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Coefficient estimates for a subclass of bi-univalent functions involving the q-derivatives operator [1]

Timilehin Gideon Shaba, Abbas Kareem Wanas

Abstract

In this current study, we introduce and investigate a new subclass of holormorphic and bi-univalent functions $\mathfrak{E}_q^{\eta,\phi}(\vartheta)$ in the unit disk $\leftthreetimes$ associated with q-derivative operator. The coefficient estimates $|b_2|$ and $|b_3|$ on the new subclass are obtained and important results are indicated.

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1 Introduction

Let $\mathfrak{A}$ be the class of all holomorphic functions of the form

$$\mathfrak{F}(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa \tag{1}$$

in the open unit disk $\leftthreetimes = \{z \in \mathfrak{C} : |z| < 1\}$.

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A function $\mathfrak{F}(z)$ which is analytic is said to be subordinate to an analytic function $\mathfrak{g}(z)$, denoted by $\mathfrak{F} \prec \mathfrak{g}$, provided there is an holomorphic (Schwarz) function $\mathfrak{w}$ with $\mathfrak{w}(0) = 0$, $|\mathfrak{w}| < 1$, for all $z \in \mathfrak{U}$ satisfying $\mathfrak{F} = \mathfrak{g}(\mathfrak{w}(z))$ for all $z \in \mathfrak{U}$.

The famous Koebe one-quarter theorem [12] ensure that the image of $\mathfrak{U}$ under every univalent function $\mathfrak{F} \in \mathfrak{A}$ contain a disk of radius $\frac{1}{4}$. Furthermore, all univalent function $\mathfrak{F}$ has an inverse $\mathfrak{F}^{-1}$ satisfying $\mathfrak{F}^{-1}(\mathfrak{F}(z)) = z$ and

$$\mathfrak{F}^{-1}(\mathfrak{F}(\mathfrak{w})) = \mathfrak{w}, \quad \left(|\mathfrak{w}| < r_0(\mathfrak{F}), r_0(\mathfrak{F}) \geq \frac{1}{4}\right)$$

where

$$\mathfrak{g}(\mathfrak{w}) = \mathfrak{F}^{-1}(\mathfrak{w}) = \mathfrak{w} - a_2\mathfrak{w}^2 + (2b_2^2 - b_3)\mathfrak{w}^3 - (5b_2^3 - 5b_2b_3 + b_4)\mathfrak{w}^4 + \cdots .$$

A function $\mathfrak{F} \in \mathfrak{A}$ denoted by $\mathfrak{E}$ is said to be bi-univalent in $\mathfrak{U}$ if both $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are univalent in $\mathfrak{U}$.

In [8], Ali et al introduced certain subclasses of the bi-univalent function class $\mathfrak{E}$ using subordination which widen the results of Brannan and Taha [10], from then on, numerous researchers as introduced subclasses of the bi-univalent function class $\mathfrak{E}$ and investigated the first two coefficients $|b_2|$ and $|b_3|$ of series expansion (1). Such as ([4, 15, 14, 11, 24, 21, 13, 22, 32, 20, 19, 23, 1, 29, 30, 31]).

Jackson [17, 18] introduced the q-derivative operator $\mathfrak{D}_q$ of a function as follows:

$$\mathfrak{D}_q\mathfrak{F}(z) = \frac{\mathfrak{F}(qz) - \mathfrak{F}(z)}{(q-1)z} \tag{2}$$

and $\mathfrak{D}_q\mathfrak{F}(z) = \mathfrak{F}'(0)$. In case $\mathfrak{F}(z) = z^\kappa$ for κ is a positive integer, the q-derivative of $\mathfrak{F}(z)$ is given by

$$\mathfrak{D}_q z^\kappa = \frac{z^\kappa - (zq)^\kappa}{(q-1)z} = [\kappa]_q z^{\kappa-1}.$$

As $q \longrightarrow 1^-$ and $\kappa \in \mathfrak{N}$, we get

$$[\kappa]_q = \frac{1-q^\kappa}{1-q} = 1 + q + \cdots + q^\kappa \longrightarrow \kappa \tag{3}$$

where $(z \neq 0,\ q \neq 0)$, for more details on the concepts of q-derivative (see [16, 3, 5, 6, 7, 26, 27, 9]).

