

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/343386509>

On a new subclass of bi-pseudo-starlike functions defined by frasin differential operator

Article in *Advances in Mathematics Scientific Journal* · August 2020

DOI: 10.37418/amsj.9.7.49

CITATIONS

8

READS

623

3 authors, including:



Timilehin Gideon Shaba
Landmark University

57 PUBLICATIONS 369 CITATIONS

[SEE PROFILE](#)



Azeez Ade Jimoh

African Institute for Mathematical Sciences Cameroon

3 PUBLICATIONS 8 CITATIONS

[SEE PROFILE](#)

**ON A NEW SUBCLASS OF BI-PSEUDO-STARLIKE FUNCTIONS DEFINED
BY FRASIN DIFFERENTIAL OPERATOR**TIMILEHIN GIDEON SHABA¹, ABDULLAHI ADINOYI IBRAHIM, AND AZEEZ ADE JIMOH

ABSTRACT. The main aim of this research is to introduce and examine new subclasses of functions class \mathfrak{E} of bi-univalent functions defined in Δ associating with γ -pseudo-starlike functions with sakaguchi type functions $\mathfrak{S}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \nu, s, t)$ and $\mathfrak{S}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$, which are defined by a differential operator of holomorphic functions with binomial series. Also, the estimate on the coefficient $|n_2|$ and $|n_3|$ for functions in these new subclasses are determined. Results acquired generalized some known consequences.

1. INTRODUCTION

We indicate by \mathcal{V} the subclass of class of function \mathcal{L} which is of the form

$$\psi(z) = z + \sum_{g=2}^{\infty} n_g z^g$$

consisting of functions which are holomorphic and univalent in the unit disk Δ . Let $\mathfrak{S}^*(\vartheta)$ and $\mathfrak{K}(\vartheta)$ indicate the familiar classes of starlike and convex function of order ϑ ($0 \leq \vartheta < 1$) respectively.

Let $\psi^{-1}(z)$ be the inverse of the function $\psi(z)$ then we have

$$\psi^{-1}(\psi(z)) = z,$$

¹corresponding author

2010 Mathematics Subject Classification. 30C45.

Key words and phrases. Analytic function, bi-univalent function, Sakaguchi type function, pseudo-starlike function, Coefficient bounds.

$$\psi(\psi^{-1}(u)) = u, \quad |u| < r_0(\psi); r_0(\psi) \geq \frac{1}{4}$$

where

$$h(u) = \psi^{-1}(u) = u - n_2u^2 + (2n_2^2 - n_3)u^3 - (5n_2^3 - 5n_2n_3 + n_4)u^4 + \dots$$

A function $\psi(z) \in \mathcal{L}$ denoted by \mathfrak{E} is said to be bi-univalent in Δ , considering that $\psi(z)$ and $\psi^{-1}(z)$ are univalent in Δ . For more details see: [7], [4], [15], [5], [10].

Definition 1.1. [3] Let $\psi(z) \in \mathcal{L}$, suppose $0 \leq \vartheta < 1$ and $\gamma \geq 1$ is real. Then $\psi(z) \in L_\gamma(\vartheta)$ of γ -pseodu-starlike function of order ϑ in Δ if and only if

$$\Re \left(\frac{z[\psi'(z)]^\gamma}{\psi(z)} \right) > \vartheta.$$

Babalola [3] verified that, all pseodu-starlike function are Bazilevic of type $(1 - \frac{1}{\gamma})$, order $\vartheta^{\frac{1}{\gamma}}$ and univalent in Δ .

A function $\psi(z) \in \mathcal{L}$ satisfying the condition

$$\Re \left(\frac{z\psi'(z)}{\psi(z) - \psi(-z)} \right) > 0.$$

is required to be a starlike functions with respect to symmetric point, which was investigated by Sakaguchi [13]. Many other authors examine bounds for numerous subclasses of bi-univalent functions, (for more details see; [9], [11], [16]).

