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FEKETE-SZEGÖ PROBLEM AND SECOND HANKEL DETERMINANT FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH FOUR LEAF DOMAIN

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Abstract. The ongoing study and the well-known idea of coefficient estimates for the classes of analytic and bi-univalent functions serve as our inspirations for this paper. We begin by outlining a brand-new subclass $\mathcal{FD}_\Sigma$ of analytical and bi-univalent functions connected to the four leaf domain. The Fekete-Szego issue is then solved for functions in class $\mathcal{FD}_\Sigma$ related to a four leaf domain, and bound estimates for the coefficients are provided. The upper bound estimate for the second hankel determinant is also calculated. We also demonstrate the sharpness of these boundaries.

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1. Introduction and Definitions

In the region $\leftthreetimes := \{z \in \mathcal{C} : |z| < 1\}$, let $\mathcal{H}$ represent the family of analytic functions. A collection of functions $f \in \mathcal{H}$ in the Taylor series form

$$f(z) - z = \sum_{\lambda=2}^{\infty} n_\lambda z^\lambda \quad (z \in \leftthreetimes) \tag{1.1}$$

is denoted by the notation $\mathcal{A}$.

Analytic functions $\sigma_1(z)$ and $\sigma_2(z)$ shall exist. If there is an analytic function $\omega(z)$ $(\omega(0) = 0$ and $|\omega(z)| < 1)$ such that $\sigma_1(z) = \sigma_2(\omega(z))$, then $\sigma_1(z)$ is said to be subordinate to $\sigma_2(z)$, denoted as $\sigma_1(z) \prec \sigma_2(z)$. We say

$$\sigma_1 \prec \sigma_2 \Longleftrightarrow \sigma_1(0) = \sigma_2(0) \text{ and } \sigma_1(\leftthreetimes) \subseteq \sigma_2(\leftthreetimes)$$

suppose $\sigma_2(z)$ is univalent in $\leftthreetimes$.

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Despite the fact that the univalent function theory was founded in 1851, the coefficient conjecture, given by Bieberbach [1] in 1916 and subsequently verified by De-Branges [2] in 1985, elevated the theory to one of the growing areas of possible research. Several academics sought to verify or refute this conjecture between 1916 and 1985, resulting in the establishment of several subclasses of the class $\mathcal{S}$ that are based on the geometry of picture domains. $\mathcal{S}^*$ and $\mathcal{K}$ are the most researched and essential subclasses of $\mathcal{S}$, containing starlike and convex functions, respectively.

In 1992, Ma and Minda [3] proposed the following generic form of the class

$$\mathcal{S}^*(\beta) = \left[f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \beta(z)\right],$$

β being an analytic function with $\Re(\beta(z)) > 0$ and $z \in \leftthreetimes$. Furthermore, the function f is symmetric about the real axis and maps $\leftthreetimes$ onto a star-shaped domain with relation to $\beta(0) = 1$.

Every function $f(z) \in \mathcal{S}$ given by (1.1) has an inverse function $f^{-1}(z)$ defined as follows:

$$f^{-1}(f(z)) = z,\ z \in \leftthreetimes, f(f^{-1}(\eta)) = \eta, \eta \in \leftthreetimes_{t_0} = \{\eta \in \mathcal{C} : |\eta| < t_0(f)\},\ 1/4 \leq t_0(f)$$

and

$$f^{-1}(\eta) = \eta + D_2\eta^2 + D_3\eta^3 + D_4\eta^4 + \cdots,\ \ ,\ \eta \in \leftthreetimes_{t_0}, \tag{1.2}$$

where

$$D_2 = -n_2,\ D_3 = 2n_2^2 - n_3,\ D_4 = -5n_2^3 + 5n_2n_3 - n_4$$

which is a well-known fact.

Additionally, it is widely known that a function $f \in \mathcal{A}$ is referred to as a bi-univalent function in $\leftthreetimes$ if both f and f^{-1} are, respectively, univalent in $\leftthreetimes$ and $\leftthreetimes_{t_0}$. Let (1.1) be the collection of bi-univalent functions in $\leftthreetimes$ denoted by Σ. See [4] for a brief history and examples of functions in the class Σ.

