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FEKETE-SZEGÖ PROBLEM AND SECOND HANKEL DETERMINANT FOR A CLASS OF τ-PSEUDO BI-UNIVALENT FUNCTIONS INVOLVING EULER POLYNOMIALS

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Abstract. In the field of geometric function theory, numerous notable authors have extensively used orthogonal polynomials. In this paper, we solve the Fekete-Szegö problem, and also give bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant for functions in the class $\mathcal{G}_\Sigma(v,\sigma)$ of analytic and bi-univalent functions involving the Euler polynomials.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of all complex valued functions $f(z)$ given by

$$f(z)=\sum_{l=2}^{\infty} s_l z^l = z+s_2z^2+s_3z^3+\cdots+s_lz^l+\cdots, \quad s_l\in\mathbb{C}, \tag{1.1}$$

which are holomorphic in the open unit disk

$$\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$$

in the complex plane. A function is called univalent in $\mathcal{U}$ if it never takes a value twice. Mathematically

$$f(z_1)\neq f(z_2) \text{ for all points } z_1 \text{ and } z_2 \text{ in } \mathcal{U} \text{ implies } z_1\neq z_2.$$

Let $\mathcal{S}$ represent the class of all univalent functions in $\mathcal{A}$ as well. As the class of starlike and convex functions of order ϕ, respectively, the classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ are some of the significant and well-researched subclasses of $\mathcal{S}$ therefore have been added here as follows (see [1,4]).

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$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \phi, \quad \phi \in [0,1), \quad z \in \mathcal{U} \right\}$$

and

$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \phi, \quad \phi \in [0,1), \quad z \in \mathcal{U} \right\}.$$

Remark 1.1. It is easy to seen that

$$\mathcal{S}^*(0) = \mathcal{S}^* \quad \text{and} \quad \mathcal{C}(0) = \mathcal{C},$$

where $\mathcal{S}^*$ and $\mathcal{C}$ are the well-known function classes of starlike and convex functions respectively.

Let the function $g(z)$ and $f(z)$ be analytic in $\mathcal{U}$. We say that the function $f(z)$ is surbordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function ω that is analytic in $\mathcal{U}$ with

$$|\omega(z)| < 1 \text{ and } \omega(0) = 0 \ \ (z \in \mathcal{U})$$

such that

$$g(\omega(z)) = f(z).$$

Beside that, if the funtion g is univalent in $\mathcal{U}$, then the following equivalence holds:

$$f(z) \prec g(z) \ \text{ if } \ g(0) = f(0)$$

and

$$f(\mathcal{U}) \subset g(\mathcal{U}).$$

For more details see [1].

It is generally known that for every $f \in \mathcal{S}$, its inverse function is defined by

$$\mathcal{F}(f(z)) = z, f(\mathcal{F}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right) \text{ and } (z, w \in \mathcal{U}),$$

and may have the following analytical extension in $\mathcal{U}$.

$$\mathcal{F}(w) = w - s_2 w^2 + (2s_2^2 - s_3)w^3 + (-5s_2^3 + 5s_2 s_3 - s_4)w^4 + \cdots. \tag{1.2}$$

An analytic function f is called bi-univalent in $\mathcal{U}$ if f and f^{-1} are both univalent $\mathcal{U}$. The classes of all such function is denoted by Σ.

The so-called "polynomials" are a significant and fascinating group of special functions, specifically orthogonal polynomials. They can be found in various disciplines of the natural sciences, such as coding theory, discrete mathematics (graph theory, combinatorics), Eulerian series, elliptic functions, theta functions, continuous fractions etc.; see [6,7], algebras and quantum groups [8–10].

Using the generating function, the Eulers polynomials $\mathcal{E}_m(v)$ are frequently defined (see, e.g., [2,5]):

$$L(v,t) = \frac{2e^{tv}}{e^t+1} = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{t^m}{m!}, \quad |t| < \pi. \tag{1.3}$$

An explicit formula for $\mathcal{E}_m(v)$ is given by

$$\mathcal{E}_n(v) = \sum_{m=0}^{n} \frac{1}{2^m} \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} (v+k)^n .$$

Now from the above equation we get $\mathcal{E}_m(v)$ in term of $\mathcal{E}_k$ as

$$\mathcal{E}_m(v) = \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \frac{\mathcal{E}_k}{2^k} \left(v - \frac{1}{2}\right)^{m-k}. \tag{1.4}$$

The initial Euler polynomials are:

$$\begin{aligned} \mathcal{E}_0(v) &= 1 \\ \mathcal{E}_1(v) &= \frac{2v-1}{2} \\ \mathcal{E}_2(v) &= v^2 - v \\ \mathcal{E}_3(v) &= \frac{4v^3-6v^2+1}{4} \\ \mathcal{E}_4(v) &= v^4 - 2v^3 + v. \end{aligned} \tag{1.5}$$

The Fekete-Szegö functional $\mathcal{L}_\beta(f) = |t_3 - \beta t_2^2|$ for $f(z) \in \mathcal{S}$ is well-known for its role as a functional in determining the sharp upper bound for functions $f(z) \in \mathcal{S}$ in geometric function theory. It was established by Fekete and Szegö [25] when they disproved the conjecture and Littlewood and Parley that the modulus of coefficients of odd functions $f \in \mathcal{S}$ are less than or equal to 1. The functional has recieved great attention (see for instance, [26,27]), particularly in many subfamilies of analytic and univalent functions. The extablishment of sharp upper bound for functional $\mathcal{L}_\beta(f)$ for any family of functions $\mathcal{S} \subset \mathcal{A}$ is what is known as the Fekete-Szegö problem of $\mathcal{S}$.

Pommerenke [28] investigated and defined below the nth-Hankel determinant, denoted by $H_s(n)(s, n \in \mathcal{N} = \{1,2,3,\cdots\})$, for any function $f \in \mathcal{S}$ in geometric function theory:

$$H_s(n) = \begin{vmatrix} t_n & t_{n+1} & \dots & t_{n+s-1} \\ t_{n+1} & t_{n+2} & \dots & t_{n+s} \\ t_{n+2} & t_{n+3} & \dots & t_{n+s+1} \\ \vdots & \vdots & \dots & \vdots \\ t_{n+s-1} & t_{n+s} & \dots & t_{n+2(s-1)} \end{vmatrix}$$

For certain s and n values,

$$H_2(1) = \begin{vmatrix} t_1 & t_2 \\ t_2 & t_3 \end{vmatrix} = |t_3 - t_2^2| \text{ and } H_2(2) = \begin{vmatrix} t_2 & t_3 \\ t_3 & t_4 \end{vmatrix} = |t_2 t_4 - t_3^2|. \tag{1.6}$$

We see that the determinant $|H_2(1)|$ corresponds with the Fekete-Szego functional $\mathcal{L}_1(f)$, implying that $\mathcal{L}_\beta(f)$ is a generalization of $|H_2(1)|$. Recent research in this area includes the papers in [29,30]. Pommerenke [28] discussed several uses of Hankel determinants in the analysis of singularities and power series with integral coefficients of analytic functions. Another area of application is in the solution of orthogonal polynomial problems (see Junod [31]).

