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COEFFICIENT BOUNDS FOR A CERTAIN FAMILIES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH Q-ANALOGUE OF WANAS OPERATOR

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Abstract. The motivation of the present paper is to define q-analogue of Wanas operator in geometric function theory. We also introduce certain families $\mathcal{T}^{\sigma,\alpha}_{\Sigma_m}(t,n,\beta,q,\delta)$ and $\mathcal{T}^{\sigma,\alpha}_{\Sigma_m}(t,n,\beta,q,\gamma)$ of holormorphic and m-fold symmetric bi-univalent functions associated with q-analogue of Wanas operator. The upper bounds for the second and third Taylor-Maclaurin coefficients for functions in each of these subfamilies are obtained. Furthermore, Several consequences of our results are pointed out based on the various special choices of the involved parameters.

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1. Introduction and Definitions

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane and let $\mathcal{A} = \{f : \mathbb{U} \to \mathbb{C} : f \text{ is holormorphic in } \mathbb{U}, f(0) = 0 = f^{'}(0) - 1\}$ be the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

Assume that $\mathcal{S}$ be the subfamily of $\mathcal{A}$ consisting of all functions f univalent in $\mathbb{U}$.

The Koebe on-quarter theorem (see [5]) state that the image of $\mathbb{U}$ under every function $f(z) \in \mathcal{S}$ contains a disk of radius 1/4. Therefore, all function $f(z) \in \mathcal{S}$ has an inverse $f^{-1}(z)$ which satisfies $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ $(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ denoted by Σ is said to be bi-univalent in $\mathbb{U}$ if both $f^{-1}(z)$ and $f(z)$ are univalent in $\mathbb{U}$ (see for details [3, 4, 7, 8, 12, 14, 16, 20, 21, 24, 27, 29, 32]).

For each function $f \in \mathcal{S}$, the function $h(z) = (f(z^m))^{1/m}$, $(z \in \mathbb{U},\ m \in \mathbb{N})$ is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [11] and [15]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}^+). \tag{3}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent function in $\mathbb{U}$, which are normalized by the series expansion (3). Also, the functions in the class $\mathcal{S}$ are one-fold symmetric.

Analogous to the concept of m-fold symmetric univalent function, here we introduced the concept of m-fold symmetric bi-univalent functions. From (3), Srivastava et al. [25] obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m-1}^2 - a_{2m+1}\right] w^{2m+1} \\ &- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots . \end{aligned} \tag{4}$$

where $f^{-1} = g$.

We denote by Σ_m the class of m-fold symmetric bi-univalent function in $\mathbb{U}$. We can note that for $m = 1$, the formular (4) coincides with the formular (2) of the class Σ. Some of the examples on m-fold symmetric bi-univalent functions are given as follows:

$$\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad [-\log(1-z^m)]^{\frac{1}{m}}, \quad \left\{\frac{z^m}{1-z^m}\right\}^{\frac{1}{m}},$$

with the corresponding inverse functions

$$\left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{1/m}, \quad \left(\frac{w^m}{1+w^m}\right)^{1/m} \quad \text{and} \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{1/m},$$

respectively. Recently, different researches related to this field investigated bounds for various subclasses of m-fold bi-univalent function (see [2, 6, 23, 26, 30]).

Jackson [9, 10] introduced the q-derivative operator $\mathcal{D}_q$ of a function as follows:

$$\mathcal{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \tag{5}$$

and $\mathcal{D}_q f(z) = f'(0)$. In case $f(z) = z^\phi$ for ϕ is a positive integer, the q-derivative of $f(z)$ is given by

$$\mathcal{D}_q z^\phi = \frac{z^\phi - (zq)^\phi}{(q-1)z} = [\phi]_q z^{\phi-1}.$$

As $q \longrightarrow 1^-$ and $\phi \in \mathbb{N}$, we get

$$[\phi]_q = \frac{1-q^\phi}{1-q} = 1 + q + \cdots + q^\phi \longrightarrow \phi \tag{6}$$

where $(z \neq 0,\ q \neq 0)$, for more details on the concepts of q-derivative (see [1, 13, 17, 22]).

