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SOBOLEV-TYPE INEQUALITIES AND COMPLETE RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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Abstract. Let M be an n-dimensional complete Riemannian manifold, we investigate the geometric nature of such M which admits any of the family of Sobolev-type inequalities with the optimal Euclidean Sobolev constant. This leads to several conditions under which M with nonnegative Ricci curvature is isometric to Euclidean space $\mathbb{R}^n$.

1. Introduction

In this paper, the investigation of geometric natures of complete Riemannian manifolds satisfying any of Sobolev-type inequalities yields several conditions under which such manifolds with nonnegative Ricci curvature are isometric to Euclidean spaces. Let (M, g) be an n-dimensional complete Riemannian manifold, g denote the Riemannian metric, dv the volume element on M and C a finite constant. The family of Sobolev-type inequalities, usually called Gagliardo-Nirenberg inequalities, is the following

$$\Big(\int_M |f|^r dv\Big)^{\frac{1}{r}} \leq C\Big(\int_M |\nabla f|^q dv\Big)^{\frac{\theta}{q}}\Big(\int_M |f|^s dv\Big)^{\frac{1-\theta}{s}}, \quad f \in C_0^\infty(M) \tag{1.1}$$

with

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{s}, \qquad p = \frac{nq}{n-q}, \qquad 1 \leq q < n, \tag{1.2}$$

for some $0 < r, s \leq +\infty$, $0 < \theta \leq 1$. Here ∇ is the gradient operator on M and $C_0^\infty(M)$ is the space of smooth functions on M with compact support.

The family of inequalities (1.1) generalizes a number of equivalent functional inequalities on a complete manifold. For instance, when $\theta = 1$ and $r = p$, it becomes the well known Sobolev inequality

$$\Big(\int_M |f|^p dv\Big)^{\frac{1}{p}} \leq C(n,q)\Big(\int_M |\nabla f|^q dv\Big)^{\frac{1}{q}}, \quad f \in C_0^\infty(M). \tag{1.3}$$

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The case $q=1$ in (1.3) is the usual L^1-Sobolev inequality which is also equivalent to the isoperimetric inequality

$$Vol_n(\Omega)^{\frac{n-1}{n}} \leq C(n,1) Vol_{n-1}(\partial\Omega), \tag{1.4}$$

where Ω is a bounded subset of M, $\partial\Omega$ its boundary, i.e., the $(n-1)$-dimensional hypersurface in M and $Vol_n(\Omega)$ is the volume of Ω. The critical value $q=2$ when $\theta=1$ gives $p=2n/(n-2)$, $n\geq 3$ and the L^2-Sobolev inequality

$$\Big(\int_M |f|^{2n/(n-2)}dv\Big)^{(n-2)/n} \leq C(n,2)^2\Big(\int_M |\nabla f|^2 dv\Big), \quad f\in C_0^\infty(M). \tag{1.5}$$

A careful choice of θ whenever $q=2=r$ in (1.1) yields another useful inequalities. For example, choosing $\theta=n/(n+2)$ yields the classical Nash inequality $(s=1)$

$$\Big(\int_M |f|^2 dv\Big)^{\frac{n+2}{n}} \leq C_2\Big(\int_M |f|dv\Big)^{\frac{4}{n}}\int_M |\nabla f|^2 dv, \quad f\in C_0^\infty(M), \tag{1.6}$$

where C_2 is also a finite constant depending on $C(n,2)$. The limiting case $\theta\to 0$ yields the corresponding Logarithmic Sobolev inequality $(s\to r,\ s<r)$

$$\int_M |f|^2\log|f|^2 dv \leq \frac{n}{2}\log\Big(C(n,2)^2\int_M|\nabla f|^2 dv\Big), \quad f\in C_0^\infty(M) \tag{1.7}$$

with $\int_M f^2 dv = 1$.