In this study, we define the new subclass introduced and studied in the present paper, denoted by $\mathcal{W}_\Sigma(\tau, v)$, consisting of bi-univalent functions satisfying a certain subordination involving Eulers polynomials. We solve the Fekete-Szegö problem for functions in the class $\mathcal{W}_\Sigma(\tau, v)$ and in the special instances, as well as provide bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant.

Definition 1.2. For $f \in \mathcal{W}_\Sigma(\tau, v)$, suppose the following subordination is true:

$$(f'(z))^\tau \prec L(v, z) = \sum_{m=0}^{\infty} \mathcal{E}_m(v) \frac{z^m}{m!} \tag{1.7}$$

and

$$(\mathcal{F}'(z))^\tau \prec L(v, w) = \sum_{m=0}^{\infty} \mathcal{E}_m(v) \frac{w^m}{m!}, \tag{1.8}$$

where $\tau > 0$, $v \in (\frac{1}{2}, 1]$, $z, w \in \mathcal{U}$, $L(v, w)$ is given by (1.3), and $\mathcal{F} = f^{-1}$ is given by (1.2). It could be seen that both the functions f and and its inverse $\mathcal{F} = f^{-1}$ are univalent in $\mathcal{U}$, so we can conclude that the function f is bi-univalent belonging to the function class $\mathcal{W}_\Sigma(\tau, v)$.

Remark 1.3. Setting $\tau = 1$ in Definition 1.2, we have the class of bounded turning functions $f \in \mathcal{R}_\Sigma^*(v)$, which fulfilled the following conditions:

$$f'(z) \prec L(v, z) = \sum_{m=0}^{\infty} \mathcal{E}_m(v) \frac{z^m}{m!} \tag{1.9}$$

and

$$\mathcal{F}'(z) \prec L(v, w) = \sum_{m=0}^{\infty} \mathcal{E}_m(v) \frac{w^m}{m!}, \tag{1.10}$$

where $z, w \in \mathcal{U}$, $L(v, w)$ is given by (1.3), and $\mathcal{F} = f^{-1}$ is given by (1.2).

Next, let $\mathcal{P}$ present the familiar Carathéodory class of functions α, analytic in an open unit disk $\mathcal{U}$, those are normalized by

$$\alpha(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n, \tag{1.11}$$

such that

$$\Re\left\{\alpha\left(z\right)\right\} > 0 \qquad \left(\forall\; z \in \mathcal{U}\right).$$

Lemma 1.4. [1] *Let the function $\alpha \in \mathcal{P}$ given by the series*

$$\alpha(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots \quad (z \in \mathcal{U}) \tag{1.12}$$

then

$$|\alpha_k| \leq 2 \quad (k \in \{1, 2, 3, \cdots\}). \tag{1.13}$$

Lemma 1.5. [32] *Let the function $\alpha \in \mathcal{P}$ given by (1.12), then*

$$2\alpha_2 = \alpha_1^2 + x(4 - \alpha_1^2) \tag{1.14}$$

and

$$4\alpha_3 = \alpha_1^3 + 2\alpha_1(4 - \alpha_1^2)x - \alpha_1(4 - \alpha_1^2)x^2 + 2(4 - \alpha_1^2)(1 - |x|^2)z \tag{1.15}$$

for some x, z, $|x| \leq 1$ and $|z| \leq 1$.

2. Coefficients Bounds for the Functions of Class $\mathcal{W}_\Sigma(\tau, v)$

Theorem 2.1. *Let $f \in \mathcal{W}_\Sigma(\tau, v)$. Then:*

$$|s_2| \leq \sqrt{\Omega_1(\tau, v)},$$

$$|s_3| \leq \frac{(2v-1)^2}{16\tau^2} + \frac{2v-1}{6\tau}$$

and

$$|s_4| \leq \frac{(46\tau^2 + 69\tau - 227)(2v-1)^3}{8832\tau^3} + \frac{5(2v-1)^2}{48\tau^2} + \frac{4v^3 - 6v^2 + 1}{96\tau}$$

where

$$\Omega_1(\tau, v) = \frac{(2v-1)^3}{|2\tau(2v-1)^2(2\tau+1) - 16\tau^2(v^2 - 3v + 1)|}. \tag{2.1}$$

Proof. Let $f \in \Sigma$ given by (1.1) be in the class $\mathcal{G}_\Sigma(v, \sigma)$. Then

$$(f'(z))^\tau = L(v, a(z)) \tag{2.2}$$

and

$$(\mathcal{F}'(z))^\tau = L(v, b(w)). \tag{2.3}$$

We define $\alpha, \delta \in \mathcal{P}$ as follows:

$$\alpha(z) = \frac{1 + a(z)}{1 - a(z)} = 1 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$$

$$\Rightarrow \quad a(z) = \frac{\alpha(z) - 1}{\alpha(z) + 1} \quad (z \in \mathcal{U}) \tag{2.4}$$

and

$$\delta(w) = \frac{1 + b(w)}{1 - b(w)} = 1 + \delta_1 w + \delta_2 w^2 + \delta^3 w^3 + \cdots$$

$$\Rightarrow \quad b(w) = \frac{\delta(w)-1}{\delta(w)+1} \quad (w \in \mathcal{U}). \tag{2.5}$$

From (2.4) and (2.5), we get

$$a(z) = \frac{\alpha_1}{2}z + \left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{4}\right)z^2 + \left(\frac{\alpha_3}{2} - \frac{\alpha_1\alpha_2}{2} + \frac{\alpha_1^3}{8}\right)z^3 + \cdots \tag{2.6}$$

and

$$b(w) = \frac{\delta_1}{2}w + \left(\frac{\delta_2}{2} - \frac{\delta_1^2}{4}\right)w^2 + \left(\frac{\delta_3}{2} - \frac{\delta_1\delta_2}{2} + \frac{\delta_1^3}{8}\right)w^3 + \cdots. \tag{2.7}$$