To proceed, Lewin [5] defined the class of bi-univalent functions and estimated $|n_2| \leq 1.51$. After that, Brannan and Clunie [6] expanded Lewin's findings to $|n_2| \leq \sqrt{2}$ for $f \in \Sigma$. As a result, Netanyahu [7] shown that $|n_2| \leq \frac{4}{3}$. Brannan and Taha [8] previously introduced some subclasses of the bi-univalent function class Σ, specifically the bi-starlike function of order ξ denoted $\mathcal{S}^*_\Sigma(\xi)$ and the bi-convex function of order ξ denoted $\mathcal{K}_\Sigma(\xi)$, which correspond to the function classes $\mathcal{S}^*(\xi)$ and $\mathcal{K}(\xi)$. In [6,8], nonsharp estimates on the first two Taylor-Maclaurin coefficients were found for each of the function classes $\mathcal{S}^*_\Sigma(\xi)$ and $\mathcal{K}_\Sigma(\xi)$. Many researchers (see [9,10]) have developed and investigated various intriguing subclasses of the bi-univalent function class, and they have discovered nonsharp estimates on the first two Taylor-Maclaurin coefficients. However, the sharp estimates for each of the Taylor-Maclaurin coefficients $|n_k|, k = 2, 3, 4, \cdots$ remain an unsolved problem (see [4,7,8] for additional information). Recently, Deniz et al. [11] achieved the upper bounds of $|H_2(2)| = |n_2n_4 - n_3^2|$ for the classes $\mathcal{S}^*_\Sigma(\xi)$ and

$\mathcal{K}_\Sigma(\xi)$. Soon after, Orhan et al. [12] revisited the study of bounds for the second Hankel determinant of the bi-univalent function subclass $M^\xi_\Sigma(\beta)$, and Mustafa et al. [13] enhanced the [11] results.

Definition 1.1. [14] Let $f \in \mathcal{A}$ has given in (1.1). We denote by $\mathcal{BT}_{4l}$ the class of analytic functions meeting the condition

$$f'(z) \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5 \quad z \in \curlywedge.$$

Gandhi developed a family of starlike functions associated with a four-leaf function defined by

$$\mathcal{S}^*_{4l} = \left[f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{6}{5}z + \frac{1}{6}z^5, \quad (z \in \curlywedge) \right]$$

and characterized it with several main attributes in [15].

Inspired by the aforementioned works, we define the subclass of bi-univalent functions represented by Σ.

Definition 1.2. $f \in \mathcal{FD}_\Sigma$, suppose the following conditions are met:

$$f'(z) \prec \Omega(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^4$$

and

$$(f^{-1}(\eta))' \prec \Omega(\eta) = 1 + \frac{5}{6}\eta + \frac{1}{6}\eta^4$$

where $z, \eta \in \Phi$ and $f^{-1}(\eta)$ is given in (1.2).

In this research, motivated by the work of Mustafa and Murugusundaramoorthy [16], we solve the Fekete-Szegö problem for the functions in the class $\mathcal{FD}_\Sigma$ related to a four leaf domain and provide bound estimates for the coefficients.

We also calculate the upper bound estimate for the second Hankel determinant.

The following lemma is required to prove our main findings.

Lemma 1.3. [17,18] *Suppose that $\mathcal{P}$ is the set of all analytic functions a of the form*

$$a(z) = 1 + \sum_{\lambda=1}^{\infty} a_\lambda z^\lambda \tag{1.3}$$

satisfying $\Re(a(z)) > 0$, $z \in \curlywedge$ and $a(0) = 1$. Then,

$$|a_\lambda| \le 2, \ \lambda = 1, 2, 3, \cdots.$$

For any value of $\lambda = 1, 2, 3, \cdots$, this inequality is sharp. For example, the function

$$a(z) = \frac{1+z}{1-z}$$

is equal for all λ.

Lemma 1.4. [17,18] *Suppose that $\mathcal{P}$ is the set of all analytic functions a of the form*

$$a(z) = 1 + \sum_{\lambda=1}^{\infty} a_\lambda z^\lambda \tag{1.4}$$

satisfying $\Re(a(z)) > 0$, $z \in \leftthreetimes$ and $a(0) = 1$. Then,

$$2a_2 = a_1^2 + (4 - a_1^2)n$$

$$4a_3 = a_1^3 + 2(4 - a_1^2)a_1 n - (4 - a_1^2)a_1 n^2 + 2(4 - a_1^2)(1 - |n|^2)z,$$

for some n, z with $|n| \leq 1$, $|z| \leq 1$.

Lemma 1.5. [18] *If and only if the Toeplitz determinants*

$$H_j = \begin{vmatrix} 2 & a_1 & a_2 & \dots & a_j \\ a_{-1} & 2 & a_1 & \dots & a_{j-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{-j} & a_{-j+1} & a_{-j+2} & \dots & 2 \end{vmatrix}, \quad j = 1, 2, 3, \cdots \tag{1.5}$$

and $a_{-j} = \bar{a_j}$ are all nonnegative, the power series given in (1.3) converges in $\leftthreetimes$ to the function $a \in \mathcal{P}$. Except for

$$a(z) = \sum_{j=1}^{j} \rho_j a_0(a^{ix_n z}),\ \rho_j > 0,\ x_j \text{ real}$$

and $x_j \neq x_k$ for $j \neq k$ in this example, they are all strictly positive. $H_j > 0$ for $j < n - 1$ and $H_j = 0$ for $j \geq n$.