Wanas [28] in 2019 introduced the following operator, which can also be called (Wanas operator) $\mathfrak{W}^{\alpha,\sigma}_{\beta,n} : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\mathfrak{W}^{\alpha,\sigma}_{\beta,n} = z + \sum_{j=2}^{\infty} [\Psi_j(\sigma, \alpha, \beta)]^n a_j z^j, \tag{7}$$

where

$$\Psi_j(\sigma, \alpha, \beta) = \sum_{c=1}^{\sigma} \binom{\sigma}{c} (-1)^{c+1} \left(\frac{\alpha^c + j\beta^c}{\alpha^c + \beta^c} \right), \tag{8}$$

$c, n \in \mathbb{N}_0,\ \beta \geqq 0,\ \alpha \in \mathcal{R}$ and $\alpha + \beta > 0$.

Special cases of this operator can be found in [31].

Now $q \longrightarrow 1^-$, $[\phi]_q \longrightarrow \phi$. For $f(z) \in \mathcal{A}$, we can define q-difference Wanas operator as given below

$$\begin{aligned}
W^{0,1}_{1,0,q} f(z) &= f(z) \\
W^{0,1}_{1,1,q} f(z) &= z\mathfrak{W}_q f(z) \\
W^{0,1}_{1,n,q} f(z) &= z\mathfrak{W}_q(\mathfrak{W}^{n-1}_q f(z)) \\
\mathfrak{W}^{\alpha,\sigma}_{\beta,n,q} f(z) &= z + \sum_{j=2}^{\infty} [\Psi_j(\sigma, \alpha, \beta)]^n_q a_j z^j
\end{aligned}$$

where

$$\Psi_j(\sigma, \alpha, \beta) = \sum_{c=1}^{\sigma} \binom{\sigma}{c} (-1)^{c+1} \left(\frac{\alpha^c + j\beta^c}{\alpha^c + \beta^c} \right), \tag{9}$$

$c, n \in \mathbb{N}_0,\ \beta \geqq 0,\ \alpha \in \mathcal{R},\ \alpha + \beta > 0,\ 0 < q < 1,\ z \in \mathbb{U}$.

Lemma 1. *Suppose $l(z) \in \mathcal{P}$, the class of functions which are holomorphic in $\mathbb{U}$ with $\Re(l(z)) > 0$, $(z \in \mathbb{U})$ and have the form $l(z) = 1 + l_1 z + l_2 z^2 + l_3 z^3 + \cdots$, $(z \in \mathbb{U})$; then $|l_n| \leq 2$ for each $n \in \mathbb{N}$.*

2. Coefficient estimates for the function class $\mathcal{T}_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\delta)$

Definition 1. *A function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{T}_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\delta)$ if it satisfies the following conditions:*

$$\left|\arg\left(\frac{\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma}f(z)}{\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma}f(z)}\right)\right| < \frac{\delta\pi}{2}, \tag{10}$$

$$\left|\arg\left(\frac{\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma}g(w)}{\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma}g(w)}\right)\right| < \frac{\alpha\pi}{2}, \tag{11}$$

where $0 < \delta \leq 1$, $n, t \in \mathbb{N}_0$, $t \geq n$ and the function $g = f^{-1}$ is given by (4). Also $\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma}f(z)$ and $\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma}f(z)$ are q-Wanas operators and have the following forms

$$\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma}f(z) = z + \sum_{j=1}^{\infty}[\Psi_{jm+1}(\sigma,\alpha,\beta)]_q^t a_{jm+1}z^{jm+1} \tag{12}$$

and

$$\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma}g(w) = w + \sum_{j=1}^{\infty}[\Psi_{jm+1}(\sigma,\alpha,\beta)]_q^n b_{jm+1}w^{jm+1}. \tag{13}$$

We state and prove the following results.

Theorem 2. *Let $f(z)$ given by (3) be in the class $\mathcal{T}_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\delta)$ $(0 < \delta \leq 1$, $n, t \in \mathbb{N}_0)$. Then*

$$|a_{m+1}| \leq \frac{2\delta}{\sqrt{\delta(m+1)\left([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n\right) - 2\delta\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t} - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n}\right) - (\delta-1)\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right)^2}} \tag{14}$$

and

$$|a_{2m+1}| \leq \frac{2\delta}{[\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n} + \frac{2(m+1)\delta^2}{([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n)^2}. \tag{15}$$

Proof. We can wirte the inequality in (10) and (11) as

$$\frac{\mathfrak{W}^{\alpha,\sigma}_{\beta,t,q}f(z)}{\mathfrak{W}^{\alpha,\sigma}_{\beta,n,q}f(z)} = [s(z)]^{\delta} \tag{16}$$

and

$$\frac{\mathfrak{W}^{\alpha,\sigma}_{\beta,t,q}g(w)}{\mathfrak{W}^{\alpha,\sigma}_{\beta,n,q}g(w)} = [t(w)]^{\delta} \tag{17}$$

respectively.