There are numerous results involving the optimal constant for which the aforementioned inequalities hold, see for examples [4]-[7], [8]-[11]. The optimal constant in the case $M=\mathbb{R}^n$ has been computed explicitly. For instance Aubin [3] and Talenti [21] give C in (1.3) as

$$C(n,q)=K(n,q)=\frac{1}{n}\left(\frac{n(q-1)}{n-q}\right)^{(q-1)/q}\left(\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right)^{1/n}, \tag{1.8}$$

where $\Gamma(\cdot)$ is the Gamma function, with $q>1$. For $q=1$

$$K(n,1)=\frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n} \tag{1.9}$$

and for $q=2$

$$K(n,2)=\sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}. \tag{1.10}$$

The best possible constant C_2 admissible in (1.6)(when $M=\mathbb{R}^n$) is due to Carlen and Loss [6]

$$C_2=\frac{2}{n\lambda_1^N(\mathbb{B}^n)\omega_n^{2/n}}\left(\frac{n+2}{2}\right)^{\frac{n+2}{2}}, \tag{1.11}$$

where $\lambda_1^N(\mathbb{B}^n)$ is the first nonzero Neumann eigenvalue of the Laplacian on the unit Euclidean ball. Here ω_n is the volume of Euclidean unit ball (ω_{n-1} is the volume of the standard unit sphere) in $\mathbb{R}^n$, precisely, $\omega_{n-1}=2\pi^{n/2}/\Gamma(n/2)$. Using Stirling's asymptotic formula, it is straightforward to compute (for n large)

$$K(n,2)^2\sim\frac{2}{\pi e n}.$$

For details on the equivalence of the above inequalities see Bakry-Coulhon-Ledoux [4] and Grigor'yan [13]. These inequalities have been applied greatly in the study of diffusion processes on general Riemannian manifold [5, 8, 22]. They can also be used to obtain certain geometric estimates on the underlying manifolds, most especially when combined with nonnegativity condition on the Ricci curvature tensor of the manifold.

Now we give some definitions to explain certain geometric estimates appearing in the results that will follow. These definitions are linked with Bishop volume comparison theorem when the Ricci curvature of M is bounded from below by $(n-1)k,\ \ k \geq 0$ (see also [1]).

Definition 1.1. **(Ricci curvature tensor)** *Let x be a point in an n-dimensional Riemannian manifold M. The Ricci curvature Ric_x is defined on the tangent space T_xM as*

$$Ric_x(v,v) := trace\{w \longrightarrow \mathcal{R}(v,w)v\}, \quad v,w \in T_xM,$$

where $\mathcal{R}(X,Y)Z$ is the Riemann curvature tensor. In other words

$$Ric_x(u,v) := \sum_{i=1}^{n} g(\mathcal{R}(u,e_i)e_i,v), \quad u,v \in T_xM,$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis for T_xM. The Ricci curvature of M is said to be **nonnegative** *(i.e., $Ric_M \geq 0$) if $Ric_x \geq 0$ for all subspaces of T_xM.*

The Ricci curvature tensor Ric is a symmetric 2-tensor obtained by contraction of Riemann curvature tensor. Thus, it can be compared with metric tensor g. Hypothesis of the type $Ric \geq kg$, for some $k \in \mathbb{R}$ turns out to be sufficient to derive important analytic and geometric results. For instance, if $Ric \geq kg$ with $k = 0$, then M must be compact. If $k \geq 0$, the volume growth on (M,g) is atmost Euclidean, that is, for all $\rho > 0$

$$Vol(B_\rho(x)) \leq \omega_n \rho^n,$$

where $Vol(B_\rho(x))$ is the volume of the geodesic ball $B_\rho(x)$ centered at x with radius $\rho > 0$. In other words, Ricci curvature determines non-Euclidean behaviour of Manifold.

Definition 1.2. *We say that M satisfies* **volume doubling property** *if there exists a constant $\eta > 0$ such that*

$$Vol(B_{2r}(p)) \leq \eta Vol(B_r(p)), \qquad \forall\ p \in M,\ r > 0.$$

Here η is no less than 2^n and doubling constant 2^n implies M is isometric to $\mathbb{R}^n$. A simple calculation also shows that the last property implies $Vol(B_r(p)) \leq c_n r^n$.