Notation 1.6. Since $a \in \mathcal{P}$, $H_j \geq 0$ and $a_{-1} = \bar{a_1} \geq 0$ are true, as stated by Lemma 1.5. This results in $H_j = \begin{vmatrix} 2 & a_1 \\ a_1 & 2 \end{vmatrix} \geq 0$ and $a_1 = \bar{a_1} = a_{-1} \geq 0$. As a result, $4 - a_1^2 \geq 0$ and $a_1 \geq 0$ are equal to $a_1 \in [0, 2]$. For these reasons, we will assume throughout our investigation that $|4 - a_1^2| = |4 - |a_1|^2| = 4 - |a_1|^2$ for a_1, the first coefficient in (1.3).

Proposition 1.7. [19] *If the function*

$$a(z) = 1 + \sum_{j \geq 1} a_j z^j, \quad v(z) = 1 + \sum_{j \geq 1} v_j z^j$$

which are in the class of $\mathcal{P}$ and $a_1 = -v_1$, then

$$a_2 - v_2 = \frac{(4 - a_1^2)(n - y)}{2} \tag{1.6}$$

$$a_2 + v_2 = h_1^2 + \frac{(4 - a_1^2)(n + y)}{2} \tag{1.7}$$

and

$$a_3 - v_3 = \frac{a_1^3}{2} + \frac{(4-a_1^2)(n+y)}{2}a_1 - \frac{(4-a_1^2)(n^2+y^2)}{4}a_1 + \frac{(4-a_1^2)[(1-|n|^2)z-(1-|y|^2)w]}{2} \tag{1.8}$$

for some $|a_1|, |v_1| \in [0,2]$*,* z, n, w, y *with* $|z|, |n|, |w|, |y| \leq 1$.

2. Coefficients bound estimates

In this section, we show the following theorem regarding upper bound estimates for the few initial coefficients of the functions belonging within the class $\mathcal{FD}_\Sigma$.

Theorem 2.1. *Let* $f \in \mathcal{FD}_\Sigma$*. Then:*

$$|n_2| \leq \frac{5}{12},$$
$$|n_3| \leq \max\left\{\frac{5}{18}, \frac{25}{144}\right\},$$
$$|n_4| \leq \max\left\{\frac{5}{24}, \frac{25}{216}\right\}.$$

The result obtained here are sharp for

$$f_n(z) = \int_0^z \left(1 + \frac{5}{6}t^n + \frac{1}{6}t^{5n}\right) \cdot dt = z + \frac{5}{6(n+1)}z^{n+1} + \frac{1}{6(5n+1)}z^{5n+1}.$$

Proof. Let $f \in \mathcal{FD}_\Sigma$. Then, there are holomorphic functions $\rho : \curlywedge \longrightarrow \curlywedge$, $\varrho : \curlywedge_{t_o} \longrightarrow \curlywedge_{t_o}$ with $\rho(0) = 0 = \varrho(0)$, $|\rho(z)| \leq 1$, $|\varrho(\varpi)| < 1$ fulfilling the following conditions

$$f'(z) = \Omega(\rho(z)) = 1 + \frac{5}{6}\rho(z) + \frac{1}{6}(\rho(z))^4, \quad z \in \curlywedge \tag{2.1}$$

and

$$(f^{-1}(\eta))' = \Omega(\varrho(\eta)) = 1 + \frac{5}{6}\varrho(\eta) + \frac{1}{6}(\varrho(\eta))^4, \quad \eta \in \curlywedge. \tag{2.2}$$

The functions $a, v \in \mathcal{P}$ are defined as follows:

$$a(z) = \frac{1+\rho(z)}{1-\rho(z)} = 1 + \sum_{\lambda=1}^{\infty} a_\lambda z^\lambda, \quad z \in \curlywedge$$

and

$$v(\eta) = \frac{1+\varrho(\eta)}{1-\varrho(\eta)} = 1 + \sum_{\lambda=1}^{\infty} v_\lambda \eta^\lambda, \quad \eta \in \curlywedge_{t_o}.$$

As a result,

$$\rho(z) = \frac{a(z)-1}{a(z)+1} = \frac{1}{2}\left[a_1 z + \left(a_2 - \frac{a_1^2}{2}\right)z^2 + \left(a_3 + a_1 a_2 + \frac{a_1^3}{4}\right)z^3 + \cdots\right], \ z \in \curlywedge \tag{2.3}$$

and

$$\varrho(\eta) = \frac{v(\eta)-1}{v(\eta)+1} = \frac{1}{2}\left[v_1\eta + \left(v_2 - \frac{v_1^2}{2}\right)\eta^2 + \left(v_3 + v_1 v_2 + \frac{v_1^3}{4}\right)\eta^3 + \cdots\right], \ \eta \in \curlywedge_{t_0}. \tag{2.4}$$

