Two-Body Spinless-Salpeter equation of unequal masses interacting with Coulomb-Hulthẻ n potential

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C.A. Onate [1*], O.S. Obaseki [1], M.C. Onyeaju [2], A.N. Ikot [2], O. Ebomwonyi[3] and N.K. Oladejo [1]

[1]Department of Physical Sciences, Landmark University, Omu-Aran, Nigeria,
[2]Physics Department, University of Port Harcourt, Port Harcourt, Nigeria
[3]Department of Physics, University of Benin, Benin City, Nigeria
Corresponding Author; onate.clement@lmu.edu.ng

Abstract

We obtained the analytical solutions of the two-body Salpeter equation via the methodology of supersymmetric quantum mechanics under a combination of Coulomb and Hulthén potentials for unequal masses. We clearly examined the energy eigenvalues for the ground state and excited states. The behaviour of energy with the sum of the masses and the screening parameter respectively, were also studied. The results showed that two bodies of unequal masses interacting within the system exhibit the same features.

Key words: Two body Salpeter equation; Supersymmetry quantum mechanics; potentials.

1. Introduction

In quantum mechanics, theoretical physicists have developed various wave equations that describe the behaviour of particles in different systems for various physical potential models. These equations are classified as either relativistic or nonrelativistic. Some of these equations are the Klein-Gordon equation which describes spin zero particle, Dirac equation that described spin-1/2 particle, Duffin-Kemmer-Petiau equation, Schrödinger equation and the Feinberg-Horodecki equation which deals mainly with the time-dependent counterpart of the Schrödinger equation for one dimensional system of quantized momentum. Another form of wave equation is the Bethe-Salpeter equation which has not been given much attention. The Salpeter equation describes in a covariant formalism, the bound states of the relativistic systems [1]. This equation is considered to be a generalization of the nonrelativistic Schrödinger equation in the relativistic scheme [2] which is non-local in nature [3]. The Salpeter equation popularly known as spinless-Salpeter equation, successfully has been applied in the description of bound states especially of quarks in hadrons [4-7], spin-averaged spectra of bound states which comprise constituents that are fermionic. Recently, some researchers have developed different traditional techniques to derive approximate analytical solutions of the spinless-Salpeter equation of different interactions for two-bodies with unequal masses [8-10]. Among the work reported for different potential models of interest are the relativistic study of the spinless-Salpeter equation with modified Hylleraas potential [11], Exact numerical solution of the spinless-Salpeter equation for the Coulomb potential in momentum space [12], On an approximation of the two-body spinless-Salpeter equation [13], Energy bounds for the spinless-Salpeter equation [14], Bound state inequality from the spinless-Salpeter equation with Yukawa potential [15], Study of Time-Evolution for approximation of Two-Body spinless-Salpeter equation in presence of time-dependent interaction [16], Analytical

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treatment of the two-body spinless-Salpeter equation with the Hulth n potential [17]. Motivated by two-body particles interacting through an exponential-type potential in the center of mass system, we intend to study spinless-Salpeter equation for a combination of Coulomb and Hulthėn potentials. The combined potential is given as

$$V(r) = -\frac{A}{r} + \frac{Ve^{-\delta r}}{1-e^{-\delta r}}, \tag{1}$$

Where A and V are potential strengths and δ is the screening parameter. As the screening parameter goes to zero, the combined potential (1) reduces to a combination of Coulomb and constant potential:

$$\lim_{\delta\to 0} V(r) = -\frac{A}{r} + V. \tag{2}$$

2. Two-Body Spinless-Salpeter Equation

The spinless-Salpeter equation for two-body systems interacting with a spherically symmetric potential in the center of mass system can be given as

$$\left[\sum_{i=1,2}\sqrt{\Delta + m_i^2} + V(r) - M\right]\chi(r) = 0, \tag{3}$$

where $\chi(r) = Y_{im}(\theta,\phi)R_{n,\ell}(r)$ and $\Delta = \nabla^2$. For heavy interacting particles, the approximation [21]

$$\sum_{i=1,2}\sqrt{\Delta + m_i^2} = m_1 + m_2 - \frac{\Delta(m_1+m_2)}{2m_1m_2} - \frac{\Delta^2(m_1+m_2)\left[(m_1+m_2)^2 - 3m_1m_2\right]}{(2m_1m_2)^3} - - -$$