Frasin [6] introduced the differential operator $D_{k,\mu}^\beta \psi(z)$ defined as follows:

$$\begin{aligned} D^0 \psi(z) &= \psi(z) \\ D_{k,\mu}^1 \psi(z) &= (1 - \mu)^k \psi(z) + (1 - (1 - \mu)^k) z \psi'(z) = D_{k,\mu} \psi(z) \\ D_{k,\mu}^\beta \psi(z) &= D_{k,\mu} (D^{\beta-1} \psi(z)) \end{aligned}$$

where $\beta \in \mathbb{N}$, then we have

$$(1.1) \quad D_{k,\mu}^\beta \psi(z) = z + \sum_{g=2}^{\infty} \left(1 + (g-1) \sum_{d=1}^k \binom{k}{d} (-1)^{d+1} \mu^d \right)^\beta n_g z^g.$$

Using (1.1), we have

$$C_d^k(\mu) z (D_{k,\mu}^\beta \psi(z))' = D_{k,\mu}^{\beta+1} \psi(z) - (1 - C_d^k(\mu)) D_{k,\mu}^\beta \psi(z)$$

where $\mu > 0, k \in \mathbb{N}, \beta \in \mathbb{N}_0$ and $C_d^k(\mu) := \sum_{d=1}^k \binom{k}{d} (-1)^{d+1} \mu^d$.

Remark 1.1. We observe that

- (1) When $k = 1$, we obtain the Al-Oboudi differential operator [2].
- (2) When $k = \mu = 1$, we obtain the Salagean operator [14].

Motivated by the earlier works of [8], [1], we introduced new subclasses $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \nu, s, t)$ and $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$ of the function class \mathfrak{E} which are defined by a differential operator of holomorphic functions comprising of binomial series in Δ . Hence, the estimate on the coefficient $|n_2|$ and $|n_3|$ for functions in these new subclasses are determined.

Lemma 1.1. [12] If $r(z) \in \mathcal{P}$ and $z \in \Delta$, then $|w_n| \leq 2$ for each n . where \mathcal{P} is the family of all function r holomorphic in Δ for which $\Re(r(z)) > 0$,

$$r(z) = 1 + w_1z + w_2z^2 + \dots$$

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \nu, s, t)$

Definition 2.1. A function $\psi(z) \in \mathcal{L}$ which gratify the condition below:

$$(2.1) \quad \left| \arg \left[\frac{(s-t)z[(D_{k,\mu}^{\beta}\psi(z))']^{\gamma}}{D_{k,\mu}^{\beta}\psi(sz) - D_{k,\mu}^{\beta}\psi(tz)} \right] \right| < \frac{\nu\pi}{2},$$

and

$$(2.2) \quad \left| \arg \left[\frac{(s-t)u[(D_{k,\mu}^{\beta}h(u))']^{\gamma}}{D_{k,\mu}^{\beta}h(su) - D_{k,\mu}^{\beta}h(tu)} \right] \right| < \frac{\nu\pi}{2}$$

where $\psi(z) \in \mathfrak{E}$, $\gamma \geq 1$, $0 < \nu \leq 1$, $s, t \in \mathbb{C}$, $z, u \in \Delta$ with $|s| \leq 1$, $|t| \leq 1$; $s \neq t$ and

$$h(u) = u - n_2u^2 + (2n_2^2 - n_3)u^3 - (5n_2^3 - 5n_2n_3 + n_4)u^4 + \dots$$

is said to be in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \nu, s, t)$.

Theorem 2.1. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \nu, s, t)$, then

$$(2.3) \quad |n_2| \leq \frac{2\nu}{\sqrt{\left| \nu(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^{\beta} - \left[2\nu(2\gamma(s+t+1-\gamma) - s^2 - t^2 - 2st) + (\nu-1)(2\gamma-s-t)^2 \right] (1 + C_d^k(\mu))^{2\beta} \right|}}$$

and

$$|n_3| \leq \frac{4\nu^2}{|(2\gamma - s - t)^2| (1 + C_d^k(\mu))^{2\beta}} + \frac{2\nu}{|(3\gamma - s^2 - t^2 - st)| (1 + 2C_d^k(\mu))^\beta}.$$

