Eigensolutions, scattering phase shift and thermodynamic properties of Hulthén-Yukawa potential

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ABSTRACT

The amendibility of a spin-0 and spin-1 particle with a combined potential in the presence of the Duffin-Kemmer-Petiau wave equation is highly recommendable. Thus, the approximate bound state of the Duffin-Kemmer-Petiau equation and Schrödinger equation were obtained with a combination of Hulthén and Yukawa potentials in the framework of asymptotic iteration method and parametric Nikiforov-Uvarov method respectively for any arbitrary angular momentum quantum number J using a suitable approximate scheme to the centrifugal term. This was done when the second-order homogeneous differential equation was transformed to a form of recurrence relation from which a quantization condition obtained was used to calculate the eigenvalue energy equation and the corresponding wave function. In other to apply more application to this work, the scattering phase shift of the Duffin-Kemmer-Petiau equation was calculated and the thermodynamic properties of the potential under consideration were also calculated in view of the Schrödinger equation. It is noted that the results obtained by varying the two strengths of the potential differs due to the effect of the screening parameter.

Introduction

In the recent years, the Duffin-Kemmer-Petiau (D.K.P.) equation as a relativistic version of a wave equation has been used to describe the interactions of the relativistic spin-0 and spin-1 bosons. The Duffin-Kemmer-Petiau equation under a vector potential possessed the same mathematical structure as that of the relativistic Klein-Gordon equation [1]. The equivalence of these two equations (Duffin-Kemmer-Petiau and Klein-Gordon) was presented by Feinberg and Pimentel [2,3] in the case of charged scalar particles interacting in a minimal way with quantized electromagnetic field. Due to the squared nature of the vector potential in the Duffin-Kemmer-Petiau equation, only a few potential models can interact with this equation and as such, only few selected potentials have been reported. Among the potentials reported with this equation are Coulomb potential [4–6], woods Saxon potential [7], deformed Hulthen potential [8], Yukawa potential [9], Varshni potential [10], Deng-Fan potential [11], Hellmann potential [12,13], Sexcillator [14], Kratzer potential [15] among others. The D.K.P. was also examined in the presence of magnetic field in (1 + 2)-dimensions by Darroodi et al. [16]. Similarly, Bouzid and Merad [17], obtained a space-time fractional D.K.P. equation and its solution. Recently, Hassanabadi et al. [18], investigated the D.K.P. equation for spin-zero system in the presence of Godel-type background space-time. In this study, we want to examine the admittance of the Hulthén-Yukawa type potential in the Duffin-Kemmer-Petiau equation and to further calculate the scattering phase shift of the potential under consideration with the same equation.

It is understood that complete information about quantum systems was obtained by the investigation of scattering state solutions with quantum mechanical potential terms [19,20]. Similarly, the scattering state can be used to analyze the interactions between various particles as well as atoms [21]. In the papers of Bohr et al. [22] and Dudek et al. [23], it was pointed out that the study of scattering state problems under Pseudo (spin) symmetry, revealed more understanding of the symmetries in Hadron and nuclear spectroscopy, deformation and superdeformation in nuclei. Thus, the understandings of more knowledge about fine-scales systems have been gained by the examination of scattering and bound state of such systems. Therefore, the study of scattering state becomes an interesting area in both the relativistic and nonrelativistic regimes. Thus, some authors studied the scattering state with some physical potential models. Among the authors and works on scattering state reported on scattering state so far include: Oluwadare and Oyewumi [10,24], obtained scattering states solutions of the Klein-Gordon equation with three physically solvable potential models and scattering state solution of the Duffin-Kemmer-Petiau equation with the varshni potential model respectively, Onyeaju et al. [25], studied

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scattering and bound states of the Klein Gordon particles with Hylleraas potential within effective mass formalism, Ikot et al. [26], in their study, investigated scattering state of Klein-Gordon particles by q-parameter Hyperbolic Pöschl-Teller potential, Yazarloo et al. [27], obtained the relativistic scattering states of the Hellmann potential. Adebimpe et al. [28], studied bound state and scattering phase shift of the Schrödinger equation with modified trigonometry scarf type potential. Eshghi and Abdi [29], studied relativistic particle scattering states. Eshghi and Mehraban [30–32], investigated the nonrelativistic particle scattering together with Pseudo-Coulomb potential plus ring-shape and energy-dependent potentials in D-dimensions, relativistic phase shift and effect of the q-deformed Pseudoscalar magnetic field and scattering state on the charge carriers in grapheme. The purpose of this work is to calculate approximate bound state, scattering phase shift of the relativistic Duffin-Kemmer-Petiau equation and thermodynamic properties of Hulthèn-Yukawa type potential. The Hulthèn-Yukawa type potential is given as

$$V(r) = -\frac{ae^{-\delta r}}{1 - e^{-\delta r}} - \frac{be^{-\delta r}}{r}, \tag{1}$$

where a and b are the strengths of the potential and δ is the screening parameter that characterized the range of the potential. This potential is a combination of Hulthèn potential and Yukawa potential. The Hulthèn potential has application in nuclear, particle, atomic, condensed matter and chemical physics [33]. The Yukawa potential on the other hand, has application in chain molecules [34], colloidal particles [35], stabilization of energy levels and the computations of bound state energies in neutral atoms [36,37]. It is therefore seen that the newly proposed potential given in Eq. (1) has many applications in science.

Asymptotic iteration method

In this section, we briefly outline the methodology of asymptotic iteration. AIM is proposed to solve the homogeneous linear second-order differential equation of the form [38,39]

$$y_n''(x) = \lambda_0(x)y_n'(x) + S_0(x)y_n(x), \tag{2}$$

where $\lambda_0(x) \neq 0$, the prime denotes the derivative with respect to x and n is the radial quantum number. The terms $S_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To get the solution, we take the derivative of the above equation with respect to x. The energy equation of any Schrödinger-like equation is obtained by transforming the equation into the form of Eq. (2) and then obtains the values of $\lambda_k(x)$ and $S_k(x)$ with $k > 0$ as follows [40–43]:

$$\lambda_k(x) = \lambda_{k-1}'(x) + S_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \tag{3}$$

$$S_k(x) = S_{k-1}'(x) + S_0(x)\lambda_{k-1}(x). \tag{4}$$

With Eqs. (3) and (4), one can obtain the quantization condition

$$\delta_k(x) = \begin{vmatrix} \lambda_k(x) & S_k(x) \\ \lambda_{k-1}(x) & S_{k-1}(x) \end{vmatrix} = 0, \ k = 1, 2, -, -, -, \tag{5}$$

The energy eigenvalues are then obtained from Eq. (5) if the problem is exactly solvable. A comprehensive/detail of the methodology can be found in the papers of Bayrak and Boztosun [38] and Falaye et al. [39]. Details of the methodology can be found in Appendix A

Bound state solutions of Dufin-Kemme-Petiau equation

To obtain energy equation for the relativistic Dufin-Kemme-Petiau wave equation of any quantum physical system, we solve the Schrödinge-like equation of the form [44,45]

$$\left[\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} - M^2 + (E_{n,J} - U_v^0)^2\right] F_{n,J}(r) = 0, \tag{6}$$

where, U_v^0 is the interacting potential, $E_{n,J}$ is the relativistic energy and