(4)

is quite reliable. In the jargon, a Hamiltonian containing relativistic corrections up to order $(v2/c2)$ is called a generalized Breit-Femi Hamiltonian, which after some transformations takes the form [18-21]

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V_{n,\ell}(r) - \frac{V_{n,\ell}^2(r)}{2\overline{m}}\right]U_{n,\ell}(r) = 0, \tag{5}$$

where

$$V_{n,\ell}(r) = V(r) - E_{n,\ell}, \tag{6}$$

$$\overline{m} = \frac{(m_1m_2)^2}{(m_1+m_2)(m_1+m_2-3)} . \tag{7}$$

It is noted that potential (1) and equation (5) do not admit the solution for $\ell = 0$ otherwise, we take care of the centrifugal barrier. Hence, we employ the following Pekeris-type of approximation [22]

$$\frac{1}{r^2} \approx \frac{\delta}{\left(1-e^{-\delta r}\right)^2} . \tag{8}$$

Which is valid for $\delta << 1$. Substituting equations (1) and (8) into equation (5), we have

$$\frac{d^2 U_{n,\ell}(r)}{dr^2} = \left[V_{eff} + E_{eff}\right] U_{n,\ell}(r) , \tag{9}$$

Where

$$E_{eff} = \ell(\ell+1)\delta^2 - \frac{\mu\left(A^2\delta^2 + 2A\delta E_{n,\ell} + E_{n,\ell}^2\right)}{\overline{m}\hbar^2} - \frac{2\mu\left(A\delta + E_{n,\ell}\right)}{\hbar^2}, \tag{10}$$

$$V_{eff} = \frac{V_1 e^{-\delta r} - V_2 e^{-2\delta r}}{\left(1-e^{-\delta r}\right)^2} + \frac{V_3 e^{-\delta r}}{1-e^{-\delta r}}, \tag{11}$$

$$V_1 = \frac{2\mu A V\delta}{\overline{m}\hbar^2} - \frac{\mu A^2\delta^2}{\overline{m}\hbar^2} + \ell(\ell+1)\delta^2 , \tag{12}$$

$$V_2 = \frac{\mu V^2}{\overline{m}\hbar^2}, \tag{13}$$

$$V_3 = \frac{2\mu V E_{n,\ell}}{\overline{m}\hbar^2} - \frac{\mu A^2\delta^2}{\overline{m}\hbar^2} - \frac{2\mu A\delta E_{n,\ell}}{\overline{m}\hbar^2} + \frac{2\mu V}{\hbar^2} - \frac{2\mu A\delta}{\hbar^2} + \ell(\ell+1)\delta^2 . \tag{14}$$

By applying the methodology of the supersymmetric quantum mechanics formalism and shape invariance technique [23-26] to solve equation (9), the state at which n = 0 can be expressed as:

$$U_{0,\ell}(r) = \exp\left(-\int W(r)dr\right), \tag{15}$$

Where $W(r)$ is known as the superpotential function in supersymmetric quantum mechanics. Equation (15) gives a solution that satisfies the Riccati equation given in equation (9). Now, considering our potential (1) and the usefulness of equations (10) - (14), our choice of superpotential function which fits equation (15) can be written as

$$W(r) = \rho_0 + \frac{\rho_1 e^{\delta r}}{e^{\delta r} - 1}, \tag{16}$$

Where ρ_0 and ρ_1 are parametric constants that will soon be determined. The basic principle and approach of this method suggests that we relate equation (16) with equation (9). Therefore, the second-order differential equation given in equation (9) relates to equation (16) as