Proof. Let $\psi(z) \in \mathfrak{H}_{\mathfrak{E}}^\beta(\mu, \gamma, \nu, s, t)$, then it follows from (2.1) and (2.2) that

$$(2.4) \quad \frac{(s - t)z[(D_{k,\mu}^\beta \psi(z))']^\gamma}{D_{k,\mu}^\beta \psi(sz) - D_{k,\mu}^\beta \psi(tz)} = [y(z)]^\nu$$

and

$$(2.5) \quad \frac{(s - t)u[(D_{k,\mu}^\beta h(u))']^\gamma}{D_{k,\mu}^\beta h(su) - D_{k,\mu}^\beta h(tu)} = [x(u)]^\nu$$

where $y(z)$ and $x(u)$ are in the class \mathcal{P} which is of the form

$$(2.6) \quad y(z) = 1 + y_1z + y_2z^2 + y_3z^3 + \dots$$

$$(2.7) \quad x(u) = 1 + x_1u + x_2u^2 + x_3u^3 + \dots$$

Hence,

$$[y(z)]^\nu = 1 + \nu y_1z + \left(\nu y_2 + \frac{\nu(\nu - 1)y_1^2}{2!} \right) z^2 + \dots$$

$$[x(u)]^\nu = 1 + \nu x_1u + \left(\nu x_2 + \frac{\nu(\nu - 1)x_1^2}{2!} \right) u^2 + \dots$$

Now, equating the coefficient in (2.4) and (2.5) we get

$$(2.8) \quad (2\gamma - s - t) (1 + C_d^k(\mu))^\beta n_2 = \nu y_1$$

$$(2.9) \quad (3\gamma - s^2 - st - t^2) (1 + 2C_d^k(\mu))^\beta n_3 - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) (1 + C_d^k(\mu))^{2\beta} n_2^2 = \nu y_2 + \frac{\nu(\nu - 1)}{2!} y_1^2$$

$$(2.10) \quad -(2\gamma - s - t) (1 + C_d^k(\mu))^\beta n_2 = \nu x_1$$

$$(2.11) \quad \left[(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) (1 + C_d^k(\mu))^{2\beta} \right] n_2^2 - \left[(3\gamma - s^2 - t^2 - st) (1 + 2C_d^k(\mu))^\beta \right] n_3 = \nu x_2 + \frac{\nu(\nu - 1)}{2!} x_1^2.$$

From (2.8) and (2.10) we get

$$(2.12) \quad y_1 = -x_1$$

and

$$(2.13) \quad 2(2\gamma - s - t)^2 (1 + C_d^k(\mu))^{2\beta} n_2^2 = \nu^2(y_1^2 + x_1^2).$$

Also from (2.9) and (2.11) we have

$$(2.14) \quad \left[(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - 2(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st) \right. \\ \left. (1 + C_d^k(\mu))^{2\beta} \right] n_2^2 = \nu(y_2 + x_2) + \frac{\nu(\nu - 1)}{2!} (y_1^2 + x_1^2).$$

From (2.14) and (2.13), we have

$$\left[\nu(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - 2\nu(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st) \right. \\ \left. (1 + C_d^k(\mu))^{2\beta} - (\nu - 1)(2\gamma - s - t)^2 (1 + C_d^k(\mu))^{2\beta} \right] n_2^2 = \nu^2(y_2 + x_2).$$

Therefore, we have

$$n_2^2 = \frac{\nu^2(y_2 + x_2)}{\left| \nu(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - 2\nu(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st) \right. \\ \left. (1 + C_d^k(\mu))^{2\beta} - (\nu - 1)(2\gamma - s - t)^2 (1 + C_d^k(\mu))^{2\beta} \right|}.$$

From Lemma 1.1, we have

$$|n_2| \leq \frac{2\nu}{\sqrt{\left| \nu(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - \left[2\nu(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st) \right. \right. \\ \left. \left. + (\nu - 1)(2\gamma - s - t)^2 \right] (1 + C_d^k(\mu))^{2\beta} \right|}}.$$