Let $\vartheta(z)$ be an holomorphic function with positive real part in $\mathfrak{U}$ such that

$$\vartheta(0) = 1, \quad \vartheta'(0) > 0$$

and $\vartheta(\mathfrak{U})$ is symmetric with respect to real axis, which is of the form:

$$\vartheta(z) = 1 + \mathfrak{B}_1 z + \mathfrak{B}_2 z^2 + \mathfrak{B}_3 z^3 + \cdots \tag{4}$$

where $\mathfrak{B}_1 > 0$.

Using the q-derivative we now introduce the following subclass of holomorphic and bi-univalent functions.

Definition 1 *A function $\mathfrak{F} \in \mathfrak{E}$, $0 \leq \eta \leq 1$, $\phi \in \mathfrak{N}_0$ and $z, \mathfrak{w} \in \mathfrak{U}$ is said to be in the class $\mathfrak{E}_q^{\eta,\phi}(\vartheta)$ if each of the following subordination condition are fulfilled:*

$$(1-\eta)\left(\mathfrak{D}_q\mathfrak{F}(z)\right)^{\phi} + \eta\left(z(\mathfrak{D}_q\mathfrak{F}(z))' + \mathfrak{D}_q\mathfrak{F}(z)\right)(\mathfrak{D}_q\mathfrak{F}(z))^{\phi-1} \prec \vartheta(z) \tag{5}$$

and

$$(1-\eta)\left(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)^{\phi} + \eta\left(\mathfrak{w}(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))' + \mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))^{\phi-1} \prec \vartheta(\mathfrak{w}), \tag{6}$$

where $\mathfrak{g}(\mathfrak{w}) = \mathfrak{F}^{-1}(\mathfrak{w})$.

Remark 1 *Putting $q \longrightarrow 1^-$, $\eta = 0$, $\phi = 1$ and*

$$\vartheta(z) = \frac{1+(1-2\varphi)}{1-z} \quad (0 \leq \varphi < 1) \quad \text{and} \quad \vartheta(z) = \left(\frac{1+z}{1-z}\right)^{\rho} \quad (0 < \rho \leq 1),$$

the class $\mathfrak{E}_q^{\eta,\phi}(\vartheta)$ reduces to the classes $\mathfrak{H}_{\mathfrak{E}}^{\rho}$ and $\mathfrak{H}_{\mathfrak{E}}(\varphi)$ introduced by Srivastava et al. [28] which are subclasses of the functions $\mathfrak{F} \in \mathfrak{E}$ satisfying

$$|\arg(\mathfrak{D}_q\mathfrak{F}(z))| < \frac{\rho\pi}{2}, \quad |\arg(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))| < \frac{\rho\pi}{2}$$

and

$$\Re(\mathfrak{D}_q\mathfrak{F}(z)) > \varphi, \quad \Re(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})) > \varphi$$

respectively.

Remark 2 *Putting*

$$\vartheta(z) = \frac{1+(1-2\varphi)}{1-z} \quad (0 \leq \varphi < 1) \quad and \quad \vartheta(z) = \left(\frac{1+z}{1-z}\right)^{\rho} \quad (0 < \rho \leq 1),$$

the class $\mathfrak{E}_q^{\eta,\phi}(\vartheta)$ *reduces to the classes* $\mathfrak{H}_{\mathfrak{E}}^{\eta,\phi}(q;\rho)$ *and* $\mathfrak{H}_{\mathfrak{E}}^{\eta,\phi}(q;\varphi)$ *introduced by Akgul [2] which are subclasses of the functions* $\mathfrak{F} \in \mathfrak{E}$ *satisfying*

$$\left|\arg\left((1-\eta)\left(\mathfrak{D}_q\mathfrak{F}(z)\right)^{\phi} + \eta\left(z(\mathfrak{D}_q\mathfrak{F}(z))' + \mathfrak{D}_q\mathfrak{F}(z)\right)(\mathfrak{D}_q\mathfrak{F}(z))^{\phi-1}\right)\right| < \frac{\rho\pi}{2},$$

$$\left|\arg\left((1-\eta)\left(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)^{\phi} + \eta\left(\mathfrak{w}(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))' + \mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))^{\phi-1})\right)\right| < \frac{\rho\pi}{2}$$

and

$$\Re\left((1-\eta)\left(\mathfrak{D}_q\mathfrak{F}(z)\right)^{\phi} + \eta\left(z(\mathfrak{D}_q\mathfrak{F}(z))' + \mathfrak{D}_q\mathfrak{F}(z)\right)(\mathfrak{D}_q\mathfrak{F}(z))^{\phi-1}\right) > \varphi,$$

$$\Re\left((1-\eta)\left(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)^{\phi} + \eta\left(\mathfrak{w}(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))' + \mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))^{\phi-1})\right) > \varphi$$

respectively.

