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Research Article

Applications of q-Derivative Operator to the Subclass of Bi-Univalent Functions Involving q-Chebyshev Polynomials

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In recent years, the usage of the q-derivative and symmetric q-derivative operators is significant. In this study, firstly, many known concepts of the q-derivative operator are highlighted and given. We then use the symmetric q-derivative operator and certain q-Chebyshev polynomials to define a new subclass of analytic and bi-univalent functions. For this newly defined functions' classes, a number of coefficient bounds, along with the Fekete–Szegö inequalities, are also given. To validate our results, we give some known consequences in form of remarks.

1. Introduction and Definitions

Let $\mathscr{H}(\mathbb{D})$ denote the class of functions which are analytic in the open unit disk:

$$\mathbb{D} = \{z: z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}. \tag{1}$$

Let $\mathscr{A}$ be the subclass of functions $f \in \mathscr{H}(\mathbb{D})$, which satisfy the normalization condition given by

$$f(0) = f'(0) - 1 = 0, \tag{2}$$

that is, which are represented by the following Taylor–Maclaurin series expansion:

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathbb{D}. \tag{3}$$

Also, let $\mathscr{S}$ be the class of functions in $\mathscr{A}$, which are univalent in $\mathbb{D}$.

It is well known that every function $f \in \mathscr{S}$ has an inverse f^{-1} defined by

$$\begin{aligned} &f^{-1}(f(z)) = z, \quad z \in \mathbb{D}, \\ &f^{-1}(f(w)) = w, \quad |w| < r_0(f);\ r_0(f) \geq \frac{1}{4}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} f^{-1}(w) = g(w) = w - b_2 w^2 + \left(2b_2^2 - b_3\right)w^3 \\ - \left(5b_2^3 - 5b_2 b_3 + b_4\right)w^4 + \cdots. \end{aligned} \tag{5}$$

A function is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$.

Let Σ denote the class of bi-univalent function in $\mathbb{D}$ given by (3). Example of functions in the class Σ is

$$\frac{z}{1-z},\ \log\frac{1}{1-z}\text{ and }\log\sqrt{\frac{1+z}{1-z}}. \tag{6}$$

However, the familiar Koebe function is an example of the class Σ. Other common examples of functions in $\mathcal{S}$, such as

$$\frac{2z-z^2}{2}\text{ and }\frac{z}{1-z^2}, \tag{7}$$

are also not members of Σ.

Lewin [1] investigated a bi-univalent functions class Σ and showed that $|b_2|<1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|b_2|<\sqrt{2}$. Netanyahu [3], on the contrary, showed that

$$\max_{f\in\Sigma}|b_2|=\frac{4}{3}. \tag{8}$$

The coefficient for each of the Taylor–Maclaurin coefficients $|a_2|$ $(n\geq 3,\ n\in\mathbb{N})$ is presumably still an open problem.

Similar to the familiar subclasses $\mathcal{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ of star-like and convex functions of order $\zeta\,(0\leq\zeta<1)$, respectively, Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ, namely, $\mathcal{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of bi-star-like functions and bi-convex functions of order $\zeta\,(0\leq\zeta<1)$, respectively. For each of the function classes $\mathcal{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$, they found nonsharp bounds on the first two Taylor–Maclaurin coefficients $|b_2|$ and $|b_3|$.

Furthermore, let s_1 and s_2 be analytic functions in open unit disc $\mathbb{D}$; then, the function s_1 is subordinated to s_2 and symbolically denoted as

$$s_1(z)\prec s_2(z),\quad z\in\mathbb{D}, \tag{9}$$

if there occurs an analytic function w with properties that

$$w(0)=0\text{ and }|w(z)|<1. \tag{10}$$

Suppose w holomorphic in $\mathbb{D}$, such that

$$s_1(z)=s_2(w(z)). \tag{11}$$

If the function s_2 is univalent in $\mathbb{D}$, then the above condition is equivalent to

$$s_1(z)\prec s_2(z)\Longleftrightarrow s_1(0)=s_2(0)\text{ and }s_1(\mathbb{D})\subset s_2(\mathbb{D}). \tag{12}$$