$F_{n,J}(r)$ is the wave function. It is noted that Eq. (6) cannot be solved for $J = 0$ due to the centrifugal barrier. Thus, we must apply an approximation scheme to deal with the centrifugal barrier. For a short potential range, the following approximation

$$\frac{J(J+1)}{r^2} \approx \frac{J(J+1)\delta^2}{(1 - e^{-\delta r})^2}, \tag{7}$$

is a good approximation scheme to the centrifugal barrier for $\delta r \ll 1$. Substituting Eqs. (1) and (7) into Eq. (6), we have

$$\left[\frac{d^2}{dr^2} - \frac{J(J+1)\delta^2}{(1 - e^{-\delta r})^2} - M^2 + \left(E_{n,J} + \frac{(a + b\delta)e^{-\delta r}}{1 - e^{-\delta r}}\right)^2\right] F_{n,J}(r) = 0, \tag{8}$$

and by making a transformation of the form $y = e^{-\delta r}$, the second order differential equation in Eq. (8) becomes

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + \frac{\xi_1 y^2 + \xi_2 y + \xi_3}{y^2(1 - y)^2}\right] F_{n,J}(y) = 0, \tag{9}$$

where

$$\xi_1 = \frac{E_{n,J}^2 - M^2 + (a + b\delta)(a + b\delta - 2E_{n\ell})}{\delta^2}, \tag{10}$$

$$\xi_2 = \frac{2(M^2 - E_{n,J}^2) + 2E_{n\ell}(a + b\delta)}{\delta^2}, \tag{11}$$

$$\xi_3 = \frac{E_{n,J}^2 - M^2}{\delta^2} - J(J+1). \tag{12}$$

In order to solve Eq. (9) using AIM, we transformed Eq. (9) into the form of Eq. (2). This makes us to write the physical wave function of the form

$$F_{n,J}(y) = y^u(1 - y)^v f(y), \tag{13}$$

where

$$u = \sqrt{J(J+1) + \frac{M^2 - E_{n,J}^2}{\delta^2}}, \tag{14}$$

$$v = 1 + J. \tag{15}$$

Inserting Eq. (13) into Eq. (9), we have a second order homogeneous linear differential equation as follow

$$f''(y) = \left[\frac{(2u+1) - (2u + 2J + 3)y}{y(1-y)}\right]f'(y) - \left[\frac{(u + 1 + J)^2 + \xi_1}{y(1-y)}\right]f(y)$$

$$= 0, \tag{16}$$

whose solution can be found by using the asymptotic iteration method. Comparing Eq. (16) with Eq. (2), we deduced the following

$$\lambda_0 = \frac{(2u + 2J + 3)y - (2u+1)}{y(1-y)}, \tag{17}$$

$$S_0 = \frac{(u + J + 1)^2 + \xi_1}{y(1-y)}. \tag{18}$$

Using Eqs. (3) and (4) and the quantization condition of Eq. (5), we can write the following relations:

$$S_0\lambda_1 = \lambda_0 S_1 \Leftrightarrow u + v = 0 + \sqrt{-\xi_1}, \tag{19}$$

$$S_1\lambda_2 = \lambda_1 S_2 \Leftrightarrow u + v = -1 + \sqrt{-\xi_1}, \tag{20}$$

$$S_2\lambda_3 = \lambda_2 S_3 \Leftrightarrow u + v = -2 + \sqrt{-\xi_1}, \tag{21}$$

$$S_3\lambda_4 = \lambda_3 S_4 \Leftrightarrow u + v = -3 + \sqrt{-\xi_1}, \tag{22}$$

$$S_n\lambda_{n+1} = \lambda_n S_{n+1} \Leftrightarrow u + v = -n + \sqrt{-\xi_1}, \tag{23}$$

When the preceding expressions are generalized, we obtain the

following equation

$$u^2 = [-(n + v) + \sqrt{-\xi_1}]^2, \qquad (24)$$

which can be fully written as

$$M^2 - E_{n,J}^2 + J(J + 1)\delta^2$$

$$= \delta^2 \left[\frac{\frac{2(a + b\delta + E_{n\ell})}{\delta^2} - J(J + 1) - (a + b\delta + n + J + 1)^2}{2(a + b\delta + n + J + 1)} \right]^2 \qquad (25)$$

Scattering phase shift

In other to calculate the phase shift, we define a variable of the form $z = 1 - e^{-\delta r}$ and substitute it into Eq. (8), we have

$$\left[(1 - z)^2 \frac{d^2}{dz^2} - (1 - z)\frac{d}{dz} + \frac{-\mathbb{R}z^2 + \sigma y - \wp}{z^2} \right] U_{n\ell}(z) = 0, \qquad (26)$$

where

$$\mathbb{R} = \frac{(a + b\delta)(a + b\delta - 2E_{nJ})}{\delta^2} - \frac{\kappa^2}{\delta^2} - J(J + 1), \qquad (27)$$

$$\sigma = \frac{2(a + b\delta)(E_{nJ} - a - b\delta)}{\delta^2}, \qquad (28)$$

$$\wp = (a + b\delta)^2 - J(J + 1), \qquad (29)$$

$$\kappa = \sqrt{-E_{n,J}^2 + M^2 - J(J + 1)\delta^2}. \qquad (30)$$

where κ is the asymptotic wave number whose parametric values are given in Eq. (30). Now introducing a wave function of the form

$$F_{nJ}(z) = z^\chi (1 - z)^{-\frac{i\kappa}{\delta}} f(z), \qquad (31)$$

and substituting it into Eq. (24), we easily obtain the following hypergeometric equation

$$z(1 - z)f''(z) + \left[2\chi - \left(2\chi + 1 - \frac{2i\kappa}{\alpha} \right)z \right]f'(z) + \left[\left(\chi - \frac{i\kappa}{\alpha} \right)^2 + \aleph \right]f(z)$$

$$= 0, \qquad (32)$$

where

$$\chi = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{4(M^2 - E_{n,J}^2)}{\delta^2} - (2J + 1)^2}. \qquad (33)$$

On this note, we can write the radial wave function for any arbitrary J wave scattering states as

$$F_{n,J}(r) = N_{n,J}(1 - e^{-\delta r})^\gamma e^{i\kappa r} \times 2F_1(\rho; \beta; \sigma; 1 - e^{-\delta r}), \qquad (34)$$

where $N_{n,J}$ is the normalization constant and the following have been used for mathematical simplicity

$$a_0 = \chi - \frac{i\kappa}{\delta} - \sqrt{\mathbb{R}}, \qquad (35)$$

$$b_0 = \chi - \frac{i\kappa}{\delta} + \sqrt{\mathbb{R}}, \qquad (36)$$

$$c = 2\chi. \qquad (37)$$

In order to completely determine the scattering phase shift by analyzing the asymptotic behaviour of the wave function, we write the following recurrence relation of the hypergeometric function

$$2F_1(a_0; b_0; c; z)$$

$$= \frac{\Gamma(c)\Gamma(c - a_0 - b_0)}{\Gamma(c - a_0)\Gamma(c - b_0)} 2F_1(a_0; b_0; 1 + a_0 + b_0 - c; 1 - z) +$$

$$(1 - z)^{c - a_0 - b_0} \frac{\Gamma(c)\Gamma(a_0 + b_0 - c)}{\Gamma(a_0)\Gamma(b_0)} 2F_1(c - a_0; c - b_0; c - a_0 - b_0 + 1; 1$$