Where $g(w) = f^{-1}$ and $s(z)$,$t(w)$ in $\mathcal{P}$ have the following series representation:

$$s(z) = 1 + s_m z^m + s_{2m} z^{2m} + s_{3m} z^{3m} + \cdots \tag{18}$$

and

$$t(w) = 1 + t_m w^m + t_{2m} w^{2m} + t_{3m} w^{3m} + \cdots \tag{19}$$

Clearly,

$$[s(z)]^{\delta} = 1 + \delta s_m z^m + \left(\delta s_{2m} + \frac{\delta(\delta-1)}{2} s_m^2\right) z^{2m} + \cdots \tag{20}$$

and

$$[t(w)]^{\delta} = 1 + \delta t_m w^m + \left(\delta t_{2m} + \frac{\delta(\delta-1)}{2} t_m^2\right) w^{2m} + \cdots \tag{21}$$

Now equating the coefficient in (10) and (11) we get

$$\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right) a_{m+1} = \delta s_m, \tag{22}$$

$$\begin{aligned}&([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n)a_{2m+1}\\ &\quad - ([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t} - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n})a_{m+1}^2 = \delta s_{2m} + \frac{\delta(\delta-1)}{2} s_m^2,\end{aligned} \tag{23}$$

$$-\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right) a_{m+1} = \delta t_m, \tag{24}$$

$$\begin{aligned}&([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n)((m+1)a_{m+1}^2 - a_{2m+1})\\ &\quad - ([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t} - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n})a_{m+1}^2 = \delta t_{2m} + \frac{\delta(\delta-1)}{2} t_m^2.\end{aligned} \tag{25}$$

From equation (22) and (24), we find that

$$s_m = -t_m \tag{26}$$

and

$$2\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right)^2 a_{m+1}^2 = \delta^2(s_m^2 + t_m^2). \tag{27}$$

Also, from (23), (25) and (27), we have

$$(m+1)\delta([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n)a_{2m+1}^2 - 2\delta([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t}$$
$$- [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n})a_{m+1}^2 = \delta(s_{2m}+t_{2m}) + \frac{\delta(\delta-1)}{2}(t_m^2 + s_m^2) = \delta^2(s_{2m}+t_{2m})$$
$$+ (\delta-1)\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right)^2 a_{m+1}^2.$$

Therefore, after simplifying and using Lemma 1 for the coefficient s_{2m} and t_{2m}, we have (14).

For us to get te bound on $|a_{2m+1}|$, we subtract (25) from (23) to have

$$[\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n$$
$$\left(2a_{2m+1} - (m+1)a_{m+1}^2\right) = \alpha(s_{2m} - t_{2m}) + \frac{\alpha(\alpha-1)}{2}(t_m^2 - s_m^2). \tag{28}$$

It follows from (26), (27) and (28)

$$a_{2m+1} = \frac{\delta(s_{2m}-t_{2m})}{[\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n}$$
$$+ \frac{(m+1)\delta^2(t_m^2 - s_m^2)}{4([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n)^2}. \tag{29}$$

Taking the absolute value of (29) and using Lemma 1 for the coefficient s_m, s_{2m}, t_m and t_{2m}, we have (15) which completes the proof of Theorem 2.