Jackson [5] introduced and studied the q-derivative operator $\mathfrak{D}_q$ of a function as follows:

$$\mathfrak{D}_q f(z)=\frac{f(z)-f(qz)}{z(1-q)}=\frac{1}{z}\left\{z+\sum_{k=2}^{\infty}\left(\frac{1-q^k}{1-q}\right)a_k z^k\right\} \tag{13}$$

and $\mathfrak{D}_q f(0)=f'(0)$. In case $f(z)=z^k$, for k is a positive integer, the q-derivative of f is given by

$$\mathfrak{D}_q z^k=\frac{(zq)^k-z^k}{z(q-1)}=\left(\frac{1-q^k}{1-q}\right)z^{k-1}, \tag{14}$$

$$\lim_{q\longrightarrow 1-}[k]_q=\lim_{q\longrightarrow 1-}\frac{1-q^k}{1-q}=k, \tag{15}$$

where $(z\neq 0,\ q\neq 0)$. For more details on the concepts of q-derivative, see [6, 7].

The quantum (or q-) calculus is an essential tool for studying diverse families of analytic functions, and its applications in mathematics and related fields have inspired researchers. Srivastava [8] was the first person to apply it in the context of univalent functions. Numerous scholars conducted substantial work on q-calculus and examined its various applications due to the applicability of q-analysis in mathematics and other domains. For example, with the help of certain higher-order q -derivative operators, Khan et al. [7] defined and studied a number of subclasses of q-star-like functions. Also, Shi et al. [9] (see also [10]) used the q-differential operator and defined a new subclass of Janowski-type multivalent q-star-like functions. In [7, 9], a number of sufficient conditions and some other interesting properties have been examined. More importantly, the convolution theory enables us to investigate various properties of analytic functions. Due to the large range of applications of q-calculus and the importance of q-operators instead of regular operators, many researchers have explored q-calculus in depth. In addition, Srivastava [11] recently published survey-cum-expository review paper which is useful for researchers and scholars (see, for example, [12, 13]) working on these subjects. Also, Srivastava's recent survey-cum-expository review article [11] further motivates the use of the q-analysis in geometric function theory, as well as commenting on the triviality of the so-called $(\mathfrak{p},q)$-analysis involving an insignificant and redundant parameter $(\mathfrak{p},q)$ (see p. 340 of [11]).

Utilizing the idea of q-derivative operator, in 2013, Brahim et al. introduced and studied the symmetric q-derivative operator $(\widetilde{\mathfrak{D}}_q f)$ for a function f as follows:

$$\left(\widetilde{\mathfrak{D}}_q f\right)(z)=\begin{cases}\dfrac{f(qz)-f\left(q^{-1}z\right)}{\left(q-q^{-1}\right)z}, & (z\neq 0),\\[2ex] f'(0), & (z=0).\end{cases} \tag{16}$$

It is easy to see that

$$\begin{aligned}\widetilde{\mathfrak{D}}_q z^k&=\widetilde{[k]}_q z^{k-1},\\ \widetilde{\mathfrak{D}}_q f(z)&=1+\sum_{k=2}^{\infty}\widetilde{[k]}_q b_k z^{k-1},\end{aligned} \tag{17}$$

where

$$\widetilde{[k]}_q=\frac{q^k-q^{-k}}{q-q^{-1}}. \tag{18}$$

The relation between q-derivative operator and symmetric q-derivative operator is given by

$$\left(\widetilde{\mathfrak{D}}_q f\right)(z) = \mathfrak{D}_{q^2} f\left(q^{-1}z\right). \tag{19}$$

Suppose f^{-1} is the inverse of f; then,

$$\begin{aligned}\left(\widetilde{\mathfrak{D}}_q f^{-1}\right)(w) = 1 - \widetilde{[2]}_q b_2 w + \widetilde{[3]}_q\left(2b_2^2 - b_3\right)w^2 \\ - \widetilde{[4]}_q\left(5b_2^3 - 5b_2b_3 + b_4\right)w^3 + \cdots.\end{aligned} \tag{20}$$

Al Salam and Ismail [14] discovered a family of polynomials that can be understood as q-analogues of the second-order bivariate Chebyshev polynomials. In 2012, Johann Cigler introduced and studied the q-Chebyshev polynomials as follows.