Definition 1.3. **(Maximum volume growth)** *Define*

$$\theta_M := \liminf_{r\to\infty} \frac{Vol(B_r(p))}{r^n \omega_n}, \quad \forall r > 0 \tag{1.12}$$

as the asymptotic volume growth of M. Then M is said to have **maximum volume growth** *property when $\theta_M > 0$, i.e., $\lim_{r\to\infty} \frac{Vol(B_r(p))}{r^n} > 0, \quad \forall r > 0$.*

Note that the constant θ_M is a global invariant of M, that is, it is independent of the base point p. It is also easy to see that whenever M has maximum volume growth

$$Vol(B_r(p)) \geq \theta_M \omega_n r^n, \qquad \forall\ p \in M, r > 0.$$

By Bishop volume comparison theorem $\theta_M = 1$ implies M is isometric to $\mathbb{R}^n$.

Definition 1.4. *The heat kernel of M is a C^∞-positive function $H(t, x, y)$ on $(0, \infty) \times M \times M$, such that*

$$e^{-t\Delta} f(x) = \int_M H(t; x, y) f(y) d\mu(y), \quad f \in C^\infty(M)$$

and

$$(\partial_t - \Delta) H(t; x, \cdot) = 0, \quad \lim_{t \mapsto 0} H(t; x, y) = \delta_y(x)$$

in the sense of distribution, where $\delta_y(\cdot)$ is the dirac mass concentrated at y, Δ is the Laplace-Beltrami operator, $\int_M H(t; x, y) dv \leq 1$ and $e^{-t\Delta}$ is the heat semigroup on M.

The inequality $\int_M H(t; x, y) dv \leq 1$ shows that the heat kernel semigroup is contractive on L^p for any $1 \leq p \leq \infty$. Also by semigroup and symmetry properties we have

$$H(t_1 + t_2; x, y) = \int_M H(t_1; x, z) H(t_2; y, z) dv(z)$$
$$H(t; x, y) = H(t; y, x), \forall x, y \in M, \ t > 0.$$

Whenever the condition $\int_M H(t; x, y) dv = 1$ holds, we have taken H to be the limit of a sequence of Neumann heat kernel, i.e., if $H_k(t; x, y)$ is the fundamental solution to the heat equation on relatively compact sub-domain Ω_k, $k = 1, 2, ...$, with smooth boundary in M. Then, H is the corresponding limit of H_k. The books [14] by Grigor'yan and [18] by Li provide detail account of this.

One of the purposes of this paper is to study further the geometry of such manifold M on which one of the Sobolev-type inequalities (1.1) is satisfied with the best constant $C \geq K(n, q)$. The results in this direction are stated as Theorems 2.1 and 2.2 (see Section 2). Theorem 2.1 proves the existence of optimal constant $C \geq K(n, q)$ for which the family of the inequalities (1.1), and of course, any of its variants holds on M and that M is isometric to Euclidean $\mathbb{R}^n$ if and only if $C = K(n, q)$. Theorem 2.2 tells us that a complete Riemannian manifold which satisfies any of the Sobolev-type inequalities has Euclidean volume growth and indeed must satisfy volume doubling property and vice versa.

Section 3 discusses diagonal upper bounds for the heat kernel on complete Riemannian manifold taking into account the geometric nature of such manifold. In particular, we discuss an on-diagonal estimate on manifold satisfying volume doubling property. We also demonstrate via straightforward computation that diagonal estimates hold on manifolds satisfying Nash inequalities. Lastly, in this section, we provide a positive answer to a proposition of Ledoux [16], which says that an n-dimesional complete Riemannian manifold with nonnegative Ricci curvature satisfying the Nash inequality is isometric to an n-Euclidean space.

2. Sobolev inequalities and volume doubling property

The following theorem proves the existence of optimal constant $C \geq K(n, q)$ for which the family of the inequalities (1.1) holds on M.

Theorem 2.1. *Let (M, g) be an n-dimensional complete Riemannian manifold and let $q \in [1, n)$ be some real number. Given some number $0 < \theta \leq 1$, $s > 1$ and* *suppose*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{s}, \qquad \frac{1}{p} = \frac{1}{q} - \frac{1}{n}, \tag{2.1}$$

then, there exists a positive constant $C \geq K(n,q)$, the Euclidean Sobolev constant (optimal) such that (1.1) holds. Moreover, if $C = K(n,q)$, then M is isometric to $\mathbb{R}^n$.