Also, subtracting (2.11) from (2.9), we get

$$(2.15) \quad \begin{aligned} & 2(3\gamma - s^2 - st - t^2) (1 + 2C_d^k(\mu))^\beta n_3 - 2(3\gamma - s^2 - t^2 - st) (1 + 2C_d^k(\mu))^\beta n_2^2 \\ & \qquad \qquad \qquad = \nu(y_2 - x_2) + \frac{\nu(\nu - 1)}{2!} (y_1^2 - x_1^2), \end{aligned}$$

it follows from (2.12), (2.13) and (2.15) that

$$\begin{aligned} & 2(3\gamma - s^2 - st - t^2) (1 + 2C_d^k(\mu))^\beta n_3 \\ & = 2(3\gamma - s^2 - t^2 - st) (1 + 2C_d^k(\mu))^\beta \frac{\nu^2(y_1^2 + x_1^2)}{2(2\gamma - s - t)^2 (1 + C_d^k(\mu))^{2\beta}} + \nu(y_2 - x_2) \end{aligned}$$

which is equivalent to,

$$n_3 = \frac{\nu^2(y_1^2 + x_1^2)}{2(2\gamma - s - t)^2 (1 + C_d^k(\mu))^{2\beta}} + \frac{\nu(y_2 - x_2)}{2(3\gamma - s^2 - t^2 - st) (1 + 2C_d^k(\mu))^\beta}.$$

Applying Lemma 1.1 for the coefficients y_1, y_2, x_1 and x_2 , we have

$$|n_3| \leq \frac{4\nu^2}{|(2\gamma - s - t)^2| (1 + C_d^k(\mu))^{2\beta}} + \frac{2\nu}{|(3\gamma - s^2 - t^2 - st)| (1 + 2C_d^k(\mu))^\beta}.$$

We get the desired estimate $|n_3|$ as asserted in (2.4). □

Putting $\beta = 0$ in Theorem 2.1, we have;

Corollary 2.1. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{S}_{\mathfrak{E}}^0(\gamma, \nu, s, t)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{\left| (6\gamma - 4\gamma(s + t + 1 - \gamma) + 2st)\nu - (\nu - 1)(2\gamma - s - t)^2 \right|}}$$

and

$$|n_3| \leq \frac{4\nu^2}{|(2\gamma - s - t)^2|} + \frac{2\nu}{|(3\gamma - s^2 - t^2 - st)|}.$$

which is the results obtain by Emeka and Opoola [8].

Putting $\gamma = 1$ in Theorem 2.1, we have;

Corollary 2.2. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, 1, \nu, s, t)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{\left| \nu(6 - 2s^2 - 2t^2 - 2st)(1 + 2C_d^k(\mu))^{\beta} - \left[2\nu(2s + 2t - s^2 - t^2 - 2st) + (\nu - 1)(2\gamma - s - t)^2 \right] (1 + C_d^k(\mu))^{2\beta} \right|}}$$

and

$$|n_3| \leq \frac{4\nu^2}{|(2 - s - t)^2| (1 + C_d^k(\mu))^{2\beta}} + \frac{2\nu}{|(3 - s^2 - t^2 - st)| (1 + 2C_d^k(\mu))^{\beta}}$$

which is the results obtain by Aldawish, Al-Hawary and Frasin [1].

Putting $t = 0, s = 1$ and $\gamma = 1$ in Corollary 2.1, we have;

Corollary 2.3. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^0(1, \nu, 1, 0)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{1 + \nu}}$$

and

$$|n_3| \leq \nu(4\nu + 1).$$

Putting $t = -1, s = 1$ and $\gamma = 1$ in Corollary 2.1, we have:

