(v)_3} m + \frac{(w)_1^4 G_1^4(t, x, q)}{16(u)_1^4(v)_1^4} m^4$$

$$\left. + \frac{3\mu^2 G_1^2(t, x, q)(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} m^2 \right\}$$

$$+ \left\{ \frac{3(w)_3(w)_1 G_1(t, x, q)[G_2(t, x, q) + G_1(t, x, q)](4 - m^2)}{8(u)_1(v)_1(u)_3(v)_3} m^2 \right.$$

$$\left. + \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - m^2)}{16(u)_1(v)_1(u)_3(v)_3} m^2 + \frac{(w)_1^2(w)_2 G_1^3(t, x, q)(4 - m^2)}{32(u)_2(v)_2(u)_1^2(v)_1^2} m^2 \right\} (\lambda_1 + \lambda_2)$$

$$+ \left\{ \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - m^2)}{32(u)_1(v)_1(u)_3(v)_3} m^2 - \frac{3(w)_1(w)_3 G_1^2(t, x, q)(4 - m^2)}{16(u)_1(v)_1(u)_3(v)_3} m \right\} (\lambda_1^2 + \lambda_2^2)$$

$$+ \frac{(w)_2^2 G_1^2(t, x, q)(4 - m^2)^2}{16(u)_2^2(v)_2^2} (\lambda_1 + \lambda_2)^2$$

and equivalently, we have:

$$|n_2 n_4 - n_3^2| \leq L_1(t, m) + L_2(t, m)(\lambda_1 + \lambda_2) + L_3(t, m)(\lambda_1^2 + \lambda_2^2)$$

$$+ L_4(t, m)(\lambda_1 + \lambda_2)^2 = Z(\lambda_1, \lambda_2) \tag{40}$$

where,

$$L_1(t,m) = \left\{ \frac{3(w)_1(w)_3 G_1^2(t,x,q)[G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)]}{8(u)_1(v)_1(u)_3(v)_3} m^4 \right.$$

$$+ \frac{3(w)_3(w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)]}{4(u)_1(v)_1(u)_3(v)_3} m^4$$

$$+ \frac{3(w)_1(w)_3 G_1^2(t,x,q)}{16(u)_1(v)_1(u)_3(v)_3} m^4 + \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{8(u)_1(v)_1(u)_3(v)_3} m$$

$$\left. + \frac{(w)_1^4 G_1^4(t,x,q)}{16(u)_1^4(v)_1^4} m^4 + \frac{3\mu^2 G_1^2(t,x,q)(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} m^2 \right\} \geq 0$$

$$L_2(t,m) = \left\{ \frac{3(w)_3(w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)](4-m^2)}{8(u)_1(v)_1(u)_3(v)_3} m^2 \right.$$

$$\left. + \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{16(u)_1(v)_1(u)_3(v)_3} m^2 + \frac{(w)_1^2(w)_2 G_1^3(t,x,q)(4-m^2)}{32(u)_2(v)_2(u)_1^2(v)_1^2} m^2 \right\} \geq 0$$

$$L_3(t,m) = \left\{ \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{32(u)_1(v)_1(u)_3(v)_3} m^2 - \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{16(u)_1(v)_1(u)_3(v)_3} m \right\} \leq 0$$

$$L_4(t,m) = \frac{(w)_2^2 G_1^2(t,x,q)(4-m^2)^2}{16(u)_2^2(v)_2^2} \geq 0$$

where $0 \leq m \leq 2$. Now, we maximize the function $Z(\lambda_1, \lambda_2)$ in the closed square:

$$\Delta = \{(\lambda_1, \lambda_2) : \lambda_1 \in [0,1], \lambda_2 \in [0,1]\} \quad for \quad m \in [0,2].$$

For a fixed value of s, the coefficients of the function $Z(\lambda_1, \lambda_2)$ in (40) are dependent on m; thus, the maximum of $Z(\lambda_1, \lambda_2)$ with regard to m must be investigated, taking into account the cases when $m = 0$, $m = 2$, and $m \in (0,2)$.

First case:
When $m = 0$,

$$Z(\lambda_1, \lambda_2) = L_4(t,0) = \frac{(w)_2^2 G_1^2(t,x,q)}{(u)_2^2(v)_2^2}(\lambda_1 + \lambda_2)^2.$$

It is obvious that the function $Z(\lambda_1, \lambda_2)$ reaches its maximum at (λ_1, λ_2) and:

$$max\{Z(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in [0,1]\} = Z(1,1) = \frac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2(v)_2^2}. \tag{41}$$