We shall need the following Lemma in our investigation.

Lemma 1 *[25] Let the function* $\mathfrak{p} \in \mathfrak{P}$ *be given by the following series:*

$$\mathfrak{p}(z) = 1 + \mathfrak{p}_1 z + \mathfrak{p}_2 z^2 + \cdots \quad (z \in \mathfrak{U}).$$

The sharp estimate given by

$$|\mathfrak{p}_n| \leq 2 \quad (n \in \mathfrak{N})$$

holds true.

The aim of the current paper is to find estimates on the coefficients $|b_2|$ and $|b_3|$ for function in this new subclass $\mathfrak{E}_q^{\eta,\phi}(\vartheta)$ of the function class $\mathfrak{E}$.

2 A Set of Main Results

Theorem 1 *Let* $\mathfrak{F} \in \mathfrak{E}_q^{\eta,\phi}(\vartheta)$ *be of the form* (1)*. Then*

$$(7)\quad |b_2| \leq \min\Bigg\{ \frac{\mathfrak{B}_1}{(\eta+\phi)[2]_q}, \frac{\sqrt{2}\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{2\mathfrak{B}_1^2(\phi+2\eta)[3]_q + \mathfrak{B}_1^2(\phi+2\eta)(\phi-1)[2]_q^2 + 2(\eta+\phi)^2[2]_q^2(\mathfrak{B}_1 - \mathfrak{B}_2)}} \Bigg\}$$

and

$$(8)\quad |b_3| \leq \min\Bigg\{ \frac{32(\phi+2\eta)(1-\phi)+8\mathfrak{B}_2}{8(\phi+2\eta)[3]_q}, \frac{\mathfrak{B}_1}{(\phi+2\eta)[3]_q} + \frac{\mathfrak{B}_1^2}{(\eta+\phi)^2[2]_q^2} \Bigg\},$$

where the coefficients $\mathfrak{B}_1$ *and* $\mathfrak{B}_1$ *are given as in* (4)*.*

Proof. Let $\mathfrak{F} \in \mathfrak{E}_q^{\eta,\phi}(\vartheta)$ and $\mathfrak{g} = \mathfrak{F}^{-1}$. Then there are holomorphic functions $s, t : \mathfrak{U} \longrightarrow \mathfrak{U}$ with $s(0) = t(0) = 0$, satisfying the following conditions:

(9)

$$(1-\eta)\Bigg(\mathfrak{D}_q\mathfrak{F}(z)\Bigg)^{\phi} + \eta\Bigg(z(\mathfrak{D}_q\mathfrak{F}(z))' + \mathfrak{D}_q\mathfrak{F}(z)\Bigg)(\mathfrak{D}_q\mathfrak{F}(z))^{\phi-1} = \vartheta(s(z)),\ z \in \mathfrak{U}$$

and

$$(10)\quad (1-\eta)\Bigg(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\Bigg)^{\phi} + \eta\Bigg(\mathfrak{w}(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))' + \mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\Bigg)(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))^{\phi-1} = \vartheta(t(\mathfrak{w})), \mathfrak{w} \in \mathfrak{U}.$$

Define the functions x and y by

$$(11)\qquad x(z) = \frac{1+s(z)}{1-s(z)} = 1 + x_1 z + x_2 z^2 + \cdots$$

and

$$(12)\qquad y(z) = \frac{1+t(z)}{1-t(z)} = 1 + y_1 z + y_2 z^2 + \cdots .$$

Then x and y are analytic in $\mathfrak{U}$ with $x(0) = y(0) = 1$. Since we have $s, t : \mathfrak{U} \longrightarrow \mathfrak{U}$, each of the functions x and y has a positive real part in $\mathfrak{U}$. So, in view of the above Lemma 1, we have

$$(13)\qquad |x_n| \leq 2 \quad \textit{and} \quad |y_n| \leq 2 \qquad (n \in \mathfrak{N}).$$

Solving for $s(z)$ and $t(z)$, we have

$$s(z)=\frac{x(z)-1}{x(z)+1}=\frac{1}{2}\left[x_1z+\left(x_2-\frac{x_1^2}{2}\right)z^2\right]+\cdots \quad (z\in\mathfrak{U}) \tag{14}$$

and

$$s(z)=\frac{y(z)-1}{y(z)+1}=\frac{1}{2}\left[y_1z+\left(y_2-\frac{y_1^2}{2}\right)z^2\right]+\cdots \quad (z\in\mathfrak{U}). \tag{15}$$