Definition 1 (see [15]). The polynomials

$$\begin{aligned}U_m(t,y,q) &= P_{m+1}(t,-1,y,q)(-q;q)_m \\ &= \sum_{k=0}^{(n/2)} q^{k^2}\begin{bmatrix} m-k \\ k\end{bmatrix}\left(1+q^{k+1}\right)\cdots\left(1+q^{m-k}\right)y^k t^{m-2k}\end{aligned} \tag{21}$$

are called q-Chebyshev polynomial of the second kind.

Theorem 1 (see [15]). *The q-Chebyshev polynomials of the second kind satisfy*

$$U_m(t,y,q) = \left(1+q^m\right)tU_{m-1}(t,y,q) + q^{m-1}yU_{m-2}(t,y,q), \tag{22}$$

with initial values

$$U_0(t,y,q) = 1 \text{ and } U_1(t,y,q) = (1+q)t. \tag{23}$$

Remark 1. It is clear that

$$U_m(t,-1,1) = U_m(t), \tag{24}$$

where $U_m(t)$ is the classical Chebyshev polynomial of the second kind.

Now, making use q-Chebyshev polynomials, we define the following.

Definition 2. Let $\mathfrak{M}(z,t,y,q)$ be defined as follows:

$$\mathfrak{M}(z,t,y,q) = 1 + \sum_{j=1}^{\infty} U_j(t,y,q)z^j. \tag{25}$$

By using the principal of subordination and the symmetric q-derivative operator $\widetilde{\mathfrak{D}}_q$, we define the following subclasses of analytic and bi-univalent functions.

Definition 3. A function $f \in \Sigma$ given by (3) is said to be in the class $\widetilde{M}_{\Sigma}^{q,y}(t)$ if the following conditions are satisfied:

$$\left(\widetilde{\mathfrak{D}}_q f(z)\right) \prec \mathfrak{M}(z,t,y,q), \quad \frac{1}{2} < t < 1,\ 0 < q < 1,\ z \in \mathbb{D}, \tag{26}$$

$$\left(\widetilde{\mathfrak{D}}_q f^{-1}(w)\right) \prec \mathfrak{M}(w,t,y,q), \quad \frac{1}{2} < t < 1,\ 0 < q < 1,\ w \in \mathbb{D}. \tag{27}$$

We note from (25) that

$$\begin{aligned}\mathfrak{M}(z,t,y,q) = 1 + U_1(t,y,q)z + U_2(t,y,q)z^2 \\ + U_3(t,y,q)z^3 + \cdots,\end{aligned} \tag{28}$$

where $z \in \mathbb{D}$ and $t \in (-1,1)$.

Also, from (22), we have the following:

$$\begin{bmatrix} U_1(t,y,q) = (1+q)t \\ U_2(t,y,q) = t^2(1+q)\left(1+q^2\right) + qy \\ U_3(t,y,q) = (1+q)\left(1+q^2\right)\left(1+q^3\right)t^3 + q(1+q)\left(1+q^2\right)yt \\ U_4(t,y,q) = (1+q)\left(1+q^2\right)\left(1+q^3\right)\left(1+q^4\right)t^4 + q(1+q)\left(1+q^2\right)\left(1+q^4+q^2\right)y^2t + q^4y \end{bmatrix}. \tag{29}$$

The goal of this research is to investigate q-Chebyshev polynomial expansions in order to derive initial coefficient estimates for some subclasses of analytic and bi-univalent functions defined by the symmetric q-derivative operator. In addition, Fekete–Szegö inequalities for the class $\widetilde{M}_{\Sigma}^{q,y}(t)$ are established.

Lemma 1 (see [16]). *Let the function* p *be given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \tag{30}$$

be in the class D *of functions with positive real part. Then,*

$$\left|p_n\right| \leq 2, \quad n \in \mathbb{N}. \tag{31}$$

This last inequality is sharp.