Second case:
When $m = 2$, $Z(\lambda_1, \lambda_2)$ is expressed as a constant function with respect to m, we have:

$$Z(\lambda_1, \lambda_2) = N_1(t,2)$$

$$= \left\{ \frac{6(w)_1(w)_3 G_1^2(t,x,q)[G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)]}{(u)_1(v)_1(u)_3(v)_3} \right.$$

$$+ \frac{3(w)_3(w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)]}{4(u)_1(v)_1(u)_3(v)_3}$$

$$+ \frac{3(w)_1(w)_3 G_1^2(t,x,q)}{16(u)_1(v)_1(u)_3(v)_3} + \frac{(w)_1^4 G_1^4(t,x,q)}{(u)_1^4(v)_1^4}$$

$$\left. + \frac{3\mu^2 G_1^2(t,x,q)(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} \right\}.$$

Third case:

When $m \in (0, 2)$, let $\lambda_1 + \lambda_2 = \sigma$ and $\lambda_1 \cdot \lambda_2 = \nu$ in this case, then (40) can be of the form:

$$Z(\lambda_1, \lambda_2) = L_1(t, m) + L_2(t, m)\sigma + (L_3(t, m) + L_4(t, m))\sigma^2 - 2L_3(t, m)\nu = V(\sigma, \nu) \quad (42)$$

where $\sigma \in [0, 2]$ *and* $\nu \in [0, 1]$. Now, we need to investigate the maximum of:

$$V(\sigma, \nu) \in \Lambda = \{(\sigma, \nu) : \sigma \in [0, 2], \nu \in [0, 1]\}. \quad (43)$$

By differentiating $V(\sigma, \nu)$ partially, we have:

$$\frac{\partial V}{\partial \sigma} = L_2(t, m) + 2(L_3(t, m) + L_4(t, m))\sigma = 0$$

$$\frac{\partial V}{\partial \nu} = -2L_3(t, m) = 0.$$

These results reveal that $V(\sigma, \nu)$ does not have a critical point in Λ, and so, $Z(\lambda_1, \lambda_2)$ does not have a critical point in the square Δ.

As a result, the function $Z(\lambda_1, \lambda_2)$ cannot have its maximum value in the interior of Δ. The maximum of $Z(\lambda_1, \lambda_2)$ on the boundary of the square Δ is investigated next.

For $\lambda_1 = 0$, $\lambda_2 \in [0, 1]$ (also, for $\lambda_2 = 0, \lambda_1 \in [0, 1]$) and:

$$Z(0, \lambda_2) = L_1(t, m) + L_2\lambda_2 + (L_3(t, m) + L_4(t, m))\lambda_2^2 = Q(\lambda_2). \quad (44)$$

Now, since $L_3(t, m) + L_4(t, m) \geq 0$, then we have:

$$Q'(\lambda_2) = L_2(t, m) + 2[L_3(t, m) + L_4(t, m)]\lambda_2 > 0$$

which implies that $Q(\lambda_2)$ is an increasing function. Therefore, for a fixed $m \in [0, 2)$ and $t \in (1/2, 1]$, the maximum occurs at $\lambda_2 = 1$. Thus, from (44),

$$\max\{G(0, \lambda_2) : \lambda_2 \in [0, 1]\} = Z(0, 1)$$
$$= L_1(t, m) + L_2(t, m) + L_3(t, m) + L_4(t, m). \quad (45)$$

For $\lambda_1 = 1, \lambda_2 \in [0, 1]$ (also for $\lambda_2 = 1, \lambda_1 \in [0, 1]$) and:

$$Z(1, \lambda_2) = L_1(t, m) + L_2(t, m) + L_3(t, m) + L_4(t, m) + [L_2(t, m)$$
$$+ 2L_4(t, m)]\lambda_2 + [L_3(t, m) + L_4(t, m)]\lambda_2^2 = D(\lambda_2) \quad (46)$$

$$D'(\lambda_2) = [L_2(t) + 2L_4(t)] + 2[L_3(t) + L_4(t)]\lambda_2. \quad (47)$$

We know that $L_3(t) + L_4(t) \geq 0$, then:

$$D'(\lambda_2) = [L_2(t) + 2L_4(t)] + 2[L_3(t) + L_4(t)]\lambda_2 > 0.$$

Therefore, the function $D(\lambda_2)$ is an increasing function, and the maximum occurs at $\lambda_2 = 1$. From (46), we have:

$$\max\{Z(1, \lambda_2) : \lambda_2 \in [0, 1]\} = Z(1, 1)$$
$$= L_1(t, m) + 2[L_2(t, m) + L_3(t, m)] + 4L_4(t, m). \quad (48)$$

Hence, for every $m \in (0, 2)$, taking it from (45) and (48), we have:

$$L_1(t, m) + 2[L_2(t, m) + L_3(t, m)] + 4L_4(t, m)$$
$$> L_1(t, m) + L_2(t, m) + L_3(t, m) + L_4(t, m).$$

Therefore,

$$
\begin{aligned}
&\max\{Z(\lambda_1, \lambda_2) : \lambda_1 \in [0,1], \lambda_2 \in [0,1]\} \\
&= \quad L_1(t,m) + 2[L_2(t,m) + L_3(t,m)] + 4L_4(t,m).
\end{aligned}
$$

Since,

$$
Q(1) \le D(1) \quad for \quad m \in [0,2] \quad and \quad t \in [1,1],
$$

then:

$$
\max\{Z(\lambda_1, \lambda_2)\} = Z(1,1)
$$

occurs on the boundary of square Δ.