Corollary 2.4. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^0(1, \nu, 1, -1)$, then*

$$|n_2| \leq \nu$$

and

$$|n_3| \leq \nu(\nu + 1).$$

3. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$

Definition 3.1. *A function $\psi(z) \in \mathcal{L}$ which gratify the condition below:*

$$(3.1) \quad \Re \left[\frac{(s - t)z[(D_{k,\mu}^{\beta}\psi(z))']^{\gamma}}{D_{k,\mu}^{\beta}\psi(sz) - D_{k,\mu}^{\beta}\psi(tz)} \right] > \phi,$$

and

$$(3.2) \quad \Re \left[\frac{(s - t)u[(D_{k,\mu}^{\beta}h(u))']^{\gamma}}{D_{k,\mu}^{\beta}h(su) - D_{k,\mu}^{\beta}h(tu)} \right] > \phi$$

where $\psi(z) \in \mathfrak{E}$, $\gamma \geq 1$, $0 \leq \phi \leq 1$, $s, t \in \mathbb{C}$, $z, u \in \Delta$ with $|s| \leq 1$, $|t| \leq 1$; $s \neq t$ and

$$h(u) = u - n_2 u^2 + (2n_2^2 - n_3)u^3 - (5n_2^3 - 5n_2 n_3 + n_4)u^4 + \dots$$

is said to be in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$.

Theorem 3.1. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$, then

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|(3\gamma - s^2 - t^2 - st)(1 + 2C_d^k(\mu))^{\beta} - (2\gamma(s+t-\gamma+1) - s^2 - 2st - t^2)(1 + C_d^k(\mu))^{2\beta}|}}$$

and

$$(3.3) \quad |n_3| \leq \frac{4(1-\phi)^2}{|(2\gamma - s - t)^2|(1 + C_d^k(\mu))^{2\beta}} + \frac{2(1-\phi)}{|(3\gamma - s^2 - st - t^2)|(1 + 2C_d^k(\mu))^{\beta}}.$$

Proof. From equation (3.1) and (3.2) we get:

$$(3.4) \quad \frac{(s-t)z[(D_{k,\mu}^{\beta}\psi(z))']^{\gamma}}{D_{k,\mu}^{\beta}\psi(sz) - D_{k,\mu}^{\beta}\psi(tz)} = \phi + (1-\phi)y(z)$$

and

$$(3.5) \quad \frac{(s-t)u[(D_{k,\mu}^{\beta}h(u))']^{\gamma}}{D_{k,\mu}^{\beta}h(su) - D_{k,\mu}^{\beta}h(tu)} = \phi + (1-\phi)x(u)$$

where $y(z)$ and $x(u)$ in \mathcal{P} given by (2.6) and (2.7), that is

$$\phi + (1-\phi)y(z) = 1 + (1-\phi)y_1z + \phi + (1-\phi)y_2z^2 + \dots$$

and

$$\phi + (1-\phi)x(u) = 1 + (1-\phi)x_1u + \phi + (1-\phi)x_2u^2 + \dots$$

Equating the coefficients of (3.4) and (3.5) we get

$$(3.6) \quad (2\gamma - s - t)(1 + C_d^k(\mu))^{\beta} n_2 = (1-\phi)y_1,$$

$$(3.7) \quad (3\gamma - s^2 - st - t^2)(1 + 2C_d^k(\mu))^{\beta} n_3 - (2\gamma(s+t-\gamma+1) - s^2 - 2st - t^2)(1 + C_d^k(\mu))^{2\beta} n_2^2 = (1-\phi)y_2,$$

$$(3.8) \quad -(2\gamma - s - t)(1 + C_d^k(\mu))^{\beta} n_2 = (1-\phi)x_1,$$

$$(3.9) \quad \left[(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) \right. \\ \left. (1 + C_d^k(\mu))^{2\beta} \right] n_2^2 - \left[(3\gamma - s^2 - t^2 - st) (1 + 2C_d^k(\mu))^\beta \right] n_3 = (1 - \phi)x_2.$$

From (3.6) and (3.8) we get

$$y_1 = -x_1$$

and

$$(3.10) \quad 2(2\gamma - s - t)^2 (1 + C_d^k(\mu))^{2\beta} n_2^2 = (1 - \phi)^2 (y_1^2 + x_1^2).$$

Now adding (3.7) and (3.9), we deduce that

$$\left[(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - (4\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) \right. \\ \left. (1 + C_d^k(\mu))^{2\beta} \right] n_2^2 = (1 - \phi)(y_2 + x_2).$$