When $m = 1$ and $\sigma = \beta = 1$ which is the one-fold symmetric bi-univalent functions, Theorem 2 gives the following corollary:

Corollary 3. *Let $f(z)$ given by* (3) *be in the class $\mathcal{T}_\Sigma^\alpha(t,n,1,q,\delta)$ $(0 < \delta \leq 1$, $n,t \in \mathbb{N}_0$, $\alpha > -1)$. Then*

$$|a_2| \leq \frac{2\delta}{\sqrt{2\delta\left(\left[\frac{2\alpha+3}{\alpha+1}\right]_q^t - \left[\frac{2\alpha+3}{\alpha+1}\right]_q^n\right) - 2\delta([2]_q^{n+t} - [2]_q^{2n}) - (1-\delta)([2]_q^t - [2]_q^n)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2\delta}{\left[\frac{2\alpha+3}{\alpha+1}\right]_q^t - \left[\frac{2\alpha+3}{\alpha+1}\right]_q^n} + \frac{4\delta^2}{([2]_q^t - [2]_q^n)^2}.$$

When $m = \sigma = 1$ and $\alpha = 1 - \beta$ which is the one-fold symmetric bi-univalent functions, Theorem 2 gives the following corollary:

Corollary 4. *Let $f(z)$ given by (3) be in the class $\mathcal{T}_{\Sigma}^{1-\beta}(t, n, q, \delta)$ $(0 < \delta \leq 1$, $n, t \in \mathbb{N}_0)$. Then*

$$|a_2| \leq \frac{2\delta}{\sqrt{2\delta\left([2+\beta]_q^t - [2+\beta]_q^n\right) - 2\delta([2]_q^{n+t} - [2]_q^{2n}) - (1-\delta)([2]_q^t - [2]_q^n)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2\delta}{[2+\beta]_q^t - [2+\beta]_q^n} + \frac{4\delta^2}{([2]_q^t - [2]_q^n)^2}.$$

Remark 1. *In Theorem 2, if we choose*

1. *$q = 1$, $\sigma = \beta = 1$ and $\alpha = 0$ then we have results determined by Seker and Taymur [[18], Theorem 2].*
2. *$m = q = 1$, $\sigma = \beta = t = 1$ and $\alpha = n = 0$ then we have results determined by Brannan and Taha [[3], Theorem 2].*
3. *$m = q = 1$, $\sigma = \beta = 1$ and $\alpha = 0$ then we have results determined by Seker [[19], Theorem 2].*

3. Coefficient estimates for the function class $\mathcal{T}_{\Sigma_m}^{\sigma,\alpha}(t, n, \beta, q, \gamma)$

Definition 2. *A function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{T}_{\Sigma_m}^{\sigma,\alpha}(t, n, \beta, q, \gamma)$ if it satisfies the following conditions:*

$$\Re\left\{\frac{\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma} f(z)}{\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma} f(z)}\right\} > \gamma, \tag{30}$$

$$\Re\left\{\frac{\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma} g(w)}{\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma} g(w)}\right\} > \gamma, \tag{31}$$

where $0 \leq \gamma < 1$, $n, t \in \mathbb{N}_0$, $t \geq n$ and the function $g = f^{-1}$ is given by (4).

We state and prove the following results.

Theorem 5. *Let $f(z)$ given by (3) be in the class $\mathcal{T}_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\gamma)$ $(0 \leq \gamma < 1$, $n,t \in \mathbb{N}_0)$. Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{1-\gamma}{(m+1)\left([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n\right) - 2\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t} - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n}\right)}} \tag{32}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\gamma)}{[\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n} + \frac{(m+1)(1-\gamma)^2}{([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n)^2}. \tag{33}$$

Proof. First of all, the argument inequality in (30) and (31) can be written in their equivalent forms as:

$$\frac{\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma} f(z)}{\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma} f(z)} = \gamma + (1-\gamma)s(z) \tag{34}$$

and

$$\frac{\mathfrak{W}_{\beta,t,q}^{\alpha,\sigma} g(w)}{\mathfrak{W}_{\beta,n,q}^{\alpha,\sigma} g(w)} = \gamma + (1-\gamma)t(w). \tag{35}$$

respectively. Where $s(z)$, $t(w) \in \mathcal{P}$ and have the forms

$$s(z) = 1 + s_m z^m + s_{2m} z^{2m} + s_{3m} z^{3m} + \cdots \tag{36}$$

and

$$t(w) = 1 + t_m w^m + t_{2m} w^{2m} + t_{3m} w^{3m} + \cdots \tag{37}$$

Clearly,

$$\gamma + (1-\beta\gamma)s(z) = 1 + (1-\gamma)s_m z^m + (1-\gamma)s_{2m} z^{2m} + \cdots \tag{38}$$

and

$$\gamma + (1-\gamma)t(w) = 1 + (1-\gamma)t_m w^m + (1-\gamma)t_{2m} w^{2m} + \cdots. \tag{39}$$