Proof. We prove by contradiction. Suppose there exists a real number $q \in [1,n)$ and constant $C < K(n,q)$ such that $f \in C_0^\infty(M)$ and

$$\Big(\int_M |f|^r dv\Big)^{\frac{1}{r}} \leq A\Big(\int_M |\nabla f|^q dv\Big)^{\frac{\theta}{q}}\Big(\int_M |f|^s dv\Big)^{\frac{1-\theta}{s}}, \quad f \in C_0^\infty(M) \tag{2.2}$$

with $1/r = \theta/p + (1-\theta)/s$, where $1/p = 1/q - 1/n$. Fixing $x \in M$, it is easy to see that for any $\eta > 0$, there exists a chart (U_x, φ) at point x and there exists $\delta > 0$ such that $\varphi(U_x) = B_\delta(0)$, the Euclidean ball of radius δ centred at the origin, and such that the metric g in this chart satisfies

$$(1+\eta)^{-1}\delta_{ij} \leq g_{ij} \leq (1+\eta)\delta_{ij}$$

as bilinear forms (see Hebey [15]). Choosing η small enough we then have by (2.2) that there exists $\delta_0 > 0$, $C' < K(n,q)$ such that $\delta \in (0, \delta_0)$ and $f \in C_0^\infty(B_{\delta_0}(0))$,

$$\Big(\int_{B_\delta(0)} |f|^r dx\Big)^{\frac{1}{r}} \leq C'\Big(\int_{B_\delta(0)} |\nabla f|^q dx\Big)^{\frac{\theta}{q}}\Big(\int_{B_\delta(0)} |f|^s dx\Big)^{\frac{1-\theta}{s}}. \tag{2.3}$$

Let $f \in C_0^\infty(\mathbb{R}^n)$. Set $f_\lambda(x) = f(\lambda x)$, for $\lambda > 0$ large enough such that $f_\lambda \in C_0^\infty(B_{\delta_0}(0))$. Hence

$$\Big(\int_{B_{\delta_0}(0)} |f_\lambda|^r dx\Big)^{\frac{1}{r}} \leq A\Big(\int_{B_{\delta_0}(0)} |\nabla f_\lambda|^q dx\Big)^{\frac{\theta}{q}}\Big(\int_{B_{\delta_0}(0)} |f_\lambda|^s dx\Big)^{\frac{1-\theta}{s}}. \tag{2.4}$$

Follow the calculation in [15, Section 4.3], we have

$$\Big(\int_{B_{\delta_0}(0)} |f_\lambda|^r dx\Big)^{\frac{1}{r}} = \lambda^{-n/r}\Big(\int_{\mathbb{R}^n} |f|^r dx\Big)^{\frac{1}{r}}$$
$$\Big(\int_{B_{\delta_0}(0)} |f_\lambda|^s dx\Big)^{\frac{(1-\theta)}{s}} = \lambda^{-n(1-\theta)/r}\Big(\int_{\mathbb{R}^n} |f|^s dx\Big)^{\frac{(1-\theta)}{s}}$$
$$\Big(\int_{B_{\delta_0}(0)} |\nabla f_\lambda|^q dx\Big)^{\frac{\theta}{q}} = \lambda^{\theta(1-n/q))}\Big(\int_{\mathbb{R}^n} |\nabla f|^q dx\Big).$$

Substituting these expressions into (2.3) yields

$$\Big(\int_{\mathbb{R}^n} |f|^r dx\Big)^{\frac{1}{r}} \leq C''\Big(\int_{\mathbb{R}^n} |\nabla f|^q dx\Big)^{\frac{\theta}{q}}\Big(\int_{\mathbb{R}^n} |f|^s dx\Big)^{\frac{1-\theta}{s}} \tag{2.5}$$

and constant

$$C'' = \lambda^{n/r - n(1-\theta)/s + \theta(1-n/q)}C'.$$

A straightforward computation using (2.1) shows that

$$\frac{n}{r} - \frac{n(1-\theta)}{s} + \theta\Big(1 - \frac{n}{q}\Big) = n\Big(\frac{1}{r} - \frac{\theta}{p} + \frac{\theta-1}{s}\Big) = 0,$$

which yields that $C'' = C'$.