Thus, we have

$$n_2^2 = \frac{(1 - \phi)(y_2 + x_2)}{(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - (4\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) (1 + C_d^k(\mu))^{2\beta}},$$

$$|n_2^2| \leq \frac{(1 - \phi)(|y_2| + |x_2|)}{|(6\gamma - 2s^2 - 2t^2 - 2st) (1 + 2C_d^k(\mu))^\beta - (4\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) (1 + C_d^k(\mu))^{2\beta}|}.$$

Applying Lemma 1.1 we have:

$$|n_2^2| \leq \frac{2(1 - \phi)}{|(3\gamma - s^2 - t^2 - st) (1 + 2C_d^k(\mu))^\beta - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) (1 + C_d^k(\mu))^{2\beta}|}$$

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|(3\gamma - s^2 - t^2 - st)(1 + 2C_d^k(\mu))^\beta - (2\gamma(s+t-\gamma+1) - s^2 - 2st - t^2)(1 + C_d^k(\mu))^{2\beta}|}}$$

Also, subtracting (3.9) from (3.7), we get

$$\begin{aligned} 2(3\gamma - s^2 - st - t^2)(1 + 2C_d^k(\mu))^\beta n_3 - 2(3\gamma - s^2 - t^2 - st)(1 + 2C_d^k(\mu))^\beta n_2^2 \\ = (1-\phi)(y_2 - x_2) \\ n_3 = n_2^2 + \frac{(1-\phi)(y_2 - x_2)}{2(3\gamma - s^2 - st - t^2)(1 + 2C_d^k(\mu))^\beta}. \end{aligned}$$

Then from (3.10), we have

$$n_3 = \frac{(1-\phi)^2(y_1^2 + x_1^2)}{2(2\gamma - s - t)^2(1 + C_d^k(\mu))^{2\beta}} + \frac{(1-\phi)(y_2 - x_2)}{2(3\gamma - s^2 - st - t^2)(1 + 2C_d^k(\mu))^\beta}.$$

Applying Lemma 1.1 for the coefficients y_1, y_2, x_1 and x_2 , we have

$$n_3 \leq \frac{4(1-\phi)^2}{|(2\gamma - s - t)^2|(1 + C_d^k(\mu))^{2\beta}} + \frac{2(1-\phi)}{|(3\gamma - s^2 - st - t^2)|(1 + 2C_d^k(\mu))^\beta}.$$

We get desired estimate on $|n_3|$ as asserted in (3.3). \square

Putting $\beta = 0$ in Theorem 3.1, we have:

Corollary 3.1. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^0(\gamma, \phi, s, t)$, then

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|3\gamma - 2\gamma(s+t-\gamma+1) + st|}}$$

and

$$|n_3| \leq \frac{4(1-\phi)^2}{|(2\gamma - s - t)^2|} + \frac{2(1-\phi)}{|(3\gamma - s^2 - st - t^2)|}$$

where $0 \leq \phi < 1$.

which is the results obtain by Emeka and Opoola [8].

Setting $\gamma = 1$ in Theorem 3.1, we have:

Corollary 3.2. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^\beta(\mu, 1, \phi, s, t)$, then

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|(3 - s^2 - t^2 - st)(1 + 2C_d^k(\mu))^\beta - (2s + 2t - s^2 - 2st - t^2)(1 + C_d^k(\mu))^{2\beta}|}}$$

and

$$|n_3| \leq \frac{4(1-\phi)^2}{|(2-s-t)^2| (1+C_d^k(\mu))^{2\beta}} + \frac{2(1-\phi)}{|(3-s^2-st-t^2)| (1+2C_d^k(\mu))^\beta}$$

where $0 \leq \phi < 1$.

which is the results obtain by Aldawish et. al. [1].

Putting $\gamma = 1$ and $\beta = 0$ in Corollary 3.2, we have:

Corollary 3.3. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^0(1, \phi, s, t)$, then

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|3-2(s+t)+st|}}$$

and

$$|n_3| \leq \frac{4(1-\phi)^2}{|(2-s-t)^2|} + \frac{2(1-\phi)}{|(3-s^2-st-t^2)|},$$

where $0 \leq \phi < 1$.