Taking it from (2.6) and (2.7), we have:

$$\begin{aligned} L(v, a(z)) = \mathcal{E}_0(v) + \frac{\mathcal{E}_1(v)}{2}\alpha_1 z + \left[\frac{\mathcal{E}_1(v)}{2}\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\alpha_1^2\right]z^2 \\ + \left[\frac{\mathcal{E}_1(v)}{2}\left(\alpha_3 - \alpha_1\alpha_2 + \frac{\alpha_1^3}{4}\right) + \frac{\mathcal{E}_2(v)}{4}\alpha_1\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\alpha_1^3\right]z^3 + \cdots \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} L(v, b(w)) = \mathcal{E}_0(v) + \frac{\mathcal{E}_1(v)}{2}\delta_1 w + \left[\frac{\mathcal{E}_1(v)}{2}\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\delta_1^2\right]w^2 \\ + \left[\frac{\mathcal{E}_1(v)}{2}\left(\delta_3 - \delta_1\delta_2 + \frac{\delta_1^3}{4}\right) + \frac{\mathcal{E}_2(v)}{4}\delta_1\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\delta_1^3\right]w^3 + \cdots. \end{aligned} \tag{2.9}$$

It follows from (2.2), (2.3), (2.8), and (2.9) that we have:

$$2\tau s_2 = \frac{\mathcal{E}_1(v)}{2}\alpha_1 \tag{2.10}$$

$$3\tau s_3 + 2\tau(\tau-1)s_2^2 = \frac{\mathcal{E}_1(v)}{2}\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\alpha_1^2 \tag{2.11}$$

$$\begin{aligned} 4\tau s_4 + 6\tau(\tau-1)s_2 s_3 + \frac{4}{3}\tau(\tau-1)(\tau-2)s_2^3 = \frac{\mathcal{E}_1(v)}{2}\left(\alpha_3 - \alpha_1\alpha_2 + \frac{\alpha_1^3}{4}\right) \\ + \frac{\mathcal{E}_2(v)}{4}\alpha_1\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\alpha_1^3 \end{aligned} \tag{2.12}$$

$$-2\tau s_2 = \frac{\mathcal{E}_1(v)}{2}\delta_1 \tag{2.13}$$

$$-3\tau s_3 + (6\tau + 2\tau(\tau-1))s_2^2 = \frac{\mathcal{E}_1(v)}{2}\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\delta_1^2 \tag{2.14}$$

$$\begin{aligned} -\frac{4}{3}\tau(\tau+2)(\tau+4)s_2^3 + 2\tau(3\tau+7)s_2 s_3 - 4\tau s_4 = \frac{\mathcal{E}_1(v)}{2}\left(\delta_3 - \delta_1\delta_2 + \frac{\delta_1^3}{4}\right) \\ + \frac{\mathcal{E}_2(v)}{4}\delta_1\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\delta_1^3. \end{aligned} \tag{2.15}$$

Adding (2.10) and (2.13) and further simplification, we have

$$\alpha_1 = -\delta_1, \quad \alpha_1^2 = \delta_1^2 \text{ and } \alpha_1^3 = -\delta_1^3. \tag{2.16}$$

When (2.10) and (2.13) are squared and added, the following result is obtained:

$$8\tau^2 s_2^2 = \frac{\mathcal{E}_1^2(v)(\alpha_1^2+\delta_1^2)}{4} \tag{2.17}$$

$$\Rightarrow s_2^2 = \frac{\mathcal{E}_1^2(v)(\alpha_1^2+\delta_1^2)}{32\tau^2}. \tag{2.18}$$

Also, adding (2.11) and (2.14) gives

$$2\tau(2\tau+1)s_2^2 = \frac{2\mathcal{E}_1(v)(\alpha_2+\delta_2)+\alpha_1^2(\mathcal{E}_2(v)-2\mathcal{E}_(v))}{4}$$

$$8\tau(2\tau+1)s_2^2 = 2\mathcal{E}_1(v)(\alpha_2+\delta_2)+\alpha_1^2(\mathcal{E}_2(v)-2\mathcal{E}_(v)). \tag{2.19}$$

Applying (2.16) in (2.17)

$$\alpha_1^2 = \frac{16\tau^2}{\mathcal{E}_1^2(v)}s_2^2. \tag{2.20}$$

In (2.19), replacing α_1^2 with the following results:

$$|s_2|^2 \leq \frac{\mathcal{E}_1^3(v)(|\alpha_2|+|\delta_2|)}{4|\tau(2\tau+1)\mathcal{E}_1^2(v)-2\tau^2[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]|}. \tag{2.21}$$

Applying Lemma 1.4 and (1.5), we get:

$$|s_2| \leq \sqrt{\Omega_1(\tau,v)}$$

where $\Omega_1(\sigma,v)$ is given by (2.1).

Subtracting (2.14) and (2.11) and with some computation, we have

$$s_3 = s_2^2 + \frac{\mathcal{E}_1(v)(\alpha_2-\delta_2)}{12\tau} \tag{2.22}$$

$$s_3 = \frac{\mathcal{E}_1^2(v)\alpha_1^2}{16\tau^2} + \frac{\mathcal{E}_1(v)(\alpha_2-\delta_2)}{12\tau}. \tag{2.23}$$

Applying Lemma 1.4 and (1.5), we get:

$$|s_3| \leq \frac{(2v-1)^2}{16\tau^2} + \frac{2v-1}{6\tau}. \tag{2.24}$$

By removing (2.15) from (2.12), we arrive at:

$$\begin{aligned} s_4 = &\frac{-(46\tau^2+69\tau-227)\mathcal{E}_1^3(v)}{8832\tau^3}\alpha_1^3 + \frac{5\mathcal{E}_1^2(v)(\alpha_2-\delta_2)}{96\tau^2}\alpha_1 + \frac{\mathcal{E}_1(v)(\alpha_3-\delta_3)}{16\tau} \\ &+ \frac{[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](\alpha_2+\delta_2)}{32\tau}\alpha_1 + \frac{[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{192\tau}\alpha_1^3. \end{aligned} \tag{2.25}$$

Applying Lemma 1.4 and (1.5), we get:

$$|s_4| \leq \frac{(46\tau^2+69\tau-227)(2v-1)^3}{8832\tau^3} + \frac{5(2v-1)^2}{48\tau^2} + \frac{4v^3-6v^2+1}{96\tau}.$$

□

Taking $\tau = 1$ in Theorem 2.1, we have the next corollary.