2. Coefficients Bounds for $f \in \widetilde{M}_{\Sigma}^{q,y}(t)$

Theorem 2. *Let* $f \in \widetilde{M}_{\Sigma}^{q,y}(t)$. *Then,*

$$\left|b_2\right| \leq \frac{(1+q)t\sqrt{(1+q)t}}{\sqrt{\left|(1+q)t^2\left[\widetilde{[3]}_q(1+q)-\left(1+q^2\right)\widetilde{[2]}_q^2\right]-qy\widetilde{[2]}_q^2+(1+q)+\widetilde{[2]}_q^2\right|}}, \tag{32}$$

$$\left|b_3\right| \leq \frac{(1+q)^2t^2}{\widetilde{[2]}_q^2}+\frac{(1+q)t}{\widetilde{[3]}_q}, \tag{33}$$

$$\begin{aligned}\left|b_4\right| \leq & \frac{5(1+q)^2t^2}{2\widetilde{[2]}_q\widetilde{[3]}_q}+\frac{(1+q)t}{\widetilde{[4]}_q}+\frac{2t(1+q)\left[t\left(1+q^2\right)-1\right]+2qy}{\widetilde{[4]}_q}\\ & +\frac{(1+q)t\left[1-2t\left(1+q^2\right)+\left(1+q^2\right)\left(1+q^3\right)t^2+q\left(1+q^2\right)y\right]-2qy}{\widetilde{[4]}_q}.\end{aligned} \tag{34}$$

Proof. Let $f \in \sigma$ given by (3) be in the class $\widetilde{M}_{\Sigma}^{q,y}(t)$. Then,

$$\left(\widetilde{\mathfrak{D}}_q f(z)\right) = \mathfrak{M}(\omega(z), t, y, q), \tag{35}$$

$$\left(\widetilde{\mathfrak{D}}_q f^{-1}(w)\right) = \mathfrak{M}(\varpi(w), t, y, q). \tag{36}$$

Let $p, y \in \mathrm{D}$ be defined as

$$\begin{aligned} p(z) &= \frac{1+\omega(z)}{1-\omega(z)} = 1+p_1z+p_2z^2+p_3z^3+\cdots\\ &\Longrightarrow \omega(z) = \frac{p(z)-1}{p(z)+1}, \quad z\in\mathbb{D}.\end{aligned} \tag{37}$$

$$\begin{aligned} y(w) &= \frac{1+\varpi(w)}{1-\varpi(w)} = 1+y_1w+y_2w^2+y_3w^3+\cdots\\ &\Longrightarrow \varpi(w) = \frac{y(w)-1}{y(w)+1}, \quad w\in\mathbb{D}.\end{aligned} \tag{38}$$

It follows that, from (37) and (38),

$$\omega(z) = \frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+\cdots\right], \tag{39}$$

$$\varpi(w) = \frac{1}{2}\left[y_1w+\left(y_2-\frac{y_1^2}{2}\right)w^2+\left(y_3-y_1y_2+\frac{y_1^3}{4}\right)w^3+\cdots\right]. \tag{40}$$

From (39) and (40), applying $\mathfrak{M}(z,t,y,q)$ as given in (25), we see that

$$\begin{aligned}\mathfrak{M}(\omega(z),t,y,q) = {} & 1+\frac{U_1(t,y,q)}{2}p_1z\\ & +\left[\frac{U_1(t,y,q)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t,y,q)}{4}p_1^2\right]z^2\\ & +\left[\frac{U_1(t,y,q)}{2}\cdot\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{U_2(t,y,q)}{2}p_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_3(t,y,q)}{8}p_1^3\right]z^3+\cdots,\end{aligned} \tag{41}$$

$$\begin{aligned}\mathfrak{M}(\varpi(w),t,y,q) = {} & 1+\frac{U_1(t,y,q)}{2}y_1w+\left[\frac{U_1(t,y,q)}{2}\left(y_2-\frac{y_1^2}{2}\right)+\frac{U_2(t,y,q)}{4}y_1^2\right]w^2\\ & +\left[\frac{U_1(t,y,q)}{2}\left(y_3-y_1y_2+\frac{y_1^3}{4}\right)+\frac{U_2(t,y,q)}{2}y_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{U_3(t,y,q)}{8}y_1^3\right]w^3+\cdots.\end{aligned} \tag{42}$$