$$- z). \qquad (38)$$

Considering the following property

$$2F_1(a_0; b_0; c; 0) = 1, \qquad (39)$$

as $r \to \infty$ and Eq. (38), we have

$$2F_1(a_0; b_0; c; 1 - e^{-\delta r})$$

$$= \Gamma(c) \left| \frac{\Gamma(c - a_0 - b_0)}{\Gamma(c - a_0)\Gamma(c - b_0)} + \frac{\Gamma(c - a_0 - b_0)}{\Gamma(a^*)\Gamma(b^*)} e^{-\delta(c - a_0 - b_0)r} \right|, \qquad (40)$$

in which the following identities have been used

$$(a + b - c)^* = c - a_0 - b_0 = \frac{2i\kappa}{\delta}, \qquad (41)$$

$$b^* = c - a_0 = \chi + \frac{i\kappa}{\delta} + \sqrt{\aleph}, \qquad (42)$$

$$a^* = c - b_0 = \chi + \frac{i\kappa}{\delta} - \sqrt{\aleph}. \qquad (43)$$

Therefore, as $r \to \infty$, the asymptotic form of the wave function can easily be obtained as

$$F_{n,J}(r) = N_{n,J}\Gamma(c) \frac{\Gamma(c - a_0 - b_0)}{\Gamma(c - a_0)\Gamma(c - b_0)} \times \sin\left(\kappa r + \frac{\pi}{2} + \delta \right). \qquad (44)$$

Taking the boundary condition into consideration: $r \to \infty \Rightarrow F_{n,J}(\infty) \to 2\sin\left(\kappa r - \frac{J\pi}{2} + \delta_J \right)$, we obtain the phase shift as

$$\delta_\ell$$

$$= 1.571(J + 1) + \arg\left[\Gamma\left(\frac{2i\kappa}{\delta} \right) \right] - \arg\left[\Gamma\left(\chi + \frac{i\kappa}{\delta} + \sqrt{\aleph} \right) \right] -$$

$$\arg\left[\Gamma\left(\chi + \frac{i\kappa}{\delta} - \sqrt{\aleph} \right) \right]. \qquad (45)$$

Schrödinger equation and the Hulthén-Yukawa type potential.

Given a radial Schrödinger equation of the form

$$\left[\frac{d^2}{dr^2} + \frac{2\mu E_{n\ell}}{\hbar^2} - \frac{2\mu V(r)}{\hbar^2} - \frac{\ell(\ell + 1)}{r^2} \right]R(r) = 0. \qquad (46)$$

Substituting Eqs. (1) and (14) into Eq. (46), and then using our previous transformation of variable, we have

$$\left[\frac{d^2}{dy^2} + \frac{1 - y}{y(1 - y)}\frac{d}{dy} + \frac{-\zeta_1 y^2 + \zeta_2 y - \zeta_3}{y^2(1 - y)^2} \right]R(y), \qquad (47)$$

where

$$\zeta_1 = \frac{2\mu(a + b\delta - E_{n\ell})}{\delta^2\hbar^2}, \qquad (48)$$

$$\zeta_2 = \frac{2\mu(a + b\delta - 2E_{n\ell})}{\delta^2\hbar^2}, \qquad (49)$$

$$\zeta_3 = \ell(\ell + 1) - \frac{2\mu E_{n\ell}}{\delta^2\hbar^2}. \qquad (50)$$

In this section, we use the parametric Nikiforov-Uvarov method presented by Tezcan and Sever [46–48]. Following their work, we write a general equation of the form

$$\left[\frac{d^2}{ds^2} + \frac{c_1 - c_2 s}{s(1 - c_3 s)}\frac{d}{ds} + \frac{-\zeta_1 s^2 + \zeta_2 s - \zeta_3}{s^2(1 - c_3 s)^2} \right]\psi(s). \qquad (51)$$

The equation above gives the condition for energy as

$$c_2 n + n(n-1)c_3 - (2n+1)c_5 + c_7 + 2c_3 c_8 + (2n+1)$$
$$(\sqrt{c_9} + c_3\sqrt{c_8}) + 2\sqrt{c_8 c_9} = 0, \tag{52}$$

and the corresponding wave function as

$$\psi(s) = N s^{c_{12}} (1-s)^{-c_{12}-c_{13}} P_n^{c_{10}-1, \frac{c_{11}}{c_3}-c_{10}-1}(1-2c_3). \tag{53}$$

The parameters in Eqs. (52) and (53), according the Tezcan and Sever [46], are obtain as follows

$$\left. \begin{array}{l} c_4 = \frac{1}{2}(1-c_1), \ c_5 = \frac{1}{2}(c_2 - 2c_3), \ c_6 = c_5^2 + \zeta_1, \ c_7 = 2c_4 c_5 - \zeta_2), \\ c_8 = c_4^2 + \zeta_3, \ c_9 = c_6 + c_3 c_7 + c_3^1 c_8, \ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \\ c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}), \ c_{12} = c_4 + \sqrt{c_8}, \ c_{13} \\ \quad = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) \end{array} \right\} \tag{54}$$

Comparing Eq. (47) and Eq. (51), the parameters in Eq. (54) are obtain as follows

$$\left. \begin{array}{l} c_1 = c_2 = c_3 = 1, \ c_4 = 0, \ c_5 = -\frac{1}{2}, \ c_6 = \frac{1}{4} + \zeta_1, \ c_7 = -\zeta_2), c_8 = \zeta_3, \ c_9 \\ \quad = \frac{1}{4}(1+2\ell)^2, \\ c_{10} = 1 + 2\sqrt{\zeta_3}, \ c_{11} = 3 + 2\ell + 2\sqrt{\zeta_3}, \ c_{12} = \sqrt{\zeta_3}, \ c_{13} = -(1+\ell) - \sqrt{\zeta_3} \end{array} \right\} \tag{55}$$

Substituting Eq. (55) into Eqs. (52) and (53) respectively, we obtain energy equation and the corresponding wave function as

$$E_{n,\ell} = \frac{\delta^2\hbar^2}{2\mu}\left[\ell(\ell+1) - \left(\frac{\frac{2\mu(a+b\delta)}{\delta^2\hbar^2} - \ell(\ell+1) - (1+n+\ell)^2}{2(1+n+\ell)}\right)^2\right], \tag{56}$$

$$R(y) = N y^{\sqrt{\zeta_3}}(1-y)^{1+\ell} P_n^{(2\sqrt{\zeta_3}, 1+2\ell)}(1-2y). \tag{57}$$

Special cases of the newly proposed potential in Eq. (1)

I: when $a = 0$, potential (1) reduces to Yukawa potential of the form

$$V_Y(r) = -\frac{be^{-\delta r}}{r}, \tag{58}$$

with energy equation of the form

$$E_{n,\ell} = \frac{\delta^2\hbar^2}{2\mu}\left[\ell(\ell+1) - \left(\frac{\frac{2\mu b\delta}{\delta^2\hbar^2} - \ell(\ell+1) - (1+n+\ell)^2}{2(1+n+\ell)}\right)^2\right]. \tag{59}$$