$$W^2(r) - \frac{dW(r)}{dr} = V_{eff} + E_{eff} , \tag{17}$$

Substituting equation (16) into equation (17) and compare the result with equation (9), we have the following

$$\rho_0^2 = E_{eff}\,, \tag{18}$$

$$\rho_1 = \frac{V_1}{\delta} - \frac{\mu V}{m\delta\hbar^2}\,, \tag{19}$$

$$\rho_0 = \frac{V_3 - \rho_1^2}{2\rho_1}\,. \tag{20}$$

To satisfied the shape invariance potential, we construct a pair of supersymmetry partner potentials $V_\pm(r) = W^2(r) \pm \frac{dW(r)}{dr}$ as

$$V_+(r) = W^2(r) + \frac{dW(r)}{dr} = \rho_0^2 + \frac{\rho_1(\rho_1 + 2\rho_0)}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1 - \delta)e^{-\delta r}}{(1-e^{-\delta r})^2}\,, \tag{21}$$

$$V_-(r) = W^2(r) - \frac{dW(r)}{dr} = \rho_0^2 + \frac{\rho_1(\rho_1 + 2\rho_0)}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1 + \delta)e^{-\delta r}}{(1-e^{-\delta r})^2}\,. \tag{22}$$

From equations (21) and (22), we discover that the family potentials $V_+(r)$ and $V_-(r)$ are shape invariant and thus, satisfied the shape invariance condition using the following [27-30]

$$V_+(a_0, r) = V_-(a_1, r) + R(a_1)\,, \tag{23}$$

Via mapping of the form $\rho_1 \to \rho_1 + \delta$, where $\rho_1 = a_0$. It is noted that $a_1 = f(a_0) = a_0 + \delta$, where a_1 is a new set of parameters specifically obtained from the old set of parameters a_0. $R(a_1)$ is a residual term that is totally independent of the variable r. Considering both the old and new set of parameters, the following recurrence relation holds: $a_2 = a_0 + 2\delta$, $a_3 = a_0 + 3\delta$, $a_4 = a_0 + 4\delta$ and subsequently, can be generalized as $a_n = a_0 + n\delta$. Using the shape invariant potentials, we establish the following relations [31-34]

$$R(a_1) = V_+(a_0, r) - V_-(a_1, r)\,, \tag{24}$$

$$R(a_2) = V_+(a_1, r) - V_-(a_2, r)\,, \tag{25}$$

$$R(a_3) = V_+(a_2, r) - V_-(a_3, r)\,, \tag{26}$$

.

$$R(a_n) = V_+(a_{n-1}, r) - V_-(a_n, r) \tag{27}$$

The energy levels of these equations can be determined via

$$E_{n,\ell} = \sum_{k=1}^{n} R(a_k) = V_+(a_0, r) - V_-(a_1, r)\,, \tag{28}$$

which on correct substitution of equations (18) - (20) with the consideration of the negative partner potential in equation (22), gives the full energy eigenvalue equation as

$$V_3 = \frac{\mu}{\hbar^2}\left(2V + \frac{E_{n,\ell}(2V + E_{n,\ell})(m_1 + m_2)(m_1 + m_2 - 3)}{(m_1 m_2)^2}\right) + \left[\frac{V_3}{2(V_1 - V_2 + n\delta)} + \frac{V_2 - V_1 - n\delta}{2}\right]^2. \quad (29)$$