By substituting (14) and (15) into (9) and (10) and applying (4), we get

$$\begin{aligned}(16)\quad &(1-\eta)\left(\mathfrak{D}_q\mathfrak{F}(z)\right)^{\phi}+\eta\left(z(\mathfrak{D}_q\mathfrak{F}(z))'+\mathfrak{D}_q\mathfrak{F}(z)\right)(\mathfrak{D}_q\mathfrak{F}(z))^{\phi-1}\\ &=\vartheta\left(\frac{x(z)-1}{x(z)+1}\right)=1+\frac{1}{2}\mathfrak{B}_1x_1z+\left[\frac{1}{2}\mathfrak{B}_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2x_1^2\right]z^2+\cdots\end{aligned}$$

and

$$\begin{aligned}(17)\quad &(1-\eta)\left(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)^{\phi}+\eta\left(\mathfrak{w}(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))'+\mathfrak{D}_q\mathfrak{g}(\mathfrak{w})\right)(\mathfrak{D}_q\mathfrak{g}(\mathfrak{w}))^{\phi-1}\\ &=\vartheta\left(\frac{y(\mathfrak{w})-1}{y(\mathfrak{w})+1}\right)=1+\frac{1}{2}\mathfrak{B}_1y_1\mathfrak{w}+\left[\frac{1}{2}\mathfrak{B}_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2y_1^2\right]\mathfrak{w}^2+\cdots\end{aligned}$$

Equating the coefficients in (9) and (10), yields

$$(\eta+\phi)[2]_qb_2=\frac{1}{2}\mathfrak{B}_1x_1, \tag{18}$$

$$(\phi+2\eta)[3]_qb_3+\frac{(\phi+2\eta)(\phi-1)[2]_q^2}{2}b_2^2=\frac{1}{2}\mathfrak{B}_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2x_1^2, \tag{19}$$

$$-(\eta+\phi)[2]_qb_2=\frac{1}{2}\mathfrak{B}_1y_1 \tag{20}$$

and

$$(\phi+2\eta)[3]_q(2b_2^2-b_3)+\frac{(\phi+2\eta)(\phi-1)[2]_q^2}{2}b_2^2=\frac{1}{2}\mathfrak{B}_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2y_1^2. \tag{21}$$

From (18) and (20), we have

$$x_1=-y_1 \tag{22}$$

and

$$2(\eta+\phi)^2[2]_q^2 b_2^2 = \frac{1}{4}\mathfrak{B}_1^2(x_1^2+y_1^2). \tag{23}$$

Also from (19) and (21), we obtain

$$2(\phi+2\eta)[3]_q b_2^2 + (\phi+2\eta)(\phi-1)[2]_q^2 b_2^2 = \frac{1}{2}\mathfrak{B}_1\left[x_2+y_2-\left(\frac{x_1^2+y_1^2}{2}\right)\right]+\frac{1}{4}\mathfrak{B}_2[x_1^2+y_1^2] \tag{24}$$

by applying (23), we have

$$b_2^2 = \frac{\mathfrak{B}_1^3(x_2+y_2)}{2[2\mathfrak{B}_1^2(\phi+2\eta)[3]_q+\mathfrak{B}_1^2(\phi+2\eta)(\phi-1)[2]_q^2+2(\eta+\phi)^2[2]_q^2(\mathfrak{B}_1-\mathfrak{B}_2)]}. \tag{25}$$

Applying Lemma 1 for the coefficients x_1, x_2, y_1, y_2 in (23) and (25), we get

$$|b_2| \leq \frac{\sqrt{2}\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{2\mathfrak{B}_1^2(\phi+2\eta)[3]_q+\mathfrak{B}_1^2(\phi+2\eta)(\phi-1)[2]_q^2+2(\eta+\phi)^2[2]_q^2(\mathfrak{B}_1-\mathfrak{B}_2)}}, \tag{26}$$

$$|b_2| \leq \frac{\mathfrak{B}_1}{(\eta+\phi)[2]_q}, \tag{27}$$

which gives the estimates of $|b_2|$.