II: Similarly, when $b = 0$, the Hulthèn-Yukawa potential turns to Hulthèn potential

$$V_H(r) = -\frac{ae^{-\delta r}}{1 - e^{-\delta r}}, \tag{60}$$

And the energy equation in Eq. (56) reduces to

$$E_{n,\ell} = \frac{\delta^2\hbar^2}{2\mu}\left[\ell(\ell+1) - \left(\frac{\frac{2\mu a}{\delta^2\hbar^2} - \ell(\ell+1) - (1+n+\ell)^2}{2(1+n+\ell)}\right)^2\right]. \tag{61}$$

Hulthèn-Yukawa type potential and the thermodynamic properties

In this section, we study the Hulthèn-Yukawa type potential with the thermodynamic properties. Most report on the various potential models with thermodynamic properties usually studied effect of $\beta\left(\frac{1}{KT}\right)$ on the thermodynamic properties [48,49]. However, Eshghi et al. [50,51], studied thermodynamic quantities applied to grapheme under magnetic fields and thermodynamic physical quantities of the Dirac-Weyl fermions in the absence of magnetic field inside grapheme

quantum dot respectively. Similarly, Eshghi and Mehraban [52], calculated the thermodynamic properties of the radial scalar power potential using the characteristic function and deduced the persistent current. In the present work, we want to study the effect of the potential parameters on the thermodynamic properties. To do this, we modified the nonrelativistic energy equation as

$$E_n = \frac{-\delta^2\hbar^2}{2\mu}\left(\frac{Q}{2(n+1)} - \frac{R}{2}\right)^2, \tag{62}$$

where $n = 0, 1, 2, 3, --- < [R \pm \sqrt{-Q}]$. In this case,

$$Q = \frac{2\mu(a_0 + b_0\delta)}{\delta^2\hbar^2}, \tag{63}$$

$$R = n + 1. \tag{64}$$

The vibrational partition function is obtained as

$$Z_{vib}(\beta) = \sum_{n=0}^{\tau} e^{-\beta E_n}, \ \beta = \frac{1}{kT} \tag{65}$$

where $\tau = -Q \pm \sqrt{-R}$ and k is a Boltzmann constant. Substituting the modified energy equation in Eq. (62) into Eq. (65), we have

$$Z_{vib}(\beta) = \sum_{n=0}^{\tau} e^{\frac{\beta\delta^2\hbar^2}{4\mu}\left(Q + \frac{Q^2}{2\mu R^2} + \frac{R^2}{2\mu}\right)}. \tag{66}$$

The sum in Eq. (66) in most cases is replaced by integral in the classical limit to easy computation and so Eq. (66) becomes

$$Z_{vib}(\beta) = \int_0^{\tau} e^{\frac{\beta\delta^2\hbar^2 Q}{4\mu} + \frac{\beta\delta^2\hbar^2 Q^2}{8\mu(n+1)^2} + \frac{\beta\delta^2\hbar^2(n+1)^2}{8\mu}} dn, \tag{67}$$

$$= \frac{\sqrt{\pi} e^{\frac{1}{2}\frac{\beta(b_0\delta+a_0)}{\mu}}\delta\hbar\left[erf\left(\frac{\frac{\beta(b_0\delta+a_0)}{\mu}(\tau+1)}{\delta\hbar\sqrt{-\frac{2\beta}{\mu}}}\right) - erf\left(\frac{\frac{\beta(b_0\delta+a_0)}{\mu}}{\delta\hbar\mu\sqrt{-\frac{2\beta}{\mu}}}\right)\right]}{\mu(b_0\delta+a_0)\sqrt{-\frac{2\beta}{\mu}}}$$

$$= -\frac{1}{\mu(b_0\delta+a_0)\sqrt{-\frac{2\beta}{\mu}}}\left[\sqrt{\pi} e^{\frac{1}{2}\frac{\beta(b_0\delta+a_0)}{\mu}}\delta\hbar\left[erf\left(\frac{\frac{\beta(b_0\delta+a_0)}{\mu}(\tau+1)}{\delta\hbar\sqrt{-\frac{2\beta}{\mu}}}\right)\right.\right.$$

$$\left.\left. - erf\left(\frac{\frac{\beta(b_0\delta+a_0)}{\mu}}{\delta\hbar\mu\sqrt{-\frac{2\beta}{\mu}}}\right)\right]\right]. \tag{68}$$

Now that we have obtained the vibrational partition function, it is convenient to compute the thermodynamic properties for the potential model under consideration (see Figs. 1 and 2).

Vibrational mean energy U

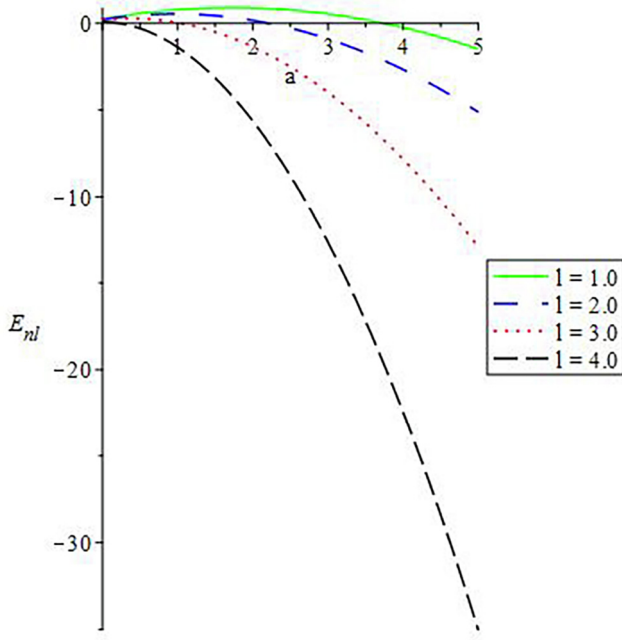$$U(\beta) = -\frac{\partial}{\partial\beta}\ln Z_{vib}(\beta)$$

Fig. 1. Energy eigenvalue $E_{n\ell}$ against a for various $l = \ell$ at the first excited state with $\mu = \hbar = b = 1$.
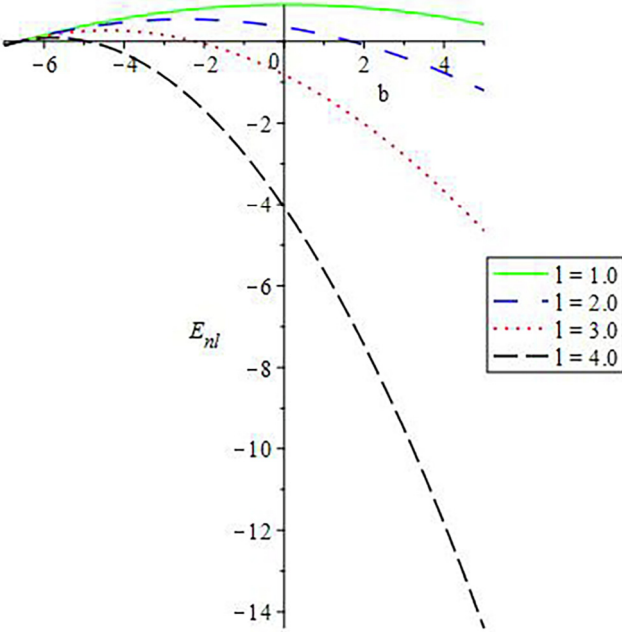