Let $T : (0,2) \to \mathbb{R}$ be defined by:

$$
T(m,t) = \max\{Z(\lambda_1, \lambda_2)\} = Z(1,1) = L_1(t,m) + 2L_2(t,m) + 2L_3(t,m) + 4L_4(t,m). \quad (49)
$$

Now, inserting the values of $L_1(t,m), L_2(t,m), L_3(t,m)$ and $L_4(t,m)$ into (49) and with some calculations, we have:

$$
\begin{aligned}
T(m,t) = &\left\{ \frac{3(w)_1(w)_3 G_1^2(t,x,q)|G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)|}{8(u)_1(v)_1(u)_3(v)_3} m^4 \right. \\
&+ \frac{3(w)_3(w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)]}{4(u)_1(v)_1(u)_3(v)_3} m^4 \\
&+ \frac{3(w)_1(w)_3 G_1^2(t,x,q)}{16(u)_1(v)_1(u)_3(v)_3} m^4 + \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{8(u)_1(v)_1(u)_3(v)_3} m \\
&\left. + \frac{(w)_1^4 G_1^4(t,x,q)}{16(u)_1^4(v)_1^4} m^4 + \frac{3\mu^2 G_1^2(t,x,q)(w)_1(w)_3}{2(u)_1(v)_1(u)_3(v)_3} m^2 \right\} \\
&+ \left\{ \frac{3(w)_3(w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)](4-m^2)}{4(u)_1(v)_1(u)_3(v)_3} m^2 \right. \\
&\left. + \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{8(u)_1(v)_1(u)_3(v)_3} m^2 + \frac{(w)_1^2(w)_2 G_1^3(t,x,q)(4-m^2)}{16(u)_2(v)_2(u)_1^2(v)_1^2} m^2 \right\} \\
&+ \left\{ \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{16(u)_1(v)_1(u)_3(v)_3} m^2 - \frac{3(w)_1(w)_3 G_1^2(t,x,q)(4-m^2)}{8(u)_1(v)_1(u)_3(v)_3} m \right\} \\
&+ \frac{(w)_2^2 G_1^2(t,x,q)(4-m^2)^2}{4(u)_2^2(v)_2^2}.
\end{aligned}
$$

By simplifying, we have:

$$
T(m,t) = \frac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2(v)_2^2} + \frac{D_1}{16(u)_1^4(v)_1^4(u)_2^2(v)_2^2(u)_3(v)_3} m^4 \quad (50)
$$

$$
+ \frac{D_2}{4(u)_1^2(v)_1^2(u)_2^2(v)_2^2(u)_3(v)_3} m^2, \quad (51)
$$

where:

$$D_1 = G_1(t,x,q)\left[6(u)_1^3(v)_1^3(u)_2^2(v)_2^2(w)_1(w)_3[G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)]\right.$$

$$+ (u)_2^2(v)_2^2(u)_3(v)_3(w)_1^4 G_1^3(t,x,q) - 6(u)_1^3(v)_1^3(u)_2^2(v)_2^2(w)_1(w)_3 G_1(t,x,q)$$

$$\left. + 4(u)_1^4(v)_1^4(u)_3(v)_3(w)_2^2 G_1(t,x,q) - (u)_1^2(v)_1^2(u)_2(v)_2(u)_3(v)_3(w)_1^2(w)_2 G_1^2(t,x,q)\right] m^4 \tag{52}$$

$$D_2 = G_1(t,x,q)\left[12(u)_2^2(v)_2^2(w)_3(w)_1(u)_1(v)_1[G_2(t,x,q) - G_1(t,x,q)]\right.$$

$$+ 9(u)_2^2(v)_2^2(u)_1(v)_1(w)_1(w)_3 G_1(t,x,q) - 8(u)_1^2(v)_1^2(u)_3(v)_3(w)_2^2 G_1(t,x,q)$$

$$\left. + (u)_2(v)_2(u)_3(v)_3(w)_1^2(w)_2 G_1^2(t,x,q) + 6(u)_1(v)_1(u)_2^2(v)_2^2\mu^2(w)_1(w)_3 G_1(t,x,q)\right] m^2. \tag{53}$$

If $T(m,t)$ has a maximum value in the interior of $m \in [0,2]$ and by applying some elementary calculus, we have:

$$T'(m,t) = \frac{D_1}{4(u)_1^4(v)_1^4(u)_2^2(v)_2^2(u)_3(v)_3} m^3 + \frac{D_2}{2(u)_1^2(v)_1^2(u)_2^2(v)_2^2(u)_3(v)_3} m.$$

Now, we need to examine the sign of the function $T'(m,t)$ depending on the signs of D_1 and D_2 as follows.