Since $C'' = C' < K(n,q)$, such an inequality in (2.5) is in contradiction to the fact that $K(n,q)$ is the optimal constant for which the inequality can hold on $\mathbb{R}^n$. This completes the proof of the theorem.

□

The next result tells us that a complete Riemannian manifold (compact or non compact) which satisfies any of the Sobolev-type inequalities has Euclidean volume growth and indeed must satisfy volume doubling property and vice versa. We note that Adriano and Xia in [2] have proved that a manifold with volume doubling property on which Sobolev-type inequalities hold must have Euclidean volume growth. As corollaries, they concluded that if the sectional curvature of M is nonnegative, the M is diffeomorphic to Euclidean space, and if the Ricci curvature of M is nonnegative, then its fundamental group is finite.

Theorem 2.2. *Let (M,g) be a complete Riemannian manifold with nonnegative Ricci curvature on which inequality (1.1) holds such that $q \in [1,n)$, $\theta \in (0,1]$, $0 < s \leq r \leq +\infty$. Then*

$$Vol(B_\rho(x)) \geq C_3 \rho^n, \quad \forall\ \rho > 0, \tag{2.6}$$

where

$$C_3 = (2^{n(n/q+(1-\theta)/s)+1} \cdot C)^{-n/\theta}.$$

In particular, if M satisfies an L^q-Sobolev inequality with $1/p = 1/q - 1/n$ for some p, then the volume growth function satisfies

$$\inf_{x\in M, \rho>0} \Big\{ \rho^{-n} Vol(B_\rho(x)) \Big\} > 0. \tag{2.7}$$

Proof. Let $d(x,p)$ be the distance function on M, $p \in M$. Note that $\|\nabla d\| = \sum_{i,j} d_i d_j \leq 1$ on $M \backslash cut\{p\}$. Since the set $cut\{p\}$ is star shaped with respect to x, we only take $d(x,p)$ to be Lipschitz continuous almost everywhere for $x \in M$.

Define a function $f : M \to \mathbb{R}$ such that

$$f(y) := \begin{cases} \rho - d(x,y), & d(x,y) \leq \rho, \\ \\ 0, & otherwise. \end{cases} \tag{2.8}$$

Notice that for $y \in B_{\rho/2}(x)$, $d(x,y) \leq \rho/2$ and $f(y) = \rho - d(x,y) \geq \rho/2$, then

$$\|f\|_r = \Big(\int_{B_\rho(x)} |f|^r \Big)^{\frac{1}{r}} \geq \Big(\int_{B_{\rho/2}(x)} |f|^r \Big)^{\frac{1}{r}}.$$

and also $|\nabla f| = |\nabla d| \leq 1$ (since gradient of distance is bounded by 1 almost everywhere). Hence

$$\|f\|_r \geq \frac{\rho}{2} Vol(B_\rho(x))^{1/r}, \ \|f\|_s \leq \rho Vol(B_{2\rho}(x))^{1/s} \ \text{ and } \ \|\nabla f\|_q \leq Vol(B_{2\rho}(x))^{1/q}.$$

Inserting these into the following inequalities

$$\|f\|_r \leq C \Big(\|\nabla f\|_q \Big)^\theta \Big(\|f\|_s \Big)^{1-\theta},$$

we have

$$Vol(B_\rho(x))^{1/r} \leq 2\rho^{-\theta} C Vol(B_{2\rho}(x))^{\theta/q + (1-\theta)/s}. \tag{2.9}$$

Recall that by [10, Theorem 5.5.1] we have

$$Vol(B_{2\rho}(x)) = 2^n Vol(B_\rho(x)). \tag{2.10}$$

Putting (2.9) and (2.10) together we have

$$Vol(B_\rho(x))^{1/r-\theta/q-(1-\theta)/s} \leq 2^{n(\theta/q+(1-\theta)/s)+1}\rho^{-\theta}C,$$

which yields (since $\theta/q + (1-\theta)/s - 1/r = \theta/n$)

$$Vol(B_\rho(x))^{\theta/n} \geq (2^{n(\theta/q+(1-\theta)/s)+1} \cdot C)^{-1}\rho^\theta$$

from where (2.6) follows.