Fig. 2. Energy eigenvalue $E_{n\ell}$ against b for various $l = \ell$ at the first excited state with $\mu = \hbar = a = 1$.

$$
= -\frac{2\left(\frac{2\mu(b\delta+a)}{\delta^2 h^2}\right)^{\frac{3}{2}} e^J \beta\delta\hbar + M\sqrt{2}\,(\beta B - 2\mu)\left(-erf\left(\frac{1}{4}\frac{\beta\delta\hbar\frac{2\mu(b\delta+a)}{\delta^2 h^2}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)\sqrt{\pi} + 2B\beta\delta\hbar(e^J + e^L)}{\delta M\sqrt{\pi}\beta\mu\left(erf\left(\frac{1}{4}\frac{\beta\frac{2\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{1}{4}\frac{\beta\delta\hbar\frac{2\mu(b\delta+a)}{\delta^2 h^2}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)}
$$

$$
- \frac{2\left(\frac{2\mu(b\delta+a)}{\delta^2 h^2}\right)^{\frac{3}{2}} e^J \beta\delta\hbar + M\sqrt{2}\,(\beta B - 2\mu)erf\left(\frac{1}{4}\frac{\beta\frac{2\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{\mu\sqrt{-\frac{\beta}{\mu}}}\right)\sqrt{\pi} + 2B\beta\delta\hbar(e^J + e^L)}{\delta M\sqrt{\pi}\beta\mu\left(erf\left(\frac{1}{4}\frac{\beta\frac{2\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{1}{4}\frac{\beta\delta\hbar\frac{2\mu(b\delta+a)}{\delta^2 h^2}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)}.
$$

(69)

Vibrational specific heat capacity C

$$
C(\beta) = \frac{\partial}{\partial T}U = -k\beta^2\frac{\partial}{\partial\beta}U
$$

$$
= -\frac{\frac{\mu\pi}{\beta}erf\left(\frac{\beta\delta\hbar\frac{\mu(b\delta+a)}{\delta^2 h^2}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)^2 + \Re - \frac{\mu\pi}{\beta}erf\left(\frac{\beta\frac{\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)^2}{2\pi\mu T\left(erf\left(\frac{\beta\frac{\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{\beta\frac{\mu(b\delta+a)}{\delta\hbar}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)^2}
$$

$$
- \frac{\delta^2 \hbar^2\left(-2n\left(\frac{2\mu(b\delta+a)}{\delta^2 h^2}\right)^{\frac{5}{2}} + \left(\frac{2\mu(b\delta+a)}{\delta^2 h^2}\right)^2\left[\left(n^2 + \frac{2\mu(b\delta+a)}{\delta^2 h^2}\right)e^{\frac{J}{2}} + e^{\frac{L}{2}}\right]\right)}{8\pi\mu T\left(erf\left(\frac{\beta\frac{\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{\beta\frac{\mu(b\delta+a)}{\delta\hbar}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)^2}
$$

$$
+ \frac{\aleph erf\left(\frac{\beta\frac{\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)}{8\pi\mu T\left(erf\left(\frac{\beta\delta\hbar\frac{\mu(b\delta+a)}{\delta^2 h^2}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{\beta\delta\hbar\frac{\mu(b\delta+a)}{\delta^2 h^2}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)^2}
$$

(70)

Vibrational mean free energy F

$$
F(\beta) = -KT In Z_{vib}(\beta)
$$

$$
= -\frac{KT}{2}\left[1.84 + 2In\right.
$$

$$
\left.\left(\frac{\left[erf\left(\frac{\beta\sqrt{2}\frac{\mu(b\delta+a)}{\delta\hbar}}{2\sqrt{-\mu\beta}}\right) - erf\left(\frac{\beta\sqrt{2}\frac{\mu(b\delta+a)}{\delta\hbar}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2 h^2}}-n\right)}{2\sqrt{-\mu\beta}}\right)\right]e^{\frac{BL}{2}}}{\frac{2(b\delta+a)}{\delta\hbar}\sqrt{-\mu\beta}}\right)\right].
$$

(71)

Vibration entropy S

$$S(\beta) = KInZ_{vib}(\beta) + KT\frac{\partial}{\partial T}InZ_{vib}(\beta)$$

$$= \frac{\left(\frac{2\mu(b\delta+a)}{\delta^2\hbar^2}\right)^{\frac{3}{2}}\sqrt{2}\,\delta\hbar e^J + 2KT\chi\pi\sqrt{-\frac{\beta}{\mu}}\left[erf\left(\frac{\frac{\beta\mu(b\delta+a)}{\delta\hbar}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2\hbar^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right]}{4\mu T\sqrt{-\frac{\beta\pi}{\mu}}\left(erf\left(\frac{\frac{\beta\mu(b\delta+a)}{\delta\hbar}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2\hbar^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{\frac{\beta 2\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)}$$

$$+ \frac{2KT\left(-erf\left(\frac{\frac{\beta\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)\chi\pi\sqrt{-\frac{\beta}{\mu}} + \sqrt{2}\,\delta\hbar\frac{2\mu(b\delta+a)}{\delta^2\hbar^2}(e^J + e^L)}{4\mu T\sqrt{-\frac{\beta\pi}{\mu}}\left(erf\left(\frac{\frac{\beta\mu(b\delta+a)}{\delta\hbar}\sqrt{2}\left(\sqrt{\frac{2\mu(b\delta+a)}{\delta^2\hbar^2}}-n\right)}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right) - erf\left(\frac{\frac{\beta 2\mu(b\delta+a)}{\delta\hbar}\sqrt{2}}{2\mu\sqrt{-\frac{\beta}{\mu}}}\right)\right)}.$$

(72)

For mathematical simplicity, we have used the following

$$B = \frac{2\mu(b\delta+a)}{\delta^2\hbar^2}, \quad A = \frac{1}{4}\frac{\beta\delta\hbar B\sqrt{2}}{\mu\sqrt{-\frac{\beta}{\mu}}}, \quad G = \frac{1}{4}\frac{\beta\delta\hbar\sqrt{2}B(\sqrt{B}-n)}{\mu\sqrt{-\frac{\beta}{\mu}}}, \quad J = \frac{1}{8}\frac{\beta\delta^2\hbar^2B^2(\sqrt{B}-n)^2}{\mu},$$

$$\Im = \left(2In\left(\frac{e^{\frac{BL}{2}}(erf(A)-erf(G))}{\delta\hbar MB}\right)+1\right), \quad \eta_0 = \left(\delta^2\hbar^2\left(\frac{1}{3}n^2+B\right)\delta^2 - \frac{4\mu}{3\beta}\right),$$

$$\aleph = \delta\hbar\sqrt{2\pi}M\left(\left(\frac{4}{3\beta}B^{\frac{3}{2}}\mu - B^{\frac{7}{2}}\delta^2\hbar^2n^2 - \frac{1}{3}B^{\frac{9}{2}}\delta^2\hbar^2 + nB\eta_0\right)e^J\right.$$

$$\left. + \frac{1}{3}e^L B\left(B^2\delta^2\hbar^2 - \frac{4\mu}{\beta}\right)\right),$$

$$\chi_0 = \left(\mu In(\pi) + \mu In(2) - \Im\mu - \frac{1}{4}B\delta^2\hbar^2\right), \quad \mathbb{Z}$$

$$= \left(-2nB^{\frac{5}{2}} + B^2(n^2+B)e^{\frac{J}{2}} + e^{\frac{L}{2}}B^2\right),$$

$$\Re = \left(\frac{32}{3\beta}\mu\pi erf(A) - \aleph\right)erf(G) - \frac{8}{3}\delta^2\hbar^2\left(B^2n - B^{\frac{5}{3}}\right)e^{(J+L)}, \quad M = \sqrt{-\frac{\beta}{\mu}}, \quad L$$

$$= \frac{1}{8}\frac{\beta\delta^2\hbar^2B^2}{\mu}.$$

Discussion

In Table 1, we presented energy eigenvalue for 2p, 3p, 3d, 4p, 4d and 4f for various δ with three values for each of the potential strengths. It is observed that the energy obtained with $b = 3$ and $a = 0$ (Yukawa

Table 1

Bound state energy eigenvalues $E_{n,\ell}$ for Hulthèn-Yukawa potential, Yukawa potential ($a = 0$) and Hulthèn potential ($b = 0$) for 2p, 3p, 3d, 4p, 4d and 4f with $\mu = \hbar = 1$.