In order to obtain the un-normalized radial wave function, we define a variable of the form $y = e^{-\delta r}$ and substitute it into equation (9) to have

$$\left(\frac{d^2}{dy^2} + \frac{1}{2y}\frac{d}{dy} + \frac{Py^2 - Qy + R}{[y(1-y)]^2}\right)U_{n,\ell}(y) = 0, \quad (30)$$

where

$$P = \frac{2\mu(V + E_{n,\ell})}{\delta^2\hbar^2} + \frac{\mu(2VE_{n,\ell} + 2V^2 + E_{n,\ell}^2)}{\overline{m}\delta^2\hbar^2}, \quad (31)$$

$$Q = \frac{2\mu(V + \delta A + 2E_{n,\ell})}{\delta^2\hbar^2} + \frac{2\mu(VE_{n,\ell} + A\delta E_{n,\ell} + \delta AV + E_{n,\ell}^2)}{\overline{m}\delta^2\hbar^2}, \quad (32)$$

$$R = \frac{2\mu(\delta A + E_{n,\ell})}{\delta^2\hbar^2} + \frac{\mu(2A\delta E_{n,\ell} + \delta^2 A^2 + E_{n,\ell}^2)}{\overline{m}\delta^2\hbar^2} - \ell(\ell+1). \quad (33)$$

Analyzing the asymptotic behaviour of equation (30) at origin and at infinity, it can be tested that as $r \to 0$ and ∞ $(y \to 1,0)$, equation (30) has a solution of the form $U_{n,\ell}(y) = (1-y)^{\lambda}$ and $U_{n,\ell}(y) = y^{\eta}$, where

$$\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{(1+2\ell)^2 + \frac{4\mu(V-A)(V-A)}{\overline{m}\hbar^2}}, \quad (34)$$

$$\eta = \sqrt{-\frac{2\mu(\delta A + E_{n,\ell})}{\delta^2\hbar^2} - \frac{\mu(2A\delta E_{n,\ell} + \delta^2 A^2 + E_{n,\ell}^2)}{\overline{m}\delta^2\hbar^2} + \ell(\ell+1)}. \quad (35)$$

Taking the wave function $U_{n,\ell}(y) = y^{\eta}(1-y)^{\lambda} f(y)$ and then substituting it into equation (30), we get

$$f''(y) + f'(y)\frac{(2\eta+1) - (2\lambda + 2\eta + 1)y}{y(1-y)} - f(y)\frac{(\lambda+\eta)^2 + P}{y(1-y)} = 0. \quad (36)$$

Equation (36) is authenticated by the hypergeometric function with a solution of the form

$$f(y) = {}_2F_1(-n, n + 2(\lambda+\eta); 2\eta+1, y). \quad (37)$$

When the function $f(y)$ is replaced by the hypergeometric function, the complete radial wave function can then be written as

$$U_{n,\ell}(y) = N_{n,\ell}\, y^{\eta}(1-y)^{\lambda}\, {}_2F_1(-n, n + 2(\lambda+\eta); 2\eta+1, y), \quad (38)$$

where $N_{n,\ell}$ is a normalization constant.