For L^q-Sobolev inequality; we take $\theta = 1$ and it then follows from (2.6) that

$$\inf_{x\in M,\rho>0} \frac{Vol(B_\rho(x))}{\rho^n} > 0,$$

□

Corollary 2.3. *With the assumption of the theorem above. A complete Riemannian manifold M satisfying (2.6) must also satisfy volume doubling property. Furthermore M has Euclidean volume growth.*

The proof of the last Corollary follows from standard argument, it is therefore omitted.

3. Heat kernel bounds

The main aim of this section is to prove that a complete Riemannian manifold with nonnegative Ricci curvature and maximal volume growth is isometric to Euclidean space, if it satisfies the Nash inequality with optimal constant. We denote the volume of a ball of radius r centred at z by $VB(z, r)$.

3.1. **Diagonal bounds.** First, we discuss diagonal upper bounds for the heat kernel on complete Riemannian manifold. The main interest here is the geometric nature of such manifold. Precisely, we show an on-diagonal estimate on manifold satisfying volume doubling property.

One of the celebrated results of Li and Yau [19] shows that Gaussian estimates for heat kernel

$$\frac{C_1(n)}{VB(x,\sqrt{t})}\exp\Big(-\frac{d^2(x,y)}{D_1 t}\Big) \leq H(t;x,y) \leq \frac{C_2(n)}{VB(x,\sqrt{t})}\exp\Big(-\frac{d^2(x,y)}{D_2 t}\Big), \quad (3.1)$$

$C_i, D_i > 0, i = 1, 2$ hold on a complete manifold with nonnegative Ricci curvature and volume doubling property. In this section our interest is to discuss estimates similar to the above. We shall do this via on-diagonal estimates of the form

$$\frac{C_1(n)}{VB(x,\sqrt{t})} \leq H(t;x,x) \leq \frac{C_2(n)}{VB(x,\sqrt{t})}, \quad \forall\ x\ \in M,\ t > 0. \quad (3.2)$$

By Cauchy-Schwarz inequality, estimates (3.2) can be written as

$$\frac{C_1(n)}{VB(x,\sqrt{t})^{\frac{1}{2}}VB(y,\sqrt{t})^{\frac{1}{2}}} \leq H(t;x,x) \leq \frac{C_2(n)}{VB(x,\sqrt{t})^{\frac{1}{2}}VB(y,\sqrt{t})^{\frac{1}{2}}}, \quad (3.3)$$

$\forall\ x,\ y\ \in M,\ t > 0.$

The result is the following (the proof is omitted).

Theorem 3.1. *Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Fix x such that*

$$VB(x, 2\rho) \leq \eta VB(x, \rho), \quad \forall \rho > 0 \quad (3.4)$$

holds for some constant $\eta > 0$. Then there exists constants C_3, C_4 depending on n, t, such that

$$H(t; x, x) \geq \frac{C_3}{VB(x, \rho(t))}, \quad \forall\ x\ \in M,\ t > 0 \tag{3.5}$$

and

$$H(t; x, x) \leq \frac{C_4}{VB(x, \rho(t))}, \quad \forall\ x\ \in M,\ t > 0. \tag{3.6}$$

The above on-diagonal estimates (3.5) and (3.6) yield full Gaussian estimates (off-diagonal). Notice that by symmetry and semigroup properties of the heat kernel we have

$$H(t; x, x) = \int_M H^2(t/2, x, y)dv(y).$$

This implies that diagonal estimate is precisely the L^2-norm estimate

$$\|H(t/2, x, \cdot)\|_2^2 \leq \frac{C}{VB(x, \rho(t))}.$$

By the semigroup property for heat kernel and application of Cauchy-Schwarz inequality

$$\begin{aligned} H(t; , x, y) &= \int_M H(t/2; x, z)H(t/2; z, y)dv(z) \\ &\leq \Big(\int_M H^2(t/2; x, z)dv(z)\Big)^{\frac{1}{2}} \Big(\int_M H^2(t/2; y, z)dv(z)\Big)^{\frac{1}{2}} \\ &= \Big(H(t; x, x)\Big)^{\frac{1}{2}} \Big(H(t; y, y)\Big)^{\frac{1}{2}}. \end{aligned}$$