State	δ	$a = 2, b = 1.$	$a = 3, b = 0.$	$b = 3, a = 0$
2p	0.30	− 5.633472220	− 10.26125000	− 0.461250000
	0.40	− 2.720000000	− 4.801250000	− 0.245000000
	0.50	− 1.281250000	− 2.281250000	− 0.031250000
3p	0.30	− 1.91126544	− 3.783472221	− 0.011250000
	0.40	− 0.642222223	− 1.400555555	0.124444445
	0.50	− 0.031250000	− 0.336805556	0.246527778
3d	0.30	− 1.360015432	− 3.066805555	0.238750000
	0.40	− 0.020000000	− 0.645000000	0.480000000
	0.50	0.663194445	0.468750000	0.718750000
4p	0.30	− 0.680868056	− 1.575312500	0.087187500
	0.40	− 0.020000000	− 0.315312500	0.148750000
	0.50	0.242187500	0.179687500	0.179687500
4d	0.30	− 0.325868056	− 1.132812500	0.267187500
	0.40	0.400000000	0.179687500	0.418750000
	0.50	0.742187500	0.742187500	0.554687500
4f	0.30	0.164444444	− 0.511250000	0.495000000
	0.40	0.955000000	0.847187500	0.748750000
	0.50	1.375000000	1.468750000	1.000000000

Table 2

Bound state energy eigenvalues for Hulthèn-Yukawa potential, Yukawa potential ($a = 0$) and Hulthèn potential ($b = 0$) for 2p, 3p, 3d, 4p, 4d and 4f with $\mu = \hbar = 1$.

State	δ	$a = 2\delta, b = 1$	$a = 3\delta, b = 0.$	$b = 3, a = 0.$
2p	0.10	− 0.901250000	− 0.901250000	− 0.901250000
	0.20	− 0.680000000	− 0.680000000	− 0.680000000
	0.30	− 0.461250000	− 0.461250000	− 0.461250000
3p	0.10	− 0.323472222	− 0.323472222	− 0.323472222
	0.20	− 0.160555556	− 0.160555556	− 0.160555556
	0.30	− 0.011250000	− 0.011250000	− 0.011250000
3d	0.10	− 0.251250000	− 0.251250000	− 0.251250000
	0.20	− 0.005000000	− 0.005000000	− 0.005000000
	0.30	0.023875000	0.023875000	0.023875000
4p	0.10	− 0.127812500	− 0.127812500	− 0.127812500
	0.20	− 0.000500000	− 0.000500000	− 0.000500000
	0.30	0.087187500	0.087187500	0.087187500
4d	0.10	− 0.080812500	− 0.080812500	− 0.080812500
	0.20	− 0.100000000	− 0.100000000	− 0.100000000
	0.30	− 0.267187500	− 0.267187500	− 0.267187500
4f	0.10	− 0.020000000	− 0.020000000	− 0.020000000
	0.20	− 0.238750000	− 0.238750000	− 0.238750000
	0.30	− 0.495000000	− 0.495000000	− 0.495000000

potential) are higher than their counter part obtained with $a = 3$ and $b = 0$ (Hulthèn potential). Similarly, the energy obtained with $a = 2$ and $b = 1$ (Hulthèn-Yukawa potential) are also higher than those obtained with $a = 3$ and $b = 0$ but lower than those obtained with $a = 0$ and $b = 3$. It is noted that as δ increases at constant n and constant ℓ respectively, the energy eigenvalue increases. To clearly study the variation in the energy eigenvalue for the various states with different values of the potential strengths, we computed other numerical results in Table 2. In Table 2, we have taken $a = \delta a$. Using the same values as used for a and b in Table 1, it is noted in Table 2 that as $a = \delta a$, the result for Hulthèn-Yukawa potential and its subsets (Hulthèn potential and Yukawa potential) are equivalent. This indicated that the differences in the results of Table 1 was caused by the screening parameter which have positive effect on the parameter b.