Corollary 2.2. *Let* $f \in \mathcal{R}^*_\Sigma(v)$. *Then:*

$$|s_2| \leq \sqrt{\Omega_1(\tau, v)},$$
$$|s_3| \leq \frac{(2v-1)^2}{16} + \frac{2v-1}{6}$$

and

$$|s_4| \leq \frac{7(2v-1)^3}{552} + \frac{5(2v-1)^2}{48} + \frac{4v^3 - 6v^2 + 1}{96}$$

where

$$\Omega_1(v) = \frac{(2v-1)^3}{|6(2v-1)^2 - 16(v^2 - 3v + 1)|}.$$

3. Fekete-Szegö Inequalities for The Functions of Class $\mathcal{W}_\Sigma(\tau, v)$

Theorem 3.1. *Let* $f \in \mathcal{W}_\Sigma(\tau, v)$. *Then, for some* $\mu \in \mathbb{R}$,

$$\left|s_3 - \mu s_2^2\right| \leq \begin{cases} 2|1-\mu|\Omega_1(\tau, v) & \left(|1-\mu|\Omega_1(\tau, v) \geq \frac{2v-1}{6\tau}\right) \\ \\ \frac{2v-1}{3} & \left(|1-\mu|\Omega_1(\tau, v) < \frac{2v-1}{6\tau}\right), \end{cases}$$

where $\Omega_1(\tau, v)$ *is given by* (2.1).

Proof. From (2.22), we get:

$$s_3 - \mu s_2^2 = s_2^2 + \frac{\mathcal{E}_1(v)(\alpha_2 - \delta_2)}{12\tau} - \mu s_2^2.$$

Applying the popular triangular inequality, we get:

$$|s_3 - \mu s_2^2| \leq \frac{2v-1}{6\tau} + |1-\mu|\Omega_1(\tau, v)$$

If:

$$|1-\mu|\Omega_1(\tau, v) \geq \frac{2v-1}{6\tau}.$$

Futhermore, we get

$$|s_3 - \mu s_2^2| \leq 2|1-\mu|\Omega_1(\tau, v)$$

where

$$|1-\mu| \geq \frac{2v-1}{6\tau \cdot \Omega_1(\tau, v)}$$

and if:

$$|1-\mu|\Omega_1(\tau, v) \leq \frac{2v-1}{6\tau}$$

then, we get:

$$|s_3 - \mu s_2^2| \leq \frac{2v-1}{3}$$

where

$$|1-\mu| \leq \frac{2v-1}{6\tau \cdot \Omega_1(\tau, v)}$$

and $\Omega_1(\tau, v)$ is given in (2.1). □

Taking $\sigma = 1$ in Theorem 3.1, we have the next corollary.

Corollary 3.2. *Let $f \in \mathcal{R}^*_\Sigma(v)$. Then, for some $\mu \in \mathbb{R}$,*

$$\left|s_3 - \mu s_2^2\right| \leq \begin{cases} 2|1-\mu|\Omega_1(\tau, v) & \left(|1-\mu|\Omega_1(\tau, v) \geq \frac{2v-1}{6}\right) \\ \\ \frac{2v-1}{3} & \left(|1-\mu|\Omega_1(\tau, v) < \frac{2v-1}{6}\right), \end{cases}$$

where

$$\Omega_1(v) = \frac{(2v-1)^3}{|6(2v-1)^2 - 16(v^2 - 3v + 1)|}.$$

4. Second Hankel Determinant for the Class $\mathcal{W}_\Sigma(\tau, v)$

Theorem 4.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{W}_\Sigma(\tau, v)$. Then:*

$$H_2(2) = \left|s_2 s_4 - s_3^2\right| \leq \begin{cases} T(2, v) & (B_1 \geq 0 \text{ and } B_2 \geq 0) \\ \\ \max\left\{\left(\frac{2v-1}{6\tau}\right)^2, T(2, v)\right\} & (B_1 > 0 \text{ and } B_2 < 0) \\ \\ \left(\frac{2v-1}{6\tau}\right)^2 & (B_1 \leq 0 \text{ and } B_2 \leq 0) \\ \\ \max\left\{T(g_0, v), T(2, v)\right\} & (B_1 < 0 \text{ and } B_2 > 0). \end{cases}$$

Where

$$T(2, v) = \frac{(46\tau^2 + 69\tau - 227)\mathcal{E}_1^4(v)}{2208\tau^4} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{48\tau^2} + \frac{\mathcal{E}_1^4(v)}{16\tau^4},$$

$$T(g_0, t) = \frac{\mathcal{E}_1^2(v)}{9\tau^2} + \frac{97336 B_2^4}{9 B_1^3} + \frac{1058 B_2^3}{9\tau B_1^2},$$

$$B_1 = \mathcal{E}_1(v)\Bigg[3\mathcal{E}_1^3(v)(64\tau^2 + 96\tau - 227) + 138\tau^2(6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v)) + 414\mathcal{E}_1^3(v)$$
$$- 276\tau\mathcal{E}_1(v) - 1656\tau^2\mathcal{E}_1(v) + 736\tau^4\mathcal{E}_1^2(v)\Bigg]g^4,$$

$$B_2 = \mathcal{E}_1(v)\Big[3\mathcal{E}_1^2(v) + 11\tau\mathcal{E}_1(v) + 9\tau(\mathcal{E}_2(v) - 2\mathcal{E}_1(v))\Big]g^2.$$

Proof. From (2.10) and (2.25), we have

$$s_2 s_4 = \frac{-(46\tau^2 + 69\tau - 227)\mathcal{E}_1^4(v)}{35328\tau^4}\alpha_1^4 + \frac{5\mathcal{E}_1^3(v)(\alpha_2 - \delta_2)}{384\tau^3}\alpha_1^2 + \frac{\mathcal{E}_1^2(v)(\alpha_3 - \delta_3)}{64\tau^2}\alpha_1$$
$$+ \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v) - 2\mathcal{E}_1(v)](\alpha_2 + \delta_2)}{128\tau^2}\alpha_1^2 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v)]}{768\tau^2}\alpha_1^4.$$