Thus, the question of finding off-diagonal bounds is now reduced to problem of finding on-diagonal estimates. For instance, if we could obtain on-diagonal estimates of the forms $H(t; x, x) \leq f(t),\ x \in M,\ H(t; y, y) \leq g(t),\ y \in M$, then

$$H(t; x, y) \leq \sqrt{f(t)}\sqrt{g(t)},\ \forall x, y \in M, t > 0.$$

To ensure this type of estimates reflect the distance between x and y (of the form Li-Yau estimates (3.1)) Grigor'yan [12, Proposition 5.1] estimated

$$H(t; x, y) \leq \sqrt{E_D(t/2, x)}\sqrt{E_D(t/2, y)} \exp\Big(-\frac{d^2(x, y)}{2Dt}\Big) \tag{3.7}$$

for any $D > 0$ and all $x, y \in M,\ t > 0$, where

$$E_D(t/2, x) := \int_M H^2(t; , x, z) \exp\Big(-\frac{d^2(x, y)}{Dt}\Big)dv(z).$$

3.2. **Nash inequality implies heat kernel bound.** Here we give an elementary argument to show that optimal heat kernel bounds hold on manifolds satisfying the Nash inequality with optimal constant. Noting that Sobolev inequality implies optimal heat kernel upper bounds of the form

$$H(t; x, x) \leq Ct^{-n/2}, \quad \forall\ x \in M,\ t > 0,$$

such that C depends on the Euclidean Sobolev constant in (1.5), Though, Sobolev inequality (1.5) is quite sensitive to the geometry of the manifold, recall that it implies Nash inequality (1.6) which holds also on manifold with Euclidean volume growth (see [20]).

Lemma 3.2. *If the Nash inequality (1.6) holds on the complete Riemannian manifold, then*

$$H(t;x,x) \le \frac{C'}{t^{n/2}}, \quad \forall\, x \in M,\ t > 0 \tag{3.8}$$

Proof. For any fixed $y \in M$, let $f = f(t,x) = H(t;x,y)$ with $f \in C_0^\infty(M)$ and $\int_M H(t;x,y)dv(x) \le 1$. Define

$$h(t) = \int_M f^2(t,x)dv.$$

Then by the heat equation and integration by parts

$$\frac{d}{dt}h(t) = \frac{\partial}{\partial t}\int_M f^2 dv = 2\int_M f\frac{\partial}{\partial t} f dv = -2\int_M |\nabla f|^2 dv.$$

Thus $h(t)$ is nonincreasing with respect to time. By using the Nash inequality (1.6) and the condition $\int_M f dv \le 1$, we obtain

$$\int_M |\nabla f|^2 dv \ge \frac{1}{C_2}\Big(\int_M |f|^2 dv\Big)^{\frac{n+2}{n}},$$

where C_2 is as defined in (1.11). Let $\alpha := 2/C_2$, we arrive at

$$\frac{d}{dt}h(t) = -2\int_M |\nabla f|^2 dv \le -\alpha h(t)^{\frac{n+2}{n}},$$

which implies

$$-\frac{n}{2}\Big(h(s)\Big)^{-\frac{2}{n}}\Bigg|_{s=0}^{s=t} \le -\alpha t.$$

Using the fact that

$$\lim_{t\to 0} h(t) = \lim_{t\to 0}\int_M f^2 dv = \int_M \lim_{t\to 0} H^2(t;x,y)dv(x) = \int_M \delta_y^2(x)dv(x) = 0,$$

we have

$$h(t) \le \Big(\frac{n}{2\alpha}\Big)^{\frac{n}{2}} t^{-\frac{n}{2}}.$$

That is (by semigroup property and Cauchy-Schwarz inequality)

$$H(2t;x,x) \le \Big(H(t;x,x)\Big)^{\frac{1}{2}}\Big(H(t;y,y)\Big)^{\frac{1}{2}} \le \Big(\frac{n}{2\alpha}\Big)^{\frac{n}{2}} t^{-\frac{n}{2}}. \tag{3.9}$$