To study the thermodynamic properties, we first obtained a high temperature partition function. This was used to compute the various thermodynamic properties in the text. The variation of these thermodynamic properties with respect to the two potential strength a and b were fully studied. In Figs. 3 and 4, we examined the variation of the vibrational mean energy $U(\beta)$ against a and b respectively for various β. It is observed in both cases that an increase in any of the potential strength result to a decreases in the vibrational mean energy. However, at the same values of the two potential strengths, the vibrational mean energies obtained with b are higher than those obtained with a. This effect is caused by the screening parameter which has more decreasing effect on the result obtained with a than with b according to Eq. (56). In Figs. 5 and 6, we examined the behavior of the vibrational specific heat against the two potential strengths. It is noted in both cases that as each of the potential strength increases, the vibrational specific heat increases. It is observed here that at $a < 1$ and $b < 1.4$, the vibrational specific heat remains constant. This is because, the values of the potential strength at this range has no significant change in the vibrational specific heat. The effect of the screening parameter was also observed here. In Figs. 7 and 8, we plotted the vibrational mean free energy against the potential strengths. It is observed that as each of the potential strength increases, the vibrational mean free energy decreases. The effect of the screening parameter on the results was also observed. In Figs. 9 and 10, we showed the variation of the vibrational entropy against the potential strength. In each case, the vibrational entropy decreases monotonically as each of the potential strengths increases.
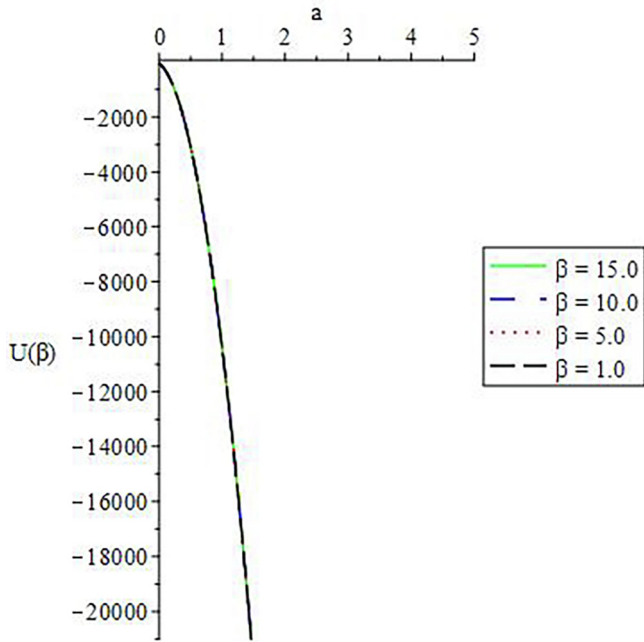
Fig. 3. Vibrational mean energy U as a function of a (potential strength) for various β with $\mu = \hbar = b = 1$.
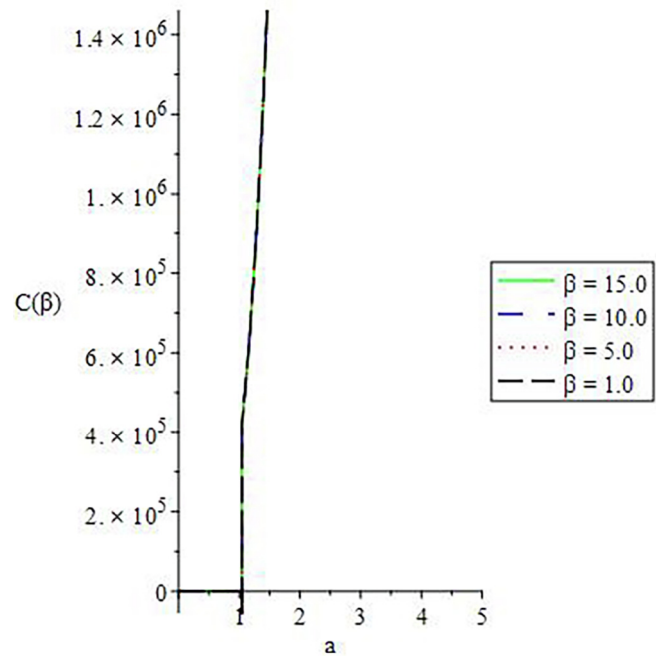
Fig. 5. Vibrational specific heat C as a function of a (potential strength) for various β with $\mu = \hbar = b = 1$.
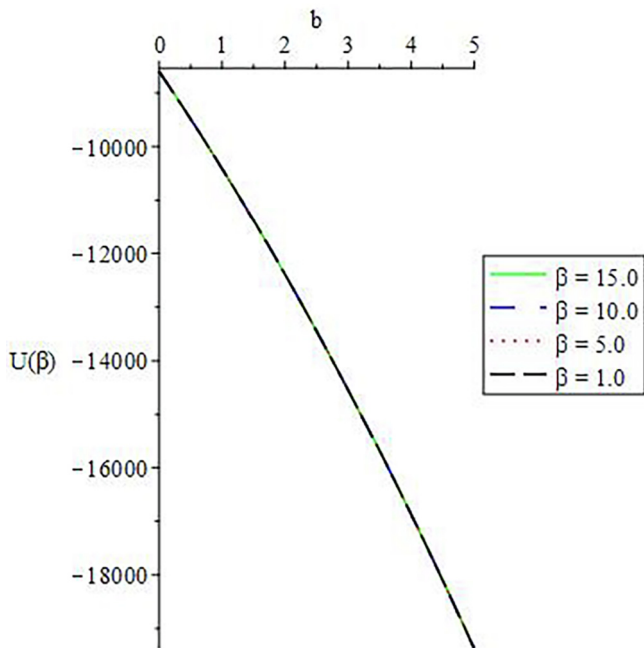
Fig. 4. Vibrational mean energy U as a function of b (potential strength) for various β with $\mu = \hbar = a = 1$.
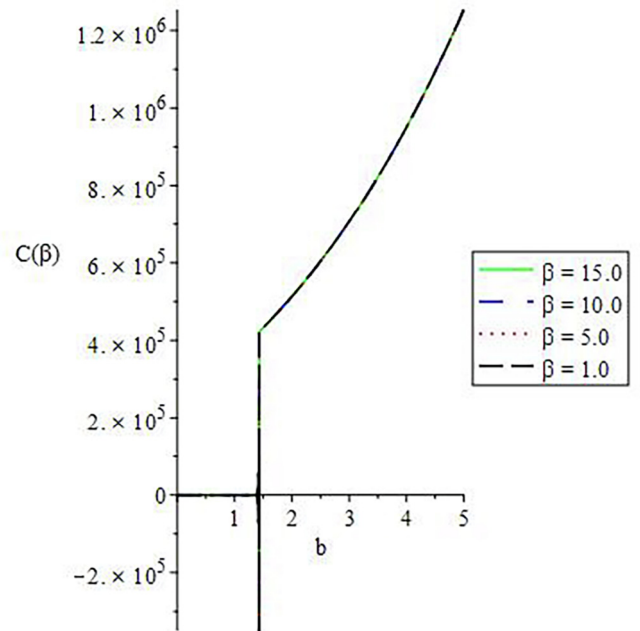
Fig. 6. Vibrational specific heat C as a function of b (potential strength) for various β with $\mu = \hbar = a = 1$.

Conclusion

The approximate relativistic bound states solutions of the Duffin-Kemmer-Petiau equation in the presence of Hulthèn-Yukawa type potential has been fully obtained using the asymptotic iteration method. The effect of the screening parameter δ on the energy eigenvalues has been numerically studied. We have mathematically analyzed the scattering phase shift of the Hulthèn-Yukawa type potential under the relativistic Duffin-Kemmer-Petiau equation. It is deduced that the screening parameter brings about a variation in the numerical results between the Hulthèn-Yukawa type potential and its subsets. Finally, we calculated the thermodynamic properties of the Hulthèn-Yukawa type potential. We know that our results will find applications in different areas of physics since the scattering state and the thermodynamic properties have been reported before in the available literature for other potentials.
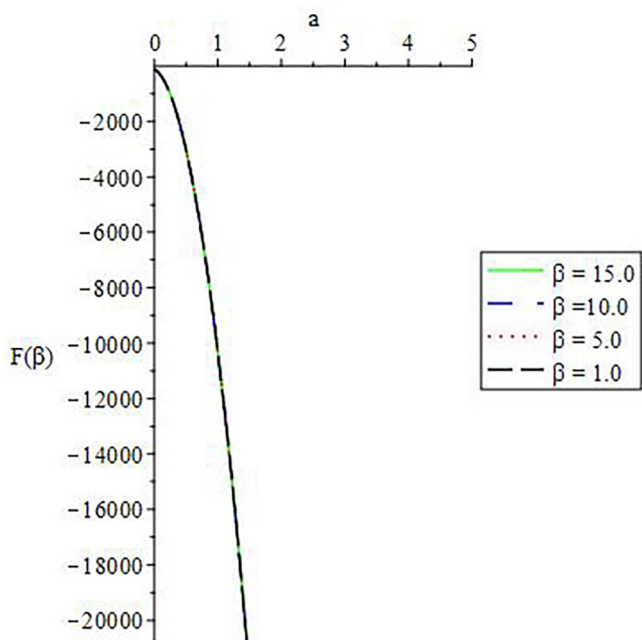
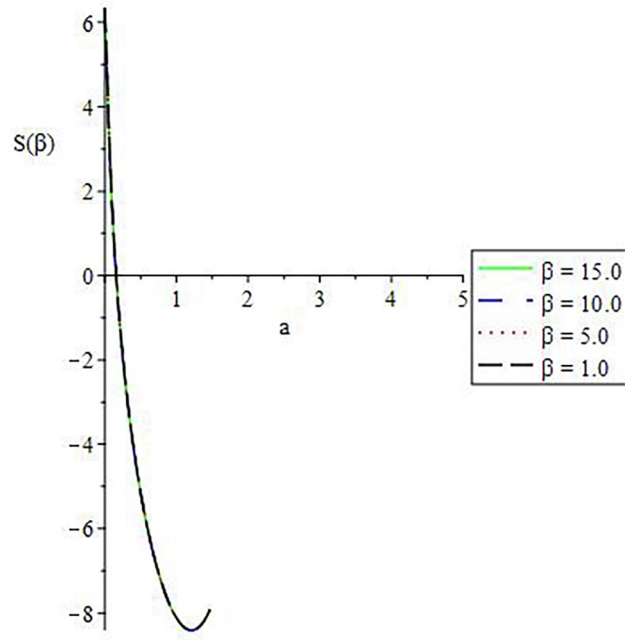**Fig. 7.** Vibrational mean free energy F as a function of a (potential strength) for various β with $\mu = \hbar = b = 1$.