First result:
Suppose $D_1 \geq 0$ and $D_2 \geq 0$, then $T'(m,t) \geq 0$. This shows that $T(m,t)$ is an increasing function on the boundary of $m \in [0,2]$, that is $m = 2$. Therefore,

$$\max\{T(m,t) : m \in (0,2)\} = \frac{6(w)_1(w)_3 G_1^2(t,x,q)[G_1(t,x,q) - 2G_2(t,x,q) + G_3(t,x,q)]}{(u)_1(v)_1(u)_3(v)_3}$$

$$+ \frac{12(w)_3(w)_1 G_1(t,x,q)[G_2(t,x,q) + G_1(t,x,q)]}{(u)_1(v)_1(u)_3(v)_3}$$

$$- \frac{15(w)_1(w)_3 G_1^2(t,x,q)}{4(u)_1(v)_1(u)_3(v)_3} + \frac{(w)_1^4 G_1^4(t,x,q)}{(u)_1^4(v)_1^4}$$

$$+ \frac{6\mu^2 G_1^2(t,x,q)(w)_1(w)_3}{(u)_1(v)_1(u)_3(v)_3} - \frac{4(w)_2^2 G_1^2(t,x,q)}{(u)_2^2(v)_2^2}.$$

Second result:
If $D_1 > 0$ and $D_2 < 0$ then,

$$T'(m,t) = \frac{D_1 m^3 + 2(u)_1^2(v)_1^2 D_2 m}{4(u)_1^4(v)_1^4(u)_2^2(v)_2^2(u)_3(v)_3} = 0 \tag{54}$$

at critical point:

$$m_0 = \sqrt{\frac{-2(u)_1^2(v)_1^2 D_2}{D_1}} \tag{55}$$

is a critical point of the function $T(m,t)$. Now,

$$T''(m_0) = \frac{-3D_2}{2(u)_1^2(v)_1^2(u)_2^2(v)_2^2(u)_3(v)_3} + \frac{D_2}{2(u)_1^2(v)_1^2(u)_2^2(v)_2^2(u)_3(v)_3} > 0.$$

Therefore, m_0 is the minimum point of the function $T(m, t)$. Hence, $T(m, t)$ cannot have a maximum.

Third result:

If $D_1 \leq 0$ and $D_2 \leq 0$, then:

$$T'(m, t) \leq 0.$$

Therefore, $T(m, t)$ is a decreasing function on the interval $(0, 2)$. Hence,

$$\max\{T(m, t) : t \in (0, 2)\} = T(0) = \frac{4(w)_2^2 G_1^2(t, x, q)}{(u)_2^2 (v)_2^2}. \tag{56}$$

Fourth result:

If $D_1 < 0$ and $D_2 > 0$

$$T''(m_0, t) = \frac{-3D_2}{2(u)_1^2 (v)_1^2 (u)_2^2 (v)_2^2 (u)_3 (v)_3} + \frac{D_2}{2(u)_1^2 (v)_1^2 (u)_2^2 (v)_2^2 (u)_3 (v)_3} < 0.$$

Therefore, $T''(m, t) < 0$. Hence, m_0 is the maximum point of the function $T(m, t)$, and the maximum value occurs at $m = m_0$. Thus,

$$max\{T(m, t) : m \in (0, 2)\} = T(m_0, 4)$$

$$T(m_0, t) = \frac{4(w)_2^2 G_1^2(t, x, q)}{(u)_2^2 (v)_2^2} + \frac{(u)_1^2 (v)_1^2 D_2^4}{(u)_2^2 (v)_2^2 (u)_3 (v)_3 D_1^3} + \frac{(u)_1^2 (v)_1^2 D_2^3}{(u)_2^2 (v)_2^2 (u)_3 (v)_3 D_1^2}.$$

□

5. Conclusions

The q-derivative operator has recently been found to be extremely useful in the disciplines of mathematics and physics. To begin, numerous well-known notions of the q-derivative operator were highlighted and explained in this study. We then developed a novel subclass of analytic and bi-univalent functions using the Hohlov operator and certain q-Chebyshev polynomials. A number of coefficient bonds, as well as the Fekete–Szegö inequalities and the second Hankel determinant were provided for these newly specified function classes.

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