We obtain by substituting the expressions for the functions $\rho(z)$ and $\varrho(\eta)$ in (2.1) and (2.2) with those in (2.3) and (2.4):

$$f'(z) = 1 + \frac{5}{12}a_1 z + \left[\frac{5}{12}a_2 - \frac{5}{24}a_1^2\right] z^2 + \left[\frac{5}{12}a_3 - \frac{5}{12}a_1 a_2 + \frac{5}{48}a_1^3\right] z^3 + \cdots \tag{2.5}$$

and

$$(f^{-1}(\eta))' = 1 + \frac{5}{12}v_1\eta + \left[\frac{5}{12}v_2 - \frac{5}{24}v_1^2\right]\eta^2 + \left[\frac{5}{12}v_3 - \frac{5}{12}v_1 v_2 + \frac{5}{48}v_1^3\right]\eta^3 + \cdots . \tag{2.6}$$

The following equations are derived for n_2, n_3, and n_4 if the operations and simplifications on the left side of (2.5) and (2.6) are made and the coefficients of the terms of the same degree are equalized.

$$2n_2 = \frac{5}{12}a_1 \tag{2.7}$$

$$3n_3 = \frac{5}{12}a_2 - \frac{5}{24}a_1^2 \tag{2.8}$$

$$4n_4 = \frac{5}{12}a_3 - \frac{5}{12}a_1 a_2 + \frac{5}{48}a_1^3 \tag{2.9}$$

and

$$-2n_2 = \frac{5}{12}v_1 \tag{2.10}$$

$$-n_3 + 6n_2^2 = \frac{5}{12}v_2 - \frac{5}{24}v_1^2 \tag{2.11}$$

$$-4n_4 + 20n_2 n_3 - 20n_2^3 = \frac{5}{12}v_3 - \frac{5}{12}v_1 v_2 + \frac{5}{48}v_1^3. \tag{2.12}$$

On the basis of equations (2.7) and (2.10), we write

$$\frac{5a_1}{24} = n_2 = \frac{-5v_1}{25} \Rightarrow n_1 = -v_1,\ n_1^2 = n_1^2,\ n_1^3 = -v_1^3. \tag{2.13}$$

The first result of the theorem is evident based on this and Lemma 1.3.

By deducting (2.11) from (2.8) and taking into account the equivalence in (2.13), we obtain

$$n_3 = n_2^2 + \frac{5[a_2 - v_2]}{72};$$

Hence,

$$n_3 = \frac{25a_1^2}{576} + \frac{5[a_2 - v_2]}{72}. \tag{2.14}$$

Additionally, by reducing (2.12) by (2.9), taking into account the equalities (2.13) and (2.14), we have

$$n_4 = \frac{5}{432}a_1^3 + \frac{125}{3456}a_1(a_2 - v_2) + \frac{5}{96}(a_3 - v_3) - \frac{5}{96}a_1(a_2 + v_2). \tag{2.15}$$

The first equivalence in Preposition 1.7 is substituted into (2.14), and the resulting expression for the coefficient n_3 is as follows:

$$n_3 = \frac{25a_1^2}{576} + \frac{5}{144}(4 - a_1^2)(n - y).$$

Take note that $|4-a_1^2| = |4-|a_1|^2| = 4-|a_1|^2 = |4-k^2| = 4-k^2$ can be written if we take $|a_1| = k$. (see, also Notation (1.6)). That is, we may assume without restriction that $a \in [0,2]$. In that instance, we can express the inequality for $|n_3|$ by using a triangle inequality and setting $|n| = \alpha$ and $|y| = \delta$.

$$|n_3| \leq \frac{25k^2}{576} + \frac{5(4-k^2)}{144}(\alpha+\delta), \quad (\alpha,\delta) \in [0,1]^2.$$

Let's now define the function $J : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as follows:

$$J(\alpha,\delta) = \frac{25k^2}{576} + \frac{5(4-k^2)}{144}(\alpha+\delta), \quad (\alpha,\delta) \in [0,1]^2.$$

The function J on the closed square $\Delta = \{(\alpha,\delta) : (\alpha,\delta) \in [0,1]^2\}$ needs to be maximized.

It is evident that the function J reaches its highest value at the closed square's Δ boundary. When we differentiate the function $J(\alpha,\delta)$ with regard to parameter α, we get:

$$J_\alpha(\alpha,\delta) = \frac{5(4-k^2)}{144}.$$

Since the function $J(\alpha,\delta)$ has a maximum value at $\alpha = 1$ and $J_\alpha(\alpha,\delta) \geq 0$, it is an increasing function with regard to α. Hence,

$$\max\{J(\alpha,\delta) : \alpha \in [0,1]\} = J(1,\delta) = \frac{25k^2}{576} + \frac{5(4-k^2)}{144}(1+\delta),$$

for each $\delta \in [0,1]$ and $k \in [0,2]$.