Putting $s = 1$ and $t = -1$ in Corollary 3.3, we have:

Corollary 3.4. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^0(1, \phi, 1, -1)$, then

$$|n_2| \leq \sqrt{1-\phi}$$

and

$$|n_3| \leq (1-\phi)(2-\phi),$$

where $0 \leq \phi < 1$.

Taking $t = 0$ and $s = 1$ in Corollary 3.3, we have:

Corollary 3.5. Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^0(1, \phi, 1, 0)$, then

$$|n_2| \leq \sqrt{2(1-\phi)}$$

and

$$|n_3| \leq (1-\phi)(5-4\phi),$$

where $0 \leq \phi < 1$.

ACKNOWLEDGMENT

The authors thank the referees(s) for their relevant contributions which improved this research.

REFERENCES

- [1] A. ALDAWISH, T. AL-HAWARY, B. A. FRASIN: *Subclasses of bi-univalent function defined by Frasin differential operator*, E. Mathematics., **8** (2020), 1–11.
- [2] F. M. AL-OBOUDI: *On univalent functions defined by a generalized Salagean operator*, Int. J. Math. Math. Sci., (2004), 1429-1436.
- [3] K. O. BABALOLA: *On λ -pseudo-starlike function*, J. Class. Anal., **3** (2013), 137-147.
- [4] D. A. BRANNAN, T. S. TAHA: *On some classes of bi-univalent functions*, In Mathematical Analysis and its Applications; Pergamon Press: Pergamon, Turkey., 2009, 53-60.
- [5] D. A. BRANNAN, J. CLUNIE, W. E. KIRWAN: *Coefficient estimate for a class of starlike functions*, Canad. J. Math., **22** (2009), 476-485.
- [6] B. A. FRASIN: *A new differential operator of analytic functions involving binomial series*, Bol. Soc. Paran. Math., **38** (2020), 205-213.
- [7] M. LEWIN: *On a coefficients problem of bi-univalent functions*, Proc. Am. Math. Soc., **18** (1967), 63-68.
- [8] E. P. MAZI, T. O. OPOOLA: *On some subclasses of bi-univalent functions associating pseudo-starlike functions with sakaguchi type functions*, General Mathematics., **25** (2017), 85-95.
- [9] G. MURUGUSUNDARAMOORTHY, N. MAGESH, V. PRAMEELA: *Coefficient bounds for certain subclasses of bi-univalent functions*, Abs. Appl. Anal., (2013), 1-3.
- [10] E. NETANYAHU: *The minimal distance of the image boundary from the origin and the second coefficient of univalent function in $|z| < 1$* , Proc. Arch. Ration. Mech. Anal., **32** (1969), 100-112.
- [11] S. O. OLATUNJI, P. T. AJAYI: *On subclasses of bi-univalent functions of Bazelevic type involving linear salagean operator*, Internat. J. Pure. Appl. Math., **92** (2014), 645-656.
- [12] C. H. POMMERENKE: *Univalent Functions*, Vandendoeck and Ruprecht, Gottingen, 1975.
- [13] K. SAKAGUCHI: *On certain a certain univalent mapping*, J. Math. Soc. Jpn., **11** (1959), 72-75.
- [14] G. S. SALAGEAN: *Subclasses of Univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin., **1013** (1983), 362–372.
- [15] T. S. TAHA: *Topics in univalent functions theory*, Ph.D. Thesis, University of London, London, UK, 1998.
- [16] Q. H. XU, Y. C. GUI, H. M. SRIVASTAVA: *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput., (2012), 11461-11465.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILORIN, ILORIN, NIGERIA
E-mail address: shaba_timilehin@yahoo.com

DEPARTMENT OF MATHEMATICS
BAZE UNIVERSITY, NIGERIA
E-mail address: abdullahi.ibrahim@bazeuniversity.edu.ng

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILORIN, ILORIN, NIGERIA
E-mail address: azeez.jimohade@gmail.com