With some calculations, we have

$$\begin{aligned} s_2s_4 - s_3^2 &= \frac{-(46\tau^2+69\tau-227)\mathcal{E}_1^4(v)}{35328\tau^4}\alpha_1^4 + \frac{\mathcal{E}_1^3(v)(\alpha_2-\delta_2)}{384\tau^3}\alpha_1^2 + \frac{\mathcal{E}_1^2(v)(\alpha_3-\delta_3)}{64\tau^2}\alpha_1 \\ &+ \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](\alpha_2+\delta_2)}{128\tau^2}\alpha_1^2 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{768\tau^2}\alpha_1^4 \\ &- \frac{\mathcal{E}_1^4(v)}{256\tau^4}\alpha_1^4 - \frac{\mathcal{E}_1^2(v)(\alpha_2-\delta_2)^2}{144\tau^2}. \end{aligned}$$

By using Lemma 1.5,

$$\alpha_2 - \delta_2 = \frac{(4-\alpha_1^2)(x-u)}{2} \tag{4.1}$$

$$\alpha_2 + \delta_2 = \alpha_1^2 + \frac{(4-\alpha_1^2)(x+u)}{2} \tag{4.2}$$

and

$$\begin{aligned} \alpha_3 - \delta_3 &= \frac{\alpha_1^3}{2} + \frac{4-\alpha_1^2}{2}\alpha_1(x+u) - \frac{4-\alpha_1^2}{4}\alpha_1(x^2+u^2) \\ &+ \frac{4-\alpha_1^2}{2}\left[(1-|x|^2z)-(1-|u|^2)w\right] \end{aligned} \tag{4.3}$$

for some x, u, z, w with $|x| \le 1,\ |u| \le 1,\ |z| \le 1,\ |w| \le 1, |\alpha_1| \in [0,2]$ and substituting $(\alpha_2+\delta_2),\ (\alpha_2-\delta_2)$ and $(\alpha_3-\delta_3)$, and after some strightforword simplifications, we have

$$\begin{aligned} s_2s_4 - s_3^2 &= \frac{-(46\tau^2+69\tau-227)\mathcal{E}_1^4(v)}{35328\tau^4}\alpha_1^4 + \frac{\mathcal{E}_1^3(v)(4-\alpha_1^2)(x-u)}{768\tau^3}\alpha_1^2 + \frac{\mathcal{E}_1^2(v)}{128\tau^2}\alpha_1^4 \\ &+ \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)(x+u)}{128\tau^2}\alpha_1^2 - \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)(x^2+u^2)}{256\tau^2}\alpha_1^2 \\ &+ \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)[(1-|x|^2z)-(1-|y|^2)w]}{128\tau^2}\alpha_1 + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]}{128\tau^2}\alpha_1^4 \\ &+ \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](4-\alpha_1^2)(x+u)}{256\tau^2}\alpha_1^2 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{768\tau^2}\alpha_1^4 \\ &- \frac{\mathcal{E}_1^4(v)}{256\tau^4}\alpha_1^4 - \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)^2(x-u)^2}{576\tau^2}. \end{aligned}$$

Let $g = \alpha_1$, assume without any restriction that $g \in [0,2],\ \eta_1 = |x| \le 1$, $\eta_2 = |u| \le 1$ and applying triangular inequality, we have

$$\begin{aligned} |s_2s_4 - s_3^2| \le &\left\{ \frac{(46\tau^2+69\tau-227)\mathcal{E}_1^4(v)}{35328\tau^4}g^4 + \frac{\mathcal{E}_1^2(v)}{128\tau^2}g^4 + \frac{\mathcal{E}_1^2(v)(4-g^2)}{64\tau^2}g \right. \\ &\left. + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]}{128\tau^2}g^4 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{768\tau^2}g^4 + \frac{\mathcal{E}_1^4(v)}{256\tau^4}g^4 \right\} \\ &+ \left\{ \frac{\mathcal{E}_1^3(v)(4-g^2)}{768\tau^3}g^2 + \frac{\mathcal{E}_1^2(v)(4-g^2)}{128\tau^2}g^2 + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](4-g^2)}{256\tau^2}g^2 \right\} \\ &(\eta_1+\eta_2) + \left\{ \frac{\mathcal{E}_1^2(v)(4-g^2)}{256\tau^2}g^2 - \frac{\mathcal{E}_1^2(v)(4-g^2)}{128\tau^2}g \right\}(\eta_1^2+\eta_2^2) + \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)^2}{576\tau^2}(\eta_1+\eta_2)^2 \end{aligned}$$

and equivalently, we have

$$|s_2s_4 - s_3^2| \leq M_1(v,g) + M_2(v,g)(\eta_1+\eta_2) + M_3(v,g)(\eta_1^2+\eta_2^2) + M_4(v,g)(\eta_1+\eta_2)^2 \quad (4.4)$$
$$= J(\eta_1,\eta_2)$$

where,

$$M_1(v,g) = \left\{ \frac{(46\tau^2+69\tau-227)\mathcal{E}_1^4(v)}{35328\tau^4}g^4 + \frac{\mathcal{E}_1^2(v)}{128\tau^2}g^4 + \frac{\mathcal{E}_1^2(v)(4-g^2)}{64\tau^2}g \right.$$
$$\left. + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]}{128\tau^2}g^4 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{768\tau^2}g^4 + \frac{\mathcal{E}_1^4(v)}{256\tau^4}g^4 \right\} \geq 0$$

$$M_2(v,g) = \left\{ \frac{\mathcal{E}_1^3(v)(4-g^2)}{768\tau^3}g^2 + \frac{\mathcal{E}_1^2(v)(4-g^2)}{128\tau^2}g^2 + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](4-g^2)}{256\tau^2}g^2 \right\} \geq 0$$

$$M_3(v,g)\left\{ \frac{\mathcal{E}_1^2(v)(4-g^2)}{256\tau^2}g^2 - \frac{\mathcal{E}_1^2(v)(4-g^2)}{128\tau^2}g \right\} \leq 0$$

$$M_4(v,g) = \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)^2}{576\tau^2} \geq 0$$

where $0 \leq g \leq 2$. Now, we maximize the function $Z(\lambda_1,\lambda_2)$ in the closed square

$$\Psi = \{(\eta_1,\eta_2) : \eta_1 \in [0,1], \eta_2 \in [0,1]\} \ for\ g \in [0,2].$$

For a fixed value of g, the coefficients of the function $J(\eta_1,\eta_2)$ in (4.4) are dependent on m, thus the maximum of $J(\eta_1,\eta_2)$ with regard to g must be investigated, taking into account the cases when $g=0,\ g=2$ and $g \in (0,2)$.