With the function $J(1,\delta)$ now differentiable, we have

$$J'(1,\delta) = \frac{5(4-k^2)}{144}.$$

The function $J(1,\varsigma)$ is an increasing function because $J'(1,\delta) \geq 0$, and maximum occurs at $\delta = 1$, therefore

$$\max\{J(1,\delta) : \delta \in [0,1]\} = J(1,1) = \frac{25k^2}{576} + \frac{5(4-k^2)}{72}, \quad k \in [0,2].$$

Thus, we get

$$J(\alpha,\delta) \leq \max\{J(\alpha,\delta) : (\alpha,\delta) \in \Delta\} = J(1,1) = \frac{25k^2}{576} + \frac{5(4-k^2)}{72}.$$

Since $|n_3| \leq J(\alpha,\delta)$, we have

$$|n_3| \leq c \times k^2 + \frac{5}{18}$$

where

$$c = \frac{5}{72}\left[\frac{5-8}{8}\right].$$

Now, let calculate the maximum of the function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ define as follows

$$\lambda(k) = c \times k^2 + \frac{5}{18}$$

in the range of $[0,2]$.

Differentiating the function $\lambda(k)$, we have $\lambda'(k) = 2ck,\ k \in [0,2]$. Since $\lambda'(k) \leq 0$ when $c \leq 0$, the function $\lambda(k)$ is a decreasing function and maximum occurs at $k = 0$, therefore

$$\max\{\lambda(k) : k \in [0,2]\} = \lambda(0) = \frac{5}{18}$$

and $\lambda'(k) \geq 0$ when $c \geq 0$, the function $\lambda(k)$ is an increasing function and maximum occurs at $k = 2$, which gives

$$\max\{\lambda(k) : k \in [0,2]\} = \lambda(2) = \frac{25}{144}.$$

As a result, we arrive at the upper bound estimate for $|n_3|$ that is provided below:

$$|n_3| \leq \max\left\{\frac{5}{18}, \frac{25}{144}\right\}.$$

The following inequality for $|n_4|$ is obtained from (2.15), using (1.6), (1.7), (1.8), and triangle inequality.

$$|n_4| \leq t_1(k) + t_2(k)(\alpha + \delta) + t_3(k)(\alpha^2 + \delta^2) := G(\alpha, \delta)$$

where

$$t_1(k) = \frac{25}{1728}k^3 + \frac{5(4-k^2)}{96},$$
$$t_2(k) = \frac{125(4-k^2)}{6912}k,$$
$$t_3(k) = \frac{5(4-k^2)(k-2)}{384}.$$

The function G for each $k \in [0,2]$ must now be maximized.

We must examine the maximum of the function G for various values of the parameter k since the coefficients $t_1(k)$, $t_2(k)$, and $t_3(k)$ of the function G depend on the parameter k.

Let $k = 0$, since $t_2(0) = 0$,

$$t_1(0) = \frac{5}{24} \text{ and}$$
$$t_3(0) = -\frac{5}{48}.$$

We then have

$$G(\alpha, \delta) = \frac{5}{24} - \frac{5}{48}(\alpha^2 + \delta^2), \quad (\alpha, \delta) \in [0,1]^2.$$

Hence, we get

$$G(\alpha, \delta) \leq \max\{Q(\alpha, \delta) : (\alpha, \delta) \in \Delta\} = G(0,0) = \frac{5}{24}.$$

Let $k = 2$. Then, since $t_2(2) = t_3(2) = 0$ and

$$t_1(2) = \frac{25}{216}.$$

The following function G is a constant.

$$G(\alpha, \delta) = t_1(2) = \frac{25}{216}.$$

We can quickly demonstrate that the function G cannot have a maximum on the Δ in the case $k \in (0, 2)$. Thus, we attain

$$|B_4| \le \max\left\{\frac{5}{24}\,,\ \frac{25}{216}\right\}.$$

The following extremal functions can be used to calculate the best possible bounds:

$$f_1(z) = \int_0^z \left(1 + \frac{5}{6}t + \frac{1}{6}t^5\right) \cdot dt = z + \frac{5}{12}z^2 + \frac{1}{36}z^6 + \cdots ,$$
$$f_2(z) = \int_0^z \left(1 + \frac{5}{6}t^2 + \frac{1}{6}t^{10}\right) \cdot dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots ,$$
$$f_3(z) == \int_0^z \left(1 + \frac{5}{6}t^3 + \frac{1}{6}t^{15}\right) \cdot dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots ,$$
$$f_4(z) == \int_0^z \left(1 + \frac{5}{6}t^4 + \frac{1}{6}t^{20}\right) \cdot dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots .$$

□

3. The second Hankel determinant and Fekete-Szegö inequality

For the function belonging to the class $\mathcal{FD}_\Sigma$ described by Definition 1.2, we provide an upper limit estimate for the second Hankel determinant and Fekete-Szegö inequality in this section.