Furthermore, in order to find the bound on $|b_3|$, we subtract (21) from (19) and also applying (22), we obtain $x_1^2 = y_1^2$, hence

$$2(\phi+2\eta)[3]_q b_3 - 2(\phi+2\eta)[3]_q b_2^2 = \frac{1}{2}\mathfrak{B}_1(x_2-y_2), \tag{28}$$

then by substituting of the value of b_2^2 from (23) into (28), gives

$$b_3 = \frac{\mathfrak{B}_1(x_2-y_2)}{4(\phi+2\eta)[3]_q} + \frac{\mathfrak{B}_1^2(x_1^2+y_1^2)}{8(\eta+\phi)^2[2]_q^2}$$

So we have

$$|b_3| \leq \frac{\mathfrak{B}_1}{(\phi+2\eta)[3]_q} + \frac{\mathfrak{B}_1^2}{(\eta+\phi)^2[2]_q^2}$$

Also, substituting the value of b_2^2 from (24) into (28), we get

$$|b_3| = \frac{4(\phi+2\eta)\mathfrak{B}_1^2(x_1^2+y_1^2)(1-\phi)+4\mathfrak{B}_1 x_2+(x_1^2+y_1^2)(\mathfrak{B}_2-\mathfrak{B}_1)}{8(\phi+2\eta)[3]_q}$$

and we have

$$|b_3| \leq \frac{32(\phi+2\eta)(1-\phi)+8\mathfrak{B}_2}{8(\phi+2\eta)[3]_q}$$

which gives us the desired estimates on the coefficient $|b_3|$.

3 Corollaries and Consequences

Putting $q \longrightarrow 1^-$ in Theorem 1, we have the following corollary.

Corollary 1 *Let the function $\mathfrak{F}(z)$ given by (1) be in the class $\mathfrak{E}^{\eta,\phi}(\vartheta)$. Then*

$$|b_2| \le \min\left\{\frac{\mathfrak{B}_1}{2(\eta+\phi)}, \frac{\mathfrak{B}_1\sqrt{\mathfrak{B}_1}}{\sqrt{3\mathfrak{B}_1^2(\phi+2\eta)+2\mathfrak{B}_1^2(\phi+2\eta)(\phi-1)+2(\eta+\phi)^2(\mathfrak{B}_1-\mathfrak{B}_2)}}\right\}$$

and

$$|b_3| \le \min\left\{\frac{32(\phi+2\eta)(1-\phi)+8\mathfrak{B}_2}{24(\phi+2\eta)}, \frac{\mathfrak{B}_1}{3(\phi+2\eta)}+\frac{\mathfrak{B}_1^2}{4(\eta+\phi)^2}\right\}.$$

Taking $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\rho = 1+2\rho z+2\rho^2 z^2+\cdots \quad (0<\rho\le 1)$ in Theorem 1, we have the following corollary.

Corollary 2 *Let the function $\mathfrak{F}(z)$ given by (1) be in the function class $\mathfrak{H}_{\mathfrak{E}}^{\eta,\phi}(q;\rho)$ where $(0<\rho\le 1)$. Then*

$$|b_2| \le \min\left\{\frac{2\rho}{(\eta+\phi)[2]_q}, \frac{2\rho}{\sqrt{2(\phi+2)[3]_q\rho+((\eta+\phi)^2-(\rho\eta(2+\eta)+\phi\rho))[2]_q^2}}\right\}$$

and

$$|b_3| \le \min\left\{\frac{2(1-\phi)\rho^2}{(\eta+\phi)^2[3]_q}+\frac{2\rho^2}{(\phi+2\eta)[3]_q}, \frac{2\rho}{(\phi+2\eta)[3]_q}+\frac{4\rho^2}{(\eta+\phi)^2[2]_q^2}\right\}.$$

Remark 3 *Corollary 2 is an improvement of the following estimates investigated by Argul [2].*

Corollary 3 *[2] Let the function $\mathfrak{F}(z)$ given by (1) be in the function class $\mathfrak{H}_{\mathfrak{E}}^{\eta,\phi}(q;\rho)$ where $(0<\rho\le 1)$. Then*

$$|b_2| \le \frac{2\rho}{\sqrt{2(\phi+2)[3]_q\rho+((\eta+\phi)^2-(\rho\eta(2+\eta)+\phi\rho))[2]_q^2}}$$

and

$$|b_3| \le \frac{2\rho}{(\phi+2\eta)[3]_q}+\frac{4\rho^2}{(\eta+\phi)^2[2]_q^2}.$$

Taking $q \longrightarrow 1^-$, $\eta = 0$ and $\phi = 1$ in Corollary 2, we have the following corollary.