Fig. 9. Vibrational Entropy S as a function of a (potential strength) for various β with $\mu = \hbar = b = 1$.
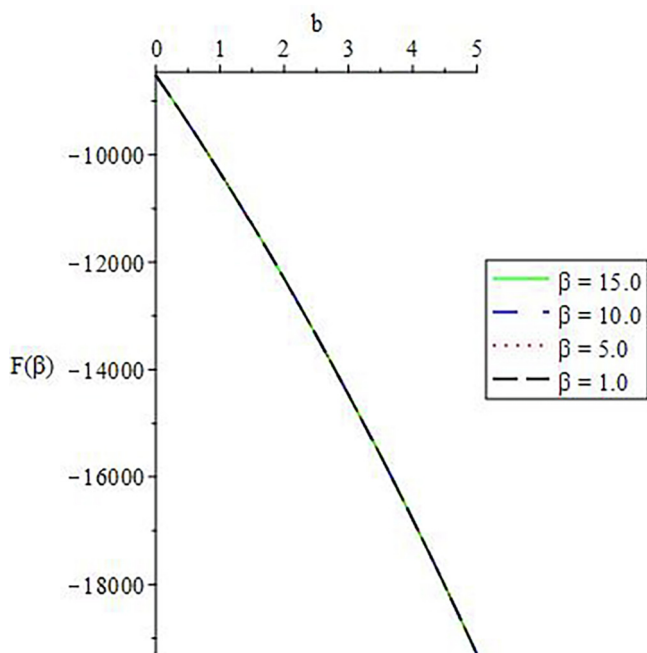

Fig. 8. Vibrational mean free energy F as a function of b (potential strength) for various β with $\mu = \hbar = a = 1$.
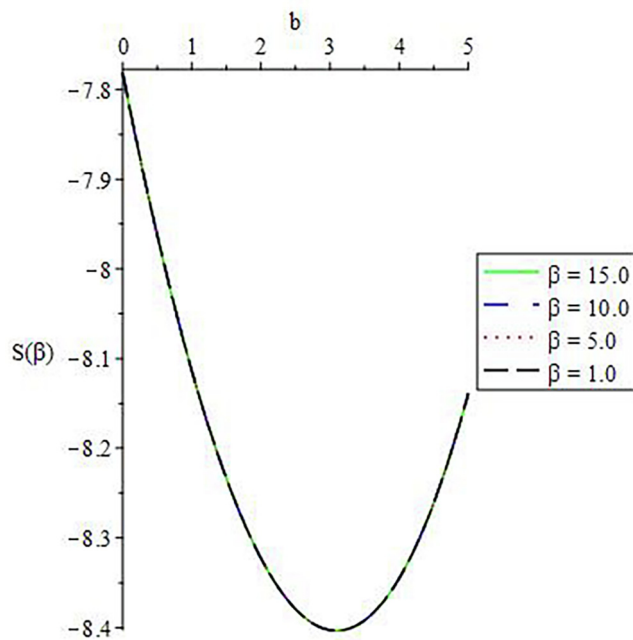

Fig. 10. Vibrational Entropy S as a function of b (potential strength) for various β with $\mu = \hbar = a = 1$.

Appendix A

Methodology of asymptotic iteration (AIM)

AIM is proposed to solve the homogeneous linear second-order differential equation of the form

$$y_n''(x) = \lambda_0(x)y_n'(x) + S_0(x)y_n(x), \tag{A1}$$

where $\lambda_0(x) \neq 0$, the prime denotes the derivative with respect to x and n is the radial quantum number. The terms $S_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To get the solution, we take the derivative of the above equation with respect to x:

$$y_n'''(x) = \lambda_1(x)y_n''(x) + s_1(x)y_n(x), \tag{A2}$$

where

$$\lambda_1(x) = y_0'(x) + s_0(x) + \lambda_0^2(x), \tag{A3}$$

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \tag{A4}$$

Taking the derivative of Eq. (A2), we have

$$y_n''(x) = \lambda_2(x)y_n'(x) + s_2(x)y_n(x), \tag{A5}$$

where

$$\lambda_2(x) = \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \tag{A6}$$

$$s_2(x) = s_1'(x) + s_0(x)\lambda_1(x). \tag{A7}$$

Thus, Eq. (A1) can be iterated to $(k+1)^{th}$ and $(k+2)^{th}$ derivatives, with $k = 1, 2, 3, 4, \dots$. Hence,

$$y_n^{(k+1)}(x) = \lambda_{k-1}(x)y_n'(x) + s_{k-1}(x)y_n(x), \tag{A8}$$

$$y_n^{(k+2)}(x) = \lambda_k(x)y_n'(x) + s_k(x)y_n(x), \tag{A9}$$

Here, we have used

$$\lambda_k(x) = \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \tag{A10}$$

$$s_k(x) = s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x). \tag{A11}$$

Taking the ratio of Eq. (A8) and Eq. (A9), we have

$$\frac{d}{dx}In(y_n^{(k+1)}(x)) = \frac{y_n^{(k+1)}(x)}{y_n^{(k+1)}(x)} = \frac{\lambda_k(x)[y_n'(x) + (s_k(x)/\lambda_k(x))y_n(x)]}{\lambda_{k-1}(x)[y_n'(x) + (s_{k-1}(x)/\lambda_{k-1}(x))y_n(x)]}. \tag{A12}$$

For a very large $k > 0$, we conveniently have

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x). \tag{A13}$$

This results to a quantization conditions

$$\delta_k(x) = \begin{vmatrix} \lambda_k(x) & S_k(x) \\ \lambda_{k-1}(x) & S_{k-1}(x) \end{vmatrix} = 0, \ k = 1, 2, -,-,-, \tag{A14}$$

Then, Eq. (A10) reduces to

$$\frac{d}{dx}[y_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{A15}$$

which yields a general solution of Eq. (A1) as

$$y(x) = \exp\left(-\int a(x')dx\right) \times \left[c_2 - c_1 \int \exp\left(\left[\int \lambda_0(x'') + 2\alpha(x'')\right]dx''\right)dx'\right] \tag{A16}$$

where c_1 and c_2 are constants of integration.

Appendix B. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.rinp.2019.102409.

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