Now equating the coefficient in (34) and (35), we get

$$\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right) a_{m+1} = (1-\gamma)s_m, \tag{40}$$

$$\begin{aligned}([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n)a_{2m+1}\\ - ([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t} - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n})a_{m+1}^2 = (1-\gamma)s_{2m},\end{aligned} \tag{41}$$

$$-\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right) a_{m+1} = (1-\gamma)t_m, \tag{42}$$

$$\begin{aligned}([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n)((m+1)a_{m+1}^2 - a_{2m+1})\\ - ([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t} - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n})a_{m+1}^2 = (1-\gamma)t_{2m}.\end{aligned} \tag{43}$$

From (40) and (42), we get

$$s_m = -t_m \tag{44}$$

and

$$2\left([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^n\right)^2 a_{m+1}^2 = (1-\gamma)^2(s_m^2 + t_m^2). \tag{45}$$

Also, adding (41) and (43), we have

$$\begin{aligned}(m+1)([\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n)a_{2m+1}^2 - 2([\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{n+t}\\ - [\Psi_{m+1}(\sigma,\alpha,\beta)]_q^{2n})a_{m+1}^2 = (1-\gamma)(s_{2m}+t_{2m})\end{aligned}$$

Therefore, after simplifying and applying Lemma 1 for the coefficient s_{2m} and t_{2m}, we obtain (32).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (43) from (41), we have

$$\begin{aligned}[\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}(\sigma,\alpha,\beta)]_q^n\\ \left(2a_{2m+1} - (m+1)a_{m+1}^2\right) = (1-\gamma)(t_{2m} - s_{2m}).\end{aligned} \tag{46}$$

Applying (45) and Lemma 1 once again for coefficients s_m, s_{2m}, t_m and t_{2m}, we have (33) which completes the proof of Theorem 5.

When $m = 1$ and $\sigma = \beta = 1$ which is the one-fold symmetric bi-univalent functions, Theorem 5 gives the following corollary:

Corollary 6. *Let $f(z)$ given by (3) be in the class $\mathcal{T}_{\Sigma}^{\alpha}(t,n,q,\gamma)$ $(0 \leq \gamma < 1,\ n,t \in \mathbb{N}_0,\ \alpha > -1)$. Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{1-\gamma}{2\left(\left[\frac{2\alpha+3}{\alpha+1}\right]_q^t - \left[\frac{2\alpha+3}{\alpha+1}\right]_q^n\right) - 2\left([2]_q^{n+t} - [2]_q^{2n}\right)}} \tag{47}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\gamma)}{\left[\frac{2\alpha+3}{\alpha+1}\right]_q^t - \left[\frac{2\alpha+3}{\alpha+1}\right]_q^n} + \frac{2(1-\gamma)^2}{([2]_q^t - [2]_q^n)^2}. \tag{48}$$

When $m = \sigma = 1$ and $\alpha = 1-\beta$ which is the one-fold symmetric bi-univalent functions, Theorem 5 gives the following corollary:

Corollary 7. *Let $f(z)$ given by (3) be in the class $\mathcal{T}_{\Sigma}^{1-\beta}(t,n,q,\gamma)$ $(0 \leq \gamma < 1,\ n,t \in \mathbb{N}_0)$. Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{1-\gamma}{2\left([2+\beta]_q^t - [2+\beta]_q^n\right) - 2\left([2]_q^{n+t} - [2]_q^{2n}\right)}} \tag{49}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\gamma)}{[2+\beta]_q^t - [2+\beta]_q^n} + \frac{2(1-\gamma)^2}{([2]_q^t - [2]_q^n)^2}. \tag{50}$$

Remark 2. *In Theorem 5, if we choose*

1. *$q = 1$, $\sigma = \beta = 1$ and $\alpha = 0$ then we have results determined by Seker and Taymur [[18], Theorem 2].*
2. *$m = q = 1$, $\sigma = \beta = t = 1$ and $\alpha = n = 0$ then we have results determined by Brannan and Taha [[3], Theorem 2].*
3. *$m = q = 1$, $\sigma = \beta = 1$ and $\alpha = 0$ then we have results determined by Seker [[19], Theorem 2].*

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