Corollary 4 *Let the function $\mathfrak{F}(z)$ given by* (1) *be in the function class $\mathfrak{E}(\rho)$, where* $(0 < \rho \leq 1)$. *Then*

$$|b_2| \leq \min\left\{\rho, \rho\sqrt{\frac{2}{\rho+2}}\right\}$$

and

$$|b_3| \leq \min\left\{\frac{2\rho^2}{3}, \frac{\rho(3\rho+2)}{3}\right\}.$$

Remark 4 *Corollary 4 is an improvement of the following estimates investigated by Srivastave et al. [28].*

Corollary 5 *[28] Let the function $\mathfrak{F}(z)$ given by* (1) *be in the function class $\mathfrak{H}_{\mathfrak{E}}(\rho)$, where* $(0 < \rho \leq 1)$. *Then*

$$|b_2| \leq \rho\sqrt{\frac{2}{\rho+2}}$$

and

$$|b_3| \leq \frac{\rho(3\rho+2)}{3}.$$

Taking $\vartheta(z) = \frac{1+(1-2\varphi)}{1-z} = 1 + 2(1-\varphi)z + 2(1-\varphi)z^2 + \cdots \quad (0 \leq \varphi < 1)$ in Theorem 1, we have the following corollary.

Corollary 6 *Let the function $\mathfrak{F}(z)$ given by* (1) *be in the function class $\mathfrak{H}_{\mathfrak{E}}^{\eta,\phi}(q;\varphi)$ where* $(0 \leq \varphi < 1)$. *Then*

$$|b_2| \leq \min\left\{\frac{2(1-\varphi)}{(\eta+\phi)[2]_q}, \sqrt{\frac{4(1-\varphi)}{2(\phi+2\eta)[3]_q + (\eta+\phi)^2(\phi-1)[2]_q^2}}\right\}$$

and

$$|b_3| \leq \frac{4(1-\varphi)^2}{(\eta+\phi)^2[2]_q^2} + \frac{2(1-\varphi)}{(\phi+2\eta)[3]_q}.$$

Remark 5 *Corollary 6 is an improvement of the following estimates investigated by Argul [2].*

Corollary 7 *[2] Let the function* $\mathfrak{F}(z)$ *given by* (1) *be in the function class* $\mathfrak{H}_{\mathfrak{E}}^{\eta,\phi}(q;\varphi)$ *where* $(0 \leq \varphi < 1)$. *Then*

$$|b_2| \leq \sqrt{\frac{4(1-\varphi)}{2(\phi+2\eta)[3]_q + (\eta+\phi)^2(\phi-1)[2]_q^2}}$$

and

$$|b_3| \leq \frac{4(1-\varphi)^2}{(\eta+\phi)^2[2]_q^2} + \frac{2(1-\varphi)}{(\phi+2\eta)[3]_q}.$$

Taking $q \longrightarrow 1^-$, $\eta = 0$ and $\phi = 1$ in Corollary 6, we have the following corollary.

Corollary 8 *Let the function* $\mathfrak{F}(z)$ *given by* (1) *be in the function class* $\mathfrak{E}(\varphi)$, *where* $(0 \leq \varphi < 1)$. *Then*

$$|b_2| \leq \min\left\{(1-\varphi), \sqrt{\frac{2(1-\varphi)}{3}}\right\}$$

and

$$|b_3| \leq \frac{2(1-\varphi)}{3}.$$

Remark 6 *Corollary 8 is an improvement of the following estimates investigated by Srivastave et al. [28].*

Corollary 9 *[28] Let the function* $\mathfrak{F}(z)$ *given by* (1) *be in the function class* $\mathfrak{H}_{\mathfrak{E}}(\varphi)$, *where* $(0 \leq \varphi < 1)$. *Then*

$$|b_2| \leq \sqrt{\frac{2(1-\varphi)}{3}}$$

and

$$|b_3| \leq \frac{(1-\varphi)(5-3\varphi)}{3}.$$

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Timilehin Gideon Shaba
University of Ilorin
Department of Mathematics
P. M. B. 1515, Ilorin, Nigeria
e-mail: shabatimilehin@gmail.com

Abbas Kareem Wanas
University of Al-Qadisiyah
College of Science
Department of Mathematics
Al Diwaniyah, Al-Qadisiyah, Iraq
e-mail: abbas.kareem.w@qu.edu.iq