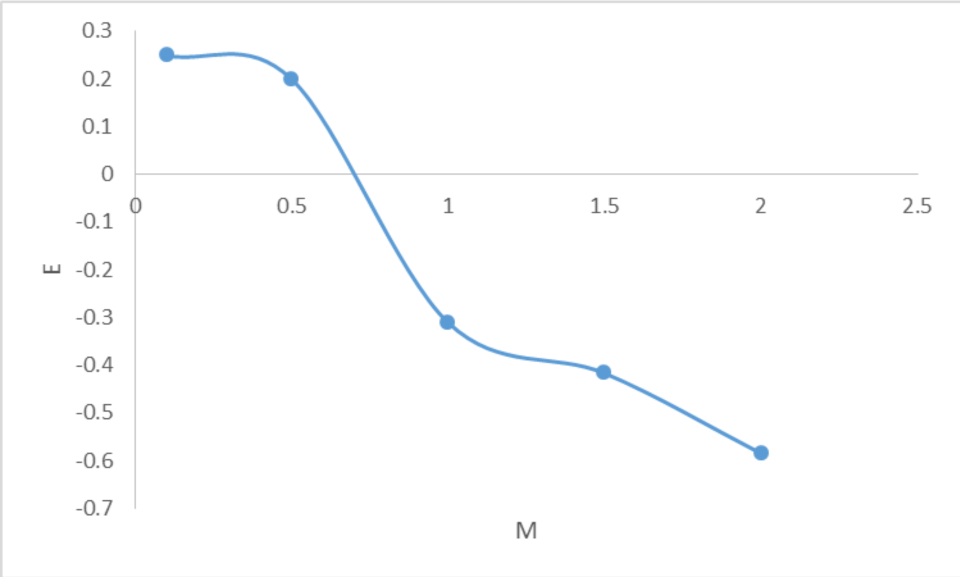

Fig 1: Energy versus mass $M=\left(m_1+m_2\right)$ with $A=2$, $V=\hbar=\mu=\ell=1$ and $\delta=0.25$ at the ground state.

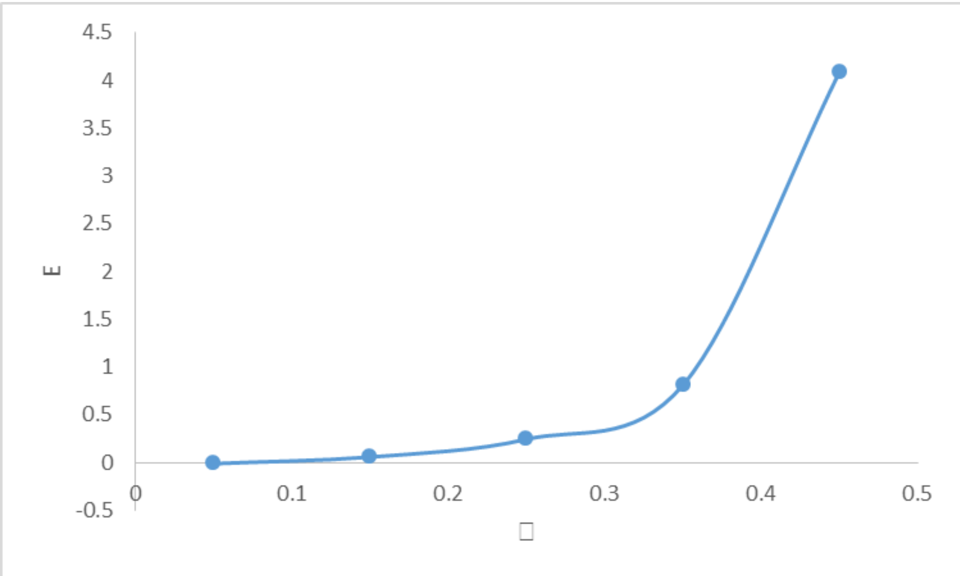

Fig 2: Energy versus the screening parameter δ at the ground state with $A=2$, $V=\mu=\hbar=\ell=1$ and $m_1=m_2=0.1$

Table 1: Bound state energy eigenvalue for unequal masses $(m_1 \neq m_2)$ with $\mu = \hbar = \ell = 1$ and $\delta = 0.25$.

$m_1 > m_2$, $m_1 = 0.2$ and $m_2 = 0.1$				$m_1 < m_2$, $m_1 = 0.1$ and $m_2 = 0.2$			
n	$A > V$	$A < V$	$A = V$	n	$A > V$	$A < V$	$A = V$
0	0.227639 0.238916	0.017804 0.021904	0.027303 0.034184	0	0.227639 0.238916	0.017804 0.021904	0.027303 0.034184
1	0.227656 0.238899	0.017811 0.021896	0.027315 0.034172	1	0.227656 0.238899	0.017811 0.021896	0.027315 0.034172
2	0.227673 0.238882	0.017818 0.021889	0.027328 0.034160	2	0.227673 0.238882	0.017818 0.021889	0.027328 0.034160
3	0.227690 0.238865	0.017826 0.021881	0.027340 0.034148	3	0.227690 0.238865	0.017826 0.021881	0.027340 0.034148
4	0.227707 0.238848	0.017833 0.021874	0.027352 0.034135	4	0.227707 0.238848	0.017833 0.021874	0.027352 0.034135
5	0.227724 0.238831	0.017841 0.021867	0.027365 0.034123	5	0.227724 0.238831	0.017841 0.021867	0.027365 0.034123