First, we establish the following theorem on the second Hankel determinant's upper bound estimate.

Theorem 3.1. *Let $f(z) \in \mathcal{FD}_\Sigma$. Then:*

$$|n_2 n_4 - n_3^2| \le \max\left\{\frac{25}{324}\,,\ \frac{1625}{20736}\right\}.$$

The result obtained here are sharp for

$$f_2(z) = \int_0^z \left(1 + \frac{5}{6}t^2 + \frac{1}{6}t^{10}\right) \cdot dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots .$$

Proof. Let $f \in \mathcal{FD}_\Sigma$. The following equality for $n_2 n_4 - n_3^2$ is therefore written from equations (2.13), (2.14), and (2.15):

$$n_2 n_4 - n_3^2 = \frac{25}{10368}a_1^4 + \frac{625a_1^2(a_2 - v_2)}{82944} + \frac{25a_1(a_3 - v_3)}{2304} - \frac{25a_1^2(a_2 + v_2)}{2304}$$
$$- \frac{625a_1^4}{331776} - \frac{125a_1^2(a_2 - v_2)}{20736} - \frac{25(a_2 - v_2)^2}{5184}.$$

We obtain the following estimate for $|n_2 n_4 - n_3^2|$ by using equalities (1.6), (1.7) and (1.8), followed by triangle inequality and setting $|n_1| = k$, $|n| = \alpha$, and $|y| = \delta$.

$$|n_2 n_4 - n_3^2| \le T_1(k) + T_2(k)(\alpha + \delta) + T_3(k)(\alpha^2 + \delta^2) + T_4(k)(\alpha + \delta)^2 \tag{3.1}$$

where

$$T_1(k) = \frac{1625}{331776}k^4 + \frac{25(4-k^2)}{2304}k \geq 0,$$
$$T_2(k) = \frac{125(4-k^2)}{165888}k^2 \geq 0,$$
$$T_3(k) = \frac{25(4-k^2)(k-2)}{9216}k \leq 0$$
$$T_4(k) = \frac{25(4-k^2)^2}{20736} \geq 0.$$

Let's now define the function $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as follows:

$$\Psi(\alpha, \delta) = T_1(d) + T_2(k)(\alpha+\delta) + T_3(k)(\alpha^2+\delta^2) + T_4(k)(\alpha+\delta)^2.\ (\alpha, \delta) \in [0,1]^2$$

for each $k \in [0, 2]$.

The function Ψ on Δ for each $k \in [0, 2]$ must now be maximized.

We must examine the maximum of the function Ψ for various values of the parameter k since the coefficients $t_1(k)$, $T_2(k)$, $T_3(k)$ and $T_4(k)$ of the function Ψ depend on the parameter k.

(1) Let $k = 0$. Since $T_1(0) = T_2(0) = T_3(0) = 0$ and

$$T_4(0) = \frac{25}{1296}$$

the function $\Psi(\alpha, \delta)$ written as follows:

$$\Psi(\alpha, \delta) = \frac{25}{1296}(\alpha+\delta)^2, \quad (\alpha, \delta) \in \Delta.$$

It is obvious that the function Ψ reaches its maximum near the closed-square boundary Δ.

Now, applying some differentiation on the function $\Psi(\alpha, \delta)$ with respect to α, we have

$$\Psi_\alpha(\alpha, \delta) = \frac{25}{648}(\alpha+\delta)$$

for each $\delta \in [0, 1]$.

The function $\Psi(\alpha, \delta)$ is an increasing function with regard to α and reaches its maximum at $\alpha = 1$ since $\Psi_\alpha(\alpha, \delta) \geq 0$. So

$$\max\{\Psi(\alpha, \delta) : \sigma \in [0,1]\} = \Psi(1, \delta) = \frac{25(1+\delta)^2}{1296}, \quad \delta \in [0, 1].$$

Taking the differentiation of the function $\Psi(1, \delta)$, we get

$$\Psi'(1, \delta) = \frac{25(1+\delta)}{648} > 0, \quad \delta \in [0, 1].$$

Since $\Psi'(1, \delta) > 0$, the function $\Psi(1, \delta)$ is an increasing function and maximum occurs at $\delta = 1$. Hence,

$$\max\{\Psi(1, \delta) : \delta \in [0,1]\} = \Psi(1, 1) = \frac{25}{324}.$$

Thus, in the instance of $d = 0$, we get

$$\Psi(\alpha, \delta) \le \max\{\Psi(\alpha, \delta) : (\alpha, \delta) \in [0, 1]^2\} = \Psi(1, 1) = \frac{25}{324}.$$

We know that $|n_2 n_4 - n_3^2| \le \Psi(\alpha, \delta)$, we can have

$$|n_2 n_4 - n_3^2| \le \frac{25}{324}.$$

(2) Now, taking $k = 2$. Since $T_2(2) = T_3(2) = T_4(2) = 0$ and

$$T_1(2) = \frac{1625}{20736}$$

the function $\Psi(\alpha, \delta)$ is a constant as follows

$$\Psi(\alpha, \delta) = T_1(2) = \frac{1625}{20736}.$$

Thus, we get

$$|n_2 n_4 - n_3^2| \le \frac{1625}{20736}.$$

in the case $k = 2$.