The First Case

When $g=0$,

$$J(\eta_1,\eta_2) = M_4(v,0) = \frac{\mathcal{E}_1^2(v)}{36\tau^2}(\eta_1+\eta_2)^2.$$

It is obvious that the function $J(\eta_1,\eta_2)$ reaches its maximum at (η_1,η_2) and

$$max\,\{J(\eta_1,\eta_2) : \eta_1,\eta_2 \in [0,1]\} = J(1,1) = \frac{\mathcal{E}_1^2(v)}{9\tau^2}. \quad (4.5)$$

The Second Case

When $g=2$, $J(\eta_1,\eta_2)$ is expressed as a constant function with respect to m, we have

$$J(\eta_1,\eta_2) = M_1(v,2) = \left\{ \frac{(46\tau^2+69\tau-227)\mathcal{E}_1^4(v)}{2208\tau^4} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{48\tau^2} + \frac{\mathcal{E}_1^4(v)}{16\tau^4} \right\}.$$

The Third Case

When $g \in (0,2)$, let $\eta_1+\eta_2 = d$ and $\eta_1 \cdot \eta_2 = q$ in this case, then (4.4) can be of the form

$$J(\eta_1,\eta_2) = M_1(v,g) + M_2(v,g)d + (M_3(v,g)+M_4(v,g))d^2 - 2M_3(v,g)l = Q(d,q) \quad (4.6)$$

where, $d \in [0, 2]$ and $q \in [0, 1]$. Now, we need to investigate the maximum of

$$Q(d, q) \in \Theta = \{(d, q) : d \in [0, 2], q \in [0, 1]\}. \tag{4.7}$$

By differentiating $Q(d, q)$ partially, we have

$$\frac{\partial Q}{\partial c} = M_2(v, g) + 2(M_3(v, g) + M_4(v, g))d = 0$$
$$\frac{\partial Q}{\partial l} = -2M_3(v, g) = 0.$$

These results reveal that $Q(d, g)$ does not have a critical point in Ψ, and so $J(\eta_1, \eta_2)$ does not have a critical point in the square Ψ.

As a result, the function $J(\eta_1, \eta_2)$ cannot have its maximum value in the interior of Ψ. The maximum of $J(\eta_1, \eta_2)$ on the boundary of the square Ψ will be investigated next.

For $\eta_1 = 0,\ \eta_2 \in [0, 1]$ (also, for $\eta_2 = 0, \eta_1 \in [0, 1]$) and

$$J(0, \eta_2) = M_1(v, g) + M_2\eta_2 + (M_3(v, g) + M_4(v, g))\eta_2^2 = D(\eta_2). \tag{4.8}$$

Now, since $M_3(v, g) + M_4(v, g) \geq 0$, then we have

$$D'(\eta_2) = M_2(v, g) + 2[M_3(v, g) + M_4(v, g)]\eta_2 > 0$$

which implies that $D(\eta_2)$ is an increasing function. Therefore, for a fixed $g \in [0, 2)$ and $v \in (1/2, 1]$, the maximum occurs at $\eta_2 = 1$. Thus, from (4.8),

$$\begin{aligned}\max\{G(0, \eta_2) : \eta_2 \in [0, 1]\} &= J(0, 1)\\ &= M_1(v, g) + M_2(v, g) + M_3(v, g) + M_4(v, g).\end{aligned} \tag{4.9}$$

For $\eta_1 = 1, \eta_2 \in [0, 1]$ (also, for $\eta_2 = 1$, $\eta_1 \in [0, 1]$) and

$$\begin{aligned}J(1, \eta_2) = M_1(v, g) + M_2(v, g) + M_3(v, g) + M_4(v, g) + [M_2(v, g)\\ +2M_4(v, g)]\eta_2 + [M_3(v, g) + M_4(v, g)]\eta_2^2 = N(\eta_2)\end{aligned} \tag{4.10}$$

$$N'(\eta_2) = [M_2(v) + 2M_4(v)] + 2[M_3(v) + M_4(v)]\eta_2. \tag{4.11}$$

We know that $M_3(v) + M_4(v) \geq 0$, then

$$N'(\eta_2) = [M_2(v) + 2M_4(v)] + 2[M_3(v) + M_4(v)]\eta_2 > 0.$$

Therefore, the function $N(\eta_2)$ is an increasing function and the maximum occurs at $\eta_2 = 1$. From (4.10), we have

$$\begin{aligned}\max\{J(1, \eta_2) : \eta_2 \in [0, 1]\} &= J(1, 1)\\ &= M_1(v, g) + 2[M_2(v, g) + M_3(v, g)] + 4M_4(v, g).\end{aligned} \tag{4.12}$$

Hence, for every $g \in (0,2)$, taking it from (4.9) and (4.12), we have

$$M_1(v,g) + 2[M_2(v,g) + M_3(v,g)] + 4M_4(v,g)$$
$$> M_1(v,g) + M_2(v,g) + M_3(v,g) + M_4(v,g).$$

Therefore,

$$\begin{aligned} & \max\{J(\eta_1,\eta_2) : \eta_1 \in [0,1], \eta_2 \in [0,1]\} \\ = \quad & M_1(v,g) + 2[M_2(v,g) + M_3(v,g)] + 4M_4(v,g). \end{aligned}$$

Since,

$$D(1) \le N(1) \quad for \quad g \in [0,2] \quad and \quad v \in [1,1],$$

then

$$\max\{J(\eta_1,\eta_2)\} = J(1,1)$$

occurs on the boundary of square Ψ.

Let $T : (0,2) \to \mathbb{R}$ defined by

$$T(v,g) = \max\{J(\eta_1,\eta_2)\} = J(1,1) = M_1(v,g) + 2M_2(v,g) + 2M_3(v,g) + 4M_4(v,g). \tag{4.13}$$

Now, inserting the values of $M_1(v,g), M_2(v,g), M_3(v,g)$ and $M_4(v,g)$ into (4.13) and with some calculations, we have

$$T(v,g) = \frac{\mathcal{E}_1^2(v)}{9\tau^2} + \frac{B_1}{105984\tau^4}g^4 + \frac{B_2}{288\tau^3}g^2, \tag{4.14}$$

where

$$\begin{aligned} B_1 = \mathcal{E}_1(v)\Bigg[& 3\mathcal{E}_1^3(v)(46\tau^2 + 69\tau - 227) + 138\tau^2(6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v)) + 414\mathcal{E}_1^3(v) \\ & - 276\tau\mathcal{E}_1(v) - 1656\tau^2\mathcal{E}_1(v) + 736\tau^4\mathcal{E}_1^2(v)\Bigg] g^4 \end{aligned}$$

$$B_2 = \mathcal{E}_1(v)\Bigg[3\mathcal{E}_1^2(v) + 11\tau\mathcal{E}_1(v) + 9\tau(\mathcal{E}_2(v) - 2\mathcal{E}_1(v))\Bigg] g^2.$$