Table 2: Bound state energy eigenvalue for unequal masses $(m_1 \neq m_2)$ with $\mu = \hbar = 1$, $\ell = 2$ and $\delta = 0.25$.

$m_1 > m_2$, $m_1 = 0.2$ and $m_2 = 0.1$				$m_1 < m_2$, $m_1 = 0.1$ and $m_2 = 0.2$			
n	$A > V$	$A < V$	$A = V$	n	$A > V$	$A < V$	$A = V$
0	0.228560 0.239809	0.018171 0.022258	0.027917 0.034777	0	0.228560 0.239809	0.018171 0.022258	0.027917 0.034777
1	0.228577 0.239792	0.018179 0.022251	0.027929 0.034765	1	0.228577 0.239792	0.018179 0.022251	0.027929 0.034765
2	0.228594 0.239775	0.018186 0.022243	0.027941 0.034753	2	0.228594 0.239775	0.018186 0.022243	0.027941 0.034753
3	0.228611 0.239758	0.018193 0.022236	0.027954 0.034740	3	0.228611 0.239758	0.018193 0.022236	0.027954 0.034740
4	0.228628 0.239741	0.018201 0.022229	0.027966 0.034728	4	0.228628 0.239741	0.018201 0.022229	0.027966 0.034728
5	0.228625 0.239724	0.018208 0.022221	0.027978 0.034715	5	0.228645 0.239724	0.018208 0.022221	0.027978 0.034715

3. Discussion

In Fig 1, we examined the behaviour of the ground state energy against the sum of the masses $M(m_1 + m_2)$ with the parameters given below the Fig. As it is observed, the energy decreases when the sum of the masses increases. The Fig also indicates negative energies for some values of sum of the masses. For $M \geq 0.75$, the interacting particles become more attractive. In Fig 2, we plotted energy against the screening parameters for the ground state with the parameters given immediately after the Fig. The Fig shows that as the screening parameter increases, the energy equally increases. It is noted that as $\delta \to 0$, the Coulomb-Hulthén potential reduces to Coulomb-Constant potential, in such a situation, a decrease in energy is expected. In Tables 1 and 2, we presented the eigenvalues for different quantum numbers with $\ell = 1$ and $\ell = 2$. In Table 1, when $\ell = 1$ with three values for each of the potential strength, it is noted that the energies obtained with $A = 2$ and $V = 1$ i.e. $(A > V)$ are greater than the energies obtained with $A = 1$ and $V = 2$ i.e. $(A < V)$ for both $m_1 > m_2$ and $m_2 > m_1$. Thus, Coulomb potential's strength is more sensitive to energy increase compared to the strength of the Hulthén potential. It is observed from Tables 1 and 2 that the energies obtained with $\ell = 1$ are lesser than the energies obtained with $\ell = 2$. The numerical results in the two Tables revealed that the energies for $m_1 > m_2$ are equal to the energies obtained with $m_2 > m_1$ for both $\ell = 1$ and $\ell = 2$. This simply means that for two heavy particles of different masses interacting in the system, they both exhibit the same characteristics/effects.

4. Conclusion

The approximate analytical solutions of two-body Salpeter equation under Coulomb-Hulthén potential have been obtained. It is noted that an increase in the screening parameter increases the energy of the system. It is also noted that the more the quantum number, the more the energy of the system. Our results showed that the differences in the masses of two bodies interacting within the system do not affect the features of the particles.

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