(3) Let's say $k \in (0, 2)$. In this instance, we must look into the maximum of the function Ψ while accounting for the sign of $\Xi(\Psi) = \Psi_{\alpha\alpha}(\alpha, \delta)\Psi_{\delta\delta}(\alpha, \delta) - (\Psi_{\alpha\delta}(\alpha, \delta))^2$.
The equation $\Xi(\Psi) = 4T_3(k)[T_3(k) + 2T_4(k)]$ is clear to see. We will look into two instances of the sign $\Xi(\Psi)$.

  (a) Let $T_3(k) + 2T_4(k) \le 0$ for same $k \in (0, 2)$. In this case, since $\Psi_{\alpha,\delta}(\alpha, \delta) = \Psi_{\delta,\alpha}(\alpha, \delta) = 2T_4(k) \ge 0$ and $\Xi(\Psi) \ge 0$, the function Ψ (having a minimum) cannot have a maximum on the square Δ according to basic calculus.

  (b) Now, let $T_3(k) + 2T_4(k) \ge 0$ for some $k \in (0, 2)$. In this case, since $\Xi(\Psi) \le 0$, the function Ψ cannot have a maximum on the square Δ.

Consequently, in light of all three instances, we write

$$|n_2 n_4 - n_3^2| \le \max\left\{\frac{25}{324}, \frac{1625}{20736}\right\}.$$

Theorem 3.1 has now been successfully proved. □

We now provide the subsequent theorem on the Fekete-Szegö inequality.

Theorem 3.2. *Let $f(z) \in \mathcal{FD}_\Sigma$, $\chi \in \mathcal{C}$. Then:*

$$\left|n_3 - \chi n_2^2\right| \le \begin{cases} \frac{5}{18} & |1 - \chi| \le \frac{5}{72} \\ \frac{25|1-\chi|}{144} & |1 - \chi| \ge \frac{5}{72}. \end{cases}$$

The result obtained here are sharp for

$$f_2(z) = \int_0^z \left(1 + \frac{5}{6}t^2 + \frac{1}{6}t^{10}\right) \cdot dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots .$$

Proof. Let $f(z) \in \mathcal{FD}_\Sigma$ and $\chi \in \mathcal{C}$. Then, from (2.13), (2.14), (1.6) and (1.7), we have the expression $n_3 - \chi n_2^2$ to be:

$$n_3 - \chi n_2^2 = (1-\chi)\frac{25a_1^2}{576} + \frac{5(4-a_1^2)}{144}(n-y) \tag{3.2}$$

for some y, n with $|y| \leq 1$ and $|n| \leq 1$.

With $|n| = \alpha$, $|y| = \delta$, $|a_1| = k$ and the triangle inequality to the equality (3.2), we can estimate the upper bound of $|n_3 - \chi n_2^2|$ as follows:

$$|n_3 - \chi n_2^2| \leq \frac{25|1-\chi|}{576}k^2 + \frac{5(4-k^2)}{144}(\alpha+\delta), \ \ (\alpha,\delta) \in \Delta, \tag{3.3}$$

for each $k \in [0,2]$.

Let's now define the function $\Lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as follows:

$$\Lambda(\alpha,\delta) = \frac{25|1-\chi|}{576}k^2 + \frac{5(4-k^2)}{144}(\alpha+\delta), \ \ (\alpha,\delta) \in \Delta,$$

for each $k \in [0,2]$. The function Λ on Δ for each $k \in [0,2]$ must now be maximized.

The function Λ clearly reaches its maximum value along the boundary of the closed square Δ.

Applying the concept of differentiation on the function $\Lambda(\alpha,\delta)$ with respect to α, we have

$$\Lambda_\alpha(\alpha,\delta) = \frac{5(4-k^2)}{144} \tag{3.4}$$

for each $k \in [0,2]$.

Since $\Lambda_\alpha(\alpha,\delta) > 0$, the function $\Lambda(\alpha,\delta)$ is an increasing function with respect to α and maximum occurs at $\delta = 1$. Hence,

$$\max\{\Lambda(\alpha,\delta) : \delta \in [0,1]\} = \Lambda(1,\delta) = \frac{|251-\chi|}{576}k^2 + \frac{5(4-k^2)}{144}(1+\delta)$$

for each $\delta \in [0,1]$ and $k \in [0,2]$.