From (35), (41) and (36), (42), we have

$$\widetilde{[2]}_q b_2 = \frac{U_1(t,y,q)}{2} p_1, \quad (43)$$

$$\widetilde{[3]}_q b_3 = \frac{U_1(t,y,q)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_2(t,y,q)}{4} p_1^2, \quad (44)$$

$$\widetilde{[4]}_q b_4 = \frac{U_1(t,y,q)}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{U_2(t,y,q)}{2} p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_3(t,y,q)}{8} p_1^3, \quad (45)$$

$$-\widetilde{[2]}_q b_2 = \frac{U_1(t,y,q)}{2} y_1, \quad (46)$$

$$\widetilde{[3]}_q\left(2b_2^2 - b_3\right) = \frac{U_1(t,y,q)}{2}\left(y_2 - \frac{y_1^2}{2}\right) + \frac{U_2(t,y,q)}{4} y_1^2, \quad (47)$$

$$-\widetilde{[4]}_q\left(5b_2^3 - 5b_2 b_3 + b_4\right) = \frac{U_1(t,y,q)}{2}\left(y_3 - y_1 y_2 + \frac{y_1^3}{4}\right) + \frac{U_2(t,y,q)}{2} y_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{U_3(t,y,q)}{8} y_1^3. \quad (48)$$

Adding (43) and (46), we have

$$p_1 = -y_1,\ p_1^2 = y_1^2 \text{ and } p_1^3 = -y_1^3, \quad (49)$$

$$b_2^2 = \frac{U_1^2(t,y,q)\left(p_1^2 + y_1^2\right)}{8\widetilde{[2]}_q^2}. \quad (50)$$

Also, adding (44) and (47) and applying (49) yields

$$2\widetilde{[3]}_q b_2^2 = \frac{U_1(t,y,q)}{2}(p_2 + y_2) - y_1^2\left(U_1(t,y,q) - U_2(t,y,q)\right). \quad (51)$$

Applying (49) in (50) gives

$$y_1^2 = \frac{4\widetilde{[2]}_q^2 b_2^2}{U_1^2(t,y,q)}. \quad (52)$$

Putting (52) into (51) with some calculations, we have

$$|b_2|^2 = \left|\frac{U_1^3(t,y,q)(p_2 + y_2)}{4\left[\widetilde{[3]}_q U_1^2(t,y,q) - \left(U_2(t,y,q) - U_1(t,y,q)\right)\widetilde{[2]}_q^2\right]}\right|. \quad (53)$$

Applying triangular inequality and Lemma 1, we have

$$|b_2| \le \frac{(1+q)t\sqrt{(1+q)t}}{\sqrt{\left|(1+q)t^2\left[\widetilde{[3]}_q(1+q) - \left(1+q^2\right)\widetilde{[2]}_q^2\right] - qy\widetilde{[2]}_q^2 + (1+q) + \widetilde{[2]}_q^2\right|}}. \quad (54)$$

Subtracting (47) from (44) with some calculations, we have

$$b_3 = b_2^2 + \frac{U_1(t,y,q)\left[p_2 - y_2\right]}{4\widetilde{[3]}_q}, \quad (55)$$

$$b_3 = \frac{U_1^2(t,y,q)p_1^2}{4\widetilde{[2]}_q^2} + \frac{U_1(t,y,q)\left[p_2 - y_2\right]}{4\widetilde{[3]}_q}. \quad (56)$$

Applying triangular inequality and Lemma 1, we have

$$|b_3| \le \frac{(1+q)^2 t^2}{\widetilde{[2]}_q^2} + \frac{(1+q)t}{\widetilde{[3]}_q}. \quad (57)$$

Subtracting (48) from (45), we have

$$2\widetilde{[4]}_q b_4 = \frac{5\widetilde{[4]}_q U_1^2(t,y,q)p_1(p_2 - y_2)}{8\widetilde{[2]}_q\widetilde{[3]}_q} + \frac{U_1(t,y,q)(p_3 - y_3)}{2} + \frac{\left[U_2(t,y,q) - U_1(t,y,q)\right]p_1(p_2 + y_2)}{2} + \frac{\left(U_1(t,y,q) - 2U_2(t,y,q) + U_3(t,y,q)\right)p_1^3}{4}. \quad (58)$$

Applying triangular inequality and Lemma 1, we have

$$|b_4| \le \frac{5(1+q)^2 t^2}{2\widetilde{[2]}_q\widetilde{[3]}_q} + \frac{(1+q)t}{\widetilde{[4]}_q} + \frac{2t(1+q)\left[t\left(1+q^2\right) - 1\right] + 2qy}{\widetilde{[4]}_q} + \frac{(1+q)t\left[1 - 2t\left(1+q^2\right) + \left(1+q^2\right)\left(1+q^3\right)t^2 + q\left(1+q^2\right)y\right] - 2qy}{\widetilde{[4]}_q}. \quad (59)$$