If $T(v,g)$ has a maximum value in the interior of $g \in [0,2]$ and by applying some elementary calculus, we have

$$T'(v,g) = \frac{B_1}{26496\tau^4}g^3 + \frac{B_2}{144\tau^3}g.$$

Now, we need to examine the sign of the function $T'(v,g)$ depending on the signs of B_1 and B_2 as follows

1^{st} ***Result:***

Suppose $B_1 \ge 0$ and $B_2 \ge 0$ then,

$T'(v,g) \geq 0$. This shows that $T(v,g)$ is an increasing function on the boundary of $g \in [0,2]$ that is $g = 2$. Therefore,

$$\max\{T(v,g) : g \in (0,2)\} = \frac{(46\tau^2 + 69\tau - 227)\mathcal{E}_1^4(v)}{2208\tau^4} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{48\tau^2} + \frac{\mathcal{E}_1^4(v)}{16\tau^4}.$$

2^{rd} ***Result:***

If $B_1 > 0$ and $B_2 < 0$ then,

$$T'(v,g) = \frac{B_1 g^3 + 184\tau B_2 g}{26496\tau^4} = 0 \tag{4.15}$$

at critical point

$$g_0 = \sqrt{\frac{-184\tau B_2}{B_1}} \tag{4.16}$$

is a critical point of the function $T(v,g)$. Now,

$$T''(g_0) = \frac{-B_2}{48\tau^3} + \frac{B_2}{144\tau^3} > 0.$$

Therefore, g_0 is the minimum point of the function $T(v,g)$. Hence, $T(v,g)$ can not have a maximum.

3^{rd} ***Result:***

If $B_1 \leq 0$ and $B_2 \leq 0$ then,

$$T'(v,g) \leq 0.$$

Therefore, $T(v,g)$ is a decreasing function on the interval $(0,2)$. Hence,

$$\max\{T(v,g) : g \in (0,2)\} = T(0) = \frac{\mathcal{E}_1^2(v)}{9\tau^2}. \tag{4.17}$$

4^{th} ***Result:***

If $B_1 < 0$ and $B_2 > 0$

$$T''(v_0,g) = \frac{-B_2}{72} < 0.$$

Therefore, $T''(v,g) < 0$. Hence, g_0 is the maximum point of the function $T(v,g)$ and the maximum value occurs at $g = g_0$. Thus,

$$max\{T(v,g) : g \in (0,2)\} = T(g_0, s)$$

$$T(g_0,t) = \frac{\mathcal{E}_1^2(v)}{9\tau^2} + \frac{97336B_2^4}{9B_1^3} + \frac{1058B_2^3}{9\tau B_1^2}.$$

□

Taking $\tau = 1$ in Theorem 4.1, we have the next corollary.

Corollary 4.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{R}^*_\Sigma(v)$. Then:*

$$H_2(2) = |s_2 s_4 - s_3^2| \le \begin{cases} T(2,v) & (B_1 \ge 0 \text{ and } B_2 \ge 0) \\ \max\left\{\left(\frac{2v-1}{6}\right)^2, T(2,v)\right\} & (B_1 > 0 \text{ and } B_2 < 0) \\ \left(\frac{2v-1}{6}\right)^2 & (B_1 \le 0 \text{ and } B_2 \le 0) \\ \max\{T(g_0,v), T(2,v)\} & (B_1 < 0 \text{ and } B_2 > 0). \end{cases}$$

Where

$$T(2,v) = \frac{13\mathcal{E}_1^4(v)}{1104} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{48},$$

$$T(g_0,t) = \frac{\mathcal{E}_1^2(v)}{9} + \frac{97336B_2^4}{9B_1^3} + \frac{1058B_2^3}{9B_1^2},$$

$$B_1 = \mathcal{E}_1(v)\left[72\mathcal{E}_1^3(v) - 828\mathcal{E}_2(v) + 138\mathcal{E}_3(v) - 1104\mathcal{E}_1(v) + 736\mathcal{E}_1^2(v)\right]g^4,$$

$$B_2 = \mathcal{E}_1(v)\left[3\mathcal{E}_1^2(v) + 9\mathcal{E}_2(v) - 7\mathcal{E}_1(v)\right]g^2.$$

5. Conclusion

According to their usefulness in numerous various fields of mathematics and other sciences, as noted in the introduction section, special functions and polynomials have recently attracted the attention of numerous well-known mathematicians. For functions in the class $\mathcal{W}_\Sigma(\tau, v)$ of analytic and bi-univalent functions involving the Euler polynomials, we solved the Fekete-Szegö problem in this paper and also obtain bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant. According to [?, 11–24], the aforementioned conclusions can be extended to a class of specific q-bounded turning functions.

References

[1] P.L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, Band 259. Springer-Verlag, New York, 1983.

[2] V. Kac, P. Cheung, Quantum calculus, Springer, New York, 2002. https://doi.org/10.1007/978-1-4613-0071-7.

[3] M.E.H. Ismail, D. Stanton, G. Viennot, The Combinatorics of q-Hermite polynomials and the Askey–Wilson Integral, Eur. J. Comb. 8 (1987), 379-392. https://doi.org/10.1016/s0195-6698(87)80046-x.

[4] H.M. Srivastava, S. Owa, Current topics in analytic function theory, World Scientific, Singapore, 1992.

[5] H.m. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Camb. Phil. Soc. 129 (2000), 77–84. https://doi.org/10.1017/s0305004100004412.

[6] G.E. Andrews, q-Series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra, Conference Series in Mathematics, Vol. 66, American Mathematical Society, Providence, RI, 1986.

[7] N.J. Fine, Basic hypergeometric series and applications, Mathematical Surveys and Monographs, Vol. 27, American Mathematical Society, Providence, RI, 1988.

[8] T.H. Koornwinder, Orthogonal polynomials in connection with quantum groups, in: P. Nevai (Ed.), Orthogonal Polynomials, Springer Netherlands, Dordrecht, 1990: pp. 257-292. https://doi.org/10.1007/978-94-009-0501-6_12.