Combining the last inequality with $h(t/2) = (n/\alpha)^{n/2}t^{-n/2}$ and $\alpha = 2/C_2$, we have

$$H(t;x,y) \le C' t^{-n/2},$$

where $C' = nC_2/2$. Since $h(t) = H(2t;x,x) = \int_M H^2(t;x,y)dv(x)$, the result then follows. □

Remark. *Sobolev inequality is not only sufficient but also necessary for the on-diagonal upper bound for heat kernel, Varopoulus* [22]. *Carlen, Kasuoka and Stroock* [5] *have proved that on-diagonal upper bound is equivalent to the Nash inequality. On-diagonal upper bound is equivalent to log-Sobolev inequality for any $f \in C_0^\infty(M), f \ge 0$, see Davies* [9, 10]. *Grigoryan proved localised on-diagonal upper estimate* [12, 13].

3.3. **Heat kernel asymptotics.** Ledoux [16] proposed that a nonnegatively Ricci curved n-dimensional Riemannian manifold M is isometric to $\mathbb{R}^n$ if M satisfies the Nash ineqaulity (1.7) with optimal constant (1.11). Our proof of this proposition relies on the optimal large time heat asymptotic proved by Li [17] and a heat kernel bound on Riemannian manifolds satisfying the Nash inquality. We show that such manifolds have maximal volume growth where the conclusion can easily follow.

Theorem 3.3. *(Li [17]) Let M be an n-dimensional complete Riemannian mannifold with nonnegative Ricci curvature. If (1.12) holds then the heat kernel $H(t,x,y)$ of M satisfies*

$$\liminf_{t\to\infty} VB(x,\sqrt{t})H(t,x,y) = \omega_n(4\pi)^{-\frac{n}{2}} \tag{3.10}$$

for any $x, y \in M$ along any path in $(0,\infty)\times M\times M$.

Li [17] asked whether or not limit of $VB(x,\sqrt{t})H(t,x,y)$ as $t\to\infty$ exists without the assumption of maximal volume growth. This has been answered in the negative by Xu [23]. The conclusion is that the limit does not necessarily exist if $\liminf_{r\to\infty} VB(x,r)/r^n = 0$. Another interesting application of the asymptotic rate is that a manifold of nonnegative Ricci curvature with maximal volume growth must have finite fundamental group.

Our result is the following

Theorem 3.4. *Let M be an n-dimensional complete Riemannian mannifold with nonnegative Ricci curvature. Suppose the Nash inequality (1.6) holds for any $f \in C_0^\infty(M)$ and optimal constant C_2 in (1.11), then M is isometric to $\mathbb{R}^n$.*

Proof. By the Nash inequality (1.6) and Lemma 3.2, there exists a positive constant C' depending on C_2, the optimal constant defined in (1.11), such that

$$\sup_{x,y\in M} H(t,x,y) \le C' t^{-n/2} \quad \forall\, t>0. \tag{3.11}$$

Applying Theorem 3.3, we combine (3.10) and (3.11) and obtain

$$\liminf_{t\to\infty} \frac{VB(x,\sqrt{t})}{t^{\frac{n}{2}}} \ge \frac{\omega_n(4\pi)^{-\frac{n}{2}}}{C'} > 0.$$

Then M has maximum volume growth. From the above one can easily see that for any $p\in M$

$$VB(p,r) \ge \theta_M \omega_n r^n, \tag{3.12}$$

where $\theta_M = \frac{(4\pi)^{-\frac{n}{2}}}{C'}$. Specifically, for $\theta_M \ge 1$ and as $r\to\infty$ we have from (3.12) that

$$VB(p,s) \ge \omega_n s^n.$$

On the other hand, by Bishop volume comparison

$$\frac{VB(p,r)}{\omega_n r^n} \le \frac{VB(p,s)}{\omega_n s^n}$$

for $s<r$ and in particular $VB(p,r)\le \omega_n r^n$ as $s\to 0$. We can therefore conclude that $VB(p,r)=\omega_n r^n$, for any $p\in M$ and $r>0$. Consequently, M is isometric to $\mathbb{R}^n$ by the equality case in Bishop's volume comparison theorem. □

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