□

3. Fekete–Szegö Inequalities for the Function Class $\widetilde{M}_{\Sigma}^{q,y}(t)$

The *nth* coefficient of a function class $\mathscr{S}$ is well known to be restricted by n, and the coefficient limits give information about the functions geometric characteristics. The famous problem solved by Fekete–Szegö [17] is to determine the greatest value of the coefficient functional $\Omega_\sigma(f)$/coloneq$|b_3-\sigma b_2^2|$ over the class $\mathscr{S}$ for each $\sigma\in[0,1]$, which was demonstrated using the Loewner technique. In this section, we aim to determine the upper bounds of the coefficient functional $|b_3-\delta b_2^2|$ for the function class $\widetilde{M}_{\Sigma}^{q,y}(t)$.

Theorem 3. *Let $f\in\widetilde{M}_{\Sigma}^{q,y}(t)$. Then, for some $\delta\in\mathbb{R}$,*

$$\left|b_3-\delta b_2^2\right|\le\begin{cases}\dfrac{(1+q)t}{\widetilde{[3]_q}}, & |\delta-1|\le\dfrac{\Lambda_q\left(q^{-1},y,t\right)}{\widetilde{[3]_q}(1+q)^2t^2},\\[2ex] \dfrac{(1+q)^3t^3|\delta-1|}{\Lambda_q\left(q^{-1},y,t\right)}, & |\delta-1|\ge\dfrac{\Lambda_q\left(q^{-1},y,t\right)}{\widetilde{[3]_q}(1+q)^2t^2},\end{cases}\tag{60}$$

where

$$\Lambda_q\left(q^{-1},y,t\right)=(1+q)t^2\left[\widetilde{[3]_q^2}(1+q)-\left(1+q^2\right)\widetilde{[2]_q^2}\right.\\ \left.-qy\widetilde{[2]_q^2}+(1+q)t\widetilde{[2]_q^2}\right].\tag{61}$$

Proof. From (51) and (55), we have

$$\begin{aligned}b_3-\delta b_2^2&=\frac{(1-\delta)U_1^3(t,y,q)\left(p_2+y_2\right)}{4\left[\widetilde{[3]_q}U_1^2(t,y,q)-\left(U_2(t,y,q)-U_1(t,y,q)\right)\widetilde{[2]_q^2}\right]}\\&\quad+\frac{U_1(t,y,q)\left[p_2-y_2\right]}{4\widetilde{[3]_q}}\\&=U_1(t,y,q)\left[\left(J(\delta)+\frac{1}{4\widetilde{[3]_q}}\right)p_2+\left(J(\delta)-\frac{1}{4\widetilde{[3]_q}}\right)y_2\right],\end{aligned}\tag{62}$$

where

$$J(\delta)=\frac{(1-\delta)U_1^2(t,y,q)}{4\left[\widetilde{[3]_q}U_1^2(t,y,q)-\left(U_2(t,y,q)-U_1(t,y,q)\right)\widetilde{[2]_q^2}\right]}.\tag{63}$$

Applying Lemma 1, we have

$$H_2(2)=\left|b_2b_4-b_3^2\right|\le\begin{cases}\dfrac{(1+q)t}{\widetilde{[3]_q}}, & 0\le|J(\delta)|\le\dfrac{1}{4\widetilde{[3]_q}},\\[2ex] 4(1+q)t|J(\delta)|, & |J(\delta)|\ge\dfrac{1}{4\widetilde{[3]_q}}.\end{cases}\tag{64}$$

□

Remark 2. Taking $q=1$ and $y=-1$ in Theorem 2 and Theorem 3, we have the results obtained by Altinkaya and Yalcin [18].

4. Conclusion

Recently, the q-derivative and symmetric q-derivative operators are particularly applicable in many diverse areas of mathematics and physics. In this study, firstly, many known concepts of the q-derivative operator have been highlighted and given. We have then used the symmetric q-derivative operator and certain q-Chebyshev Polynomials and have defined a new subclass of analytic and bi-univalent functions. For these newly defined functions' classes, a number of coefficients bonds, along with the Fekete–Szeg ö inequalities, have also been given. To validate our results, we have given some known consequence in the form of Remarks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors jointly worked on the results, and they have read and approved the final manuscript.

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