[9] T.H. Koornwinder, Compact quantum groups and q-special functions, in: V. Baldoni, M.A. Picardello (Eds.), Representations of Lie Groups and Quantum Groups, Pitman Research Notes in Mathematics Series, Vol. 311, Longman Scienti1c & Technical, New York, 1994, 46–128.

[10] N.Ya. Vilenkin, A.U. Klimyk, Representations of Lie groups and special functions, Vol. I, II, III, Kluwer Academic Publishers, Dordrecht, 1992.

[11] Q. Hu, T.G. Shaba, J. Younis, B. Khan, W.K. Mashwani, M. Çağlar, Applications of q-derivative operator to subclasses of bi-univalent functions involving Gegenbauer polynomials, Appl. Math. Sci. Eng. 30 (2022), 501-520. https://doi.org/10.1080/27690911.2022.2088743.

[12] S. Riaz, T.G. Shaba, Q. Xin, F. Tchier, B. Khan, S.N. Malik, Fekete–Szegö problem and second Hankel determinant for a class of bi-univalent functions involving Euler polynomials, Fractal Fract. 7 (2023), 295. https://doi.org/10.3390/fractalfract7040295.

[13] A.B. Patil, T.G. Shaba, Sharp initial coefficient bounds and the Fekete–Szegö problem for some certain subclasses of analytic and bi-univalent functions, Ukr. Mat. Zhurn. 75 (2023), 198-206. https://doi.org/10.37863/umzh.v75i2.6602.

[14] I. Al-shbeil, J. Gong, T.G. Shaba, Coefficients inequalities for the bi-univalent functions related to q-Babalola convolution operator, Fractal Fract. 7 (2023), 155. https://doi.org/10.3390/fractalfract7020155.

[15] C. Zhang, B. Khan, T.G. Shaba, J.S. Ro, S. Araci, M.G. Khan, Applications of q-Hermite polynomials to subclasses of analytic and bi-univalent functions, Fractal Fract. 6 (2022), 420. https://doi.org/10.3390/fractalfract6080420.

[16] I. Al-Shbeil, T.G. Shaba, A. Cătaş, Second Hankel determinant for the subclass of bi-univalent functions using q-Chebyshev polynomial and Hohlov operator, Fractal Fract. 6 (2022), 186. https://doi.org/10.3390/fractalfract6040186.

[17] B. Khan, Z.-G. Liu, T.G. Shaba, S. Araci, N. Khan, M.G. Khan, Applications of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials, J. Math. 2022 (2022), 8162182. https://doi.org/10.1155/2022/8162182.

[18] H. Orhan, T.G. Shaba, M. Çağlar, $(\mathfrak{P}, \mathcal{Q})$-Lucas polynomial coefficient relations of bi-univalent functions defined by the combination of Opoola and Babalola differential operators, Afr. Mat. 33 (2022), 11. https://doi.org/10.1007/s13370-021-00953-y.

[19] B. Khan, Z.-G. Liu, H.M. Srivastava, N. Khan, M. Darus, M. Tahir, A study of some families of multlivalent q-starlike functions involving higher-order q-derivatives, Mathematics. 8 (2020), 1470. https://doi.org/10.3390/math8091470.

[20] B. Khan, Z.-G. Liu, H.M. Srivastava, S. Araci, N. Khan, Q.Z. Ahmad, Higher-order q-derivatives and their applications to subclasses of multivalent Janowski type q-starlike functions, Adv. Differ. Equ. 2021 (2021), 440. https://doi.org/10.1186/s13662-021-03611-6.

[21] Y. Taj, S. Zainab, Q. Xin, F.M.O. Tawfiq, M. Raza, S.N. Malik, Certain coefficient problems for q-starlike functions associated with q-analogue of sine function, Symmetry. 14 (2022), 2200. https://doi.org/10.3390/sym14102200.

[22] L. Shi, M. Arif, J. Iqbal, K. Ullah, S.M. Ghufran, Sharp bounds of Hankel determinant on logarithmic coefficients for functions starlike with exponential function, Fractal Fract. 6 (2022), 645. https://doi.org/10.3390/fractalfract6110645.

[23] S. Riaz, U.A. Nisar, Q. Xin, S.N. Malik, A. Raheem, On starlike functions of negative order defined by q-fractional derivative, Fractal Fract. 6 (2022), 30. https://doi.org/10.3390/fractalfract6010030.

[24] I. Al-shbeil, J. Gong, S. Khan, N. Khan, A. Khan, M.F. Khan, Hankel and symmetric toeplitz determinants for a new subclass of q-starlike functions, Fractal Fract. 6 (2022), 658. https://doi.org/10.3390/fractalfract6110658.

[25] M. Fekete, G. Szegö, Eine bemerkung über ungerade schlichte funktionen, J. London Math. Soc. s1-8 (1933), 85-89. https://doi.org/10.1112/jlms/s1-8.2.85.

[26] H.M. Srivastava, T.G. Shaba, G. Murugusundaramoorthy, A.K. Wanas, G.I. Oros, The Fekete-Szegö functional and the Hankel determinant for a certain class of analytic functions involving the Hohlov operator, AIMS Math. 8 (2023), 340–360. https://doi.org/10.3934/math.2023016.

[27] A. Saliu, I. Al-Shbeil, J. Gong, S.N. Malik, N. Aloraini, Properties of q-symmetric starlike functions of Janowski type, Symmetry. 14 (2022), 1907. https://doi.org/10.3390/sym14091907.

[28] Ch. Pommerenke, On the coefficients and hankel determinants of univalent functions, J. London Math. Soc. s1-41 (1966), 111–122. https://doi.org/10.1112/jlms/s1-41.1.111.

[29] H.Y. Zhang, R. Srivastava, H. Tang, Third-order Hankel and Toeplitz determinants for starlike functions connected with the sine function, Mathematics. 7 (2019), 404. https://doi.org/10.3390/math7050404.

[30] B. Khan, I. Aldawish, S. Araci, M.G. Khan, Third Hankel determinant for the logarithmic coefficients of starlike functions associated with sine function, Fractal Fract. 6 (2022), 261. https://doi.org/10.3390/fractalfract6050261.

[31] A. Junod, Hankel determinants and orthogonal polynomials, Exposit. Math. 21 (2003), 63-74. https://doi.org/10.1016/s0723-0869(03)80010-5.

[32] R.J. Libera, E.J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative, Proc. Amer. Math. Soc. 87 (1983), 251–257.