Now, differentiating the function $\Lambda(1,\delta)$, we get

$$\Lambda'(1,\delta) = \frac{5(4-k^2)}{144}$$

for each $k \in [0,2]$.

Since $\Lambda'(1,\delta) > 0$, the function $\Lambda(1,\delta)$ is an increasing function and maximum occurs at $\delta = 1$. Hence,

$$\max\{\Lambda(1,\delta) : \delta \in [0,1]\} = \Lambda(1,1) = \frac{25|1-\chi|}{576}k^2 + \frac{5(4-k^2)}{72}, \ \delta \in [0,2].$$

Thus, we have

$$\Lambda(\alpha,\delta) \leq \max\{\Lambda(\alpha,\delta) : \chi \in [0,1]\} = \Lambda(1,1) = \frac{25|1-\chi|}{576}k^2 + \frac{5(4-k^2)}{72}.$$

Since $|n_3 - \chi n_2^2| \leq \Lambda(\alpha, \delta)$, we have the following estimate

$$|n_3 - \chi n_2^2| \leq \frac{25}{576}\left[|1-\chi| - \frac{576}{25}L\right]k^2 + 4L,$$

where

$$L = \frac{5}{72}.$$

In such instance, finding the maximum of the following function, $\vartheta : [0,2] \longrightarrow \mathbb{R}$, would be appropriate.

Differenting the function $\vartheta(k)$, we have

$$\vartheta'(k) = \frac{25\left[|1-\chi| - \frac{576}{25}L\right]}{288}k, \quad k \in [0,2].$$

If $|1-\chi| \leq L$ and maximum occur at $k = 0$, then the function $\vartheta(k)$ is a decreasing function since $\vartheta'(k) \leq 0$,

$$\max\{\vartheta(k) : \varsigma \in [0,2]\} = \vartheta(0) = 4L$$

and $\vartheta'(k) \geq 0$, the function $\vartheta(k)$ is an increasing function. If $|1-\chi| \geq L$ and maximum occurs at $d = 2$, so

$$\max\{\vartheta(k) : \varsigma \in [0,2]\} = \vartheta(2) = \frac{25|1-\chi|}{144}.$$

Consequently, we achieve

$$|n_3 - \chi n_2^2| \leq \begin{cases} 4L & |1-\chi| \leq L \\ \\ \frac{25|1-\chi|}{144} & |1-\chi| \geq L. \end{cases}$$

The result reached in this instance is sharp for $|1-\chi| \geq \frac{5}{72}$.

If we set the function $f(z)$ as follows:

$$f_2(z) = \int_0^z \left(1 + \frac{5}{6}t^2 + \frac{1}{6}t^{10}\right) \cdot dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots.$$

□

Theorem 3.2 is presented in the following manner for the case $\chi \in \mathbb{R}$.

Theorem 3.3. *Let $f(z) \in \mathcal{FD}_\Sigma$, $\chi \in \mathbb{R}$. Then:*

$$|n_3 - \chi n_2^2| \leq \begin{cases} \frac{25(1-\chi)}{144} & \text{if } \chi \leq \frac{67}{72} \\ \\ \frac{5}{18} & \text{if } \frac{67}{72} \leq \chi \leq \frac{77}{72} \\ \\ \frac{25(\chi-1)}{144} & \text{if } \frac{77}{72} \leq \chi. \end{cases} \tag{3.5}$$

Proof. Let $f(z) \in \mathcal{FD}_\Sigma$ and $\chi \in \mathbb{R}$. Since in the case $\chi \in \mathbb{R}$, the inequalities $|1-\chi| \geq L$ and $|1-\chi| \leq L$ are equivalent to:

$$\chi \leq 1 - L \ \text{ or } \ \chi \geq 1 + L$$

and

$$1 - L \leq \chi \leq 1 + L,$$

respectively. The conclusion of the theorem is derived from Theorem 3.2. □

Furthermore, we get the following conclusions for $\chi = 1$ from Theorem 3.3.

Corollary 3.4. *Let $f(z) \in \mathcal{FD}_\Sigma$. Then:*

$$|n_3 - n_2^2| \leq \frac{5}{18}.$$

4. Conclusion

Recently, well-known mathematicians have been drawn to special domains and polynomials in Geometric Functions Theory due to their usefulness in various fields of mathematics and other sciences. In our paper, we addressed the Fekete-Szegö problem and provided solutions for functions belonging to the class $f \in \mathcal{FD}_\Sigma$, which consists of analytic and bi-univalent functions that involve the four leaf domain. We also derived estimates for the coefficients and an upper bound estimate for the second Hankel determinant. The conclusions mentioned above, supported by references [20–39], can be expanded to include a specific class of analytic and bi-univalent functions.

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