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Approximate solutions of the Dirac equation with Coulomb-Hulthén-like tensor interaction

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ABSTRACT

The solutions of spin and pseudospin symmetries with Trigonometric scarf potential are obtained in the presence of a new tensor interaction which breaks down the energy degeneracies in both symmetries leading to atomic stability. The nonrelativistic equation is obtained by taking the nonrelativistic limit of the spin symmetry. A simplified condition for energy equation for Formula method for special case is also given. Finally, we calculated Fisher information for both position and momentum spaces which can be used to determine the accuracy of predicting the localization of a particle.

Introduction

In the recent time, the study of the effect of tensor interaction in the concept of Dirac equation has drawn many attentions in theoretical physics. This is because, the inclusion of tensor term in Dirac equation removes energy degeneracies between states in both the spin and pseudospin symmetries [1–4]. The popularly used tensor term over the years is the Coulomb-like tensor potential. However, few years ago, a Yukawa-like tensor potential was introduced. In both tensor potentials used, there are still some energy degeneracies between states in both symmetries. Motivated by this, the authors intend to use another tensor potential called Coulomb-Hulthén-like tensor potential. This tensor potential is a combination of the Coulomb potential and Hulthén potential. To the best of our knowledge, there is no report on this tensor potential. It is our belief that this tensor potential will break down the energy degeneracies which could not be split by the Coulomb-like and Yukawa-like tensors. The pseudospin doublet was based on the small energy difference between nuclear energy levels [5,6]. This doublet structure is expressed in terms of a pseudo orbital angular momentum and the average of the orbital angular momentum of two states in doublet. In order to explain the different patterns for nucleon spectrum in nuclei in the relativistic quantum mechanics, the spin and pseudospin symmetries of the Dirac equation become so significant. Hence, various attentions are drawn in the relativistic quantum mechanics. The spin and pseudospin symmetries respectively, are exact under the condition $d[V(r) - S(r)]/dr = 0$ and $d[V(r) + S(r)]/dr = 0$ [7–9].

The solutions of the Dirac equation under spin and pseudospin symmetries over the years have been obtained using the traditional methodologies such as Nikiforov-Uvarov method [10–13], factorization method [14,15], asymptotic iteration method [16–19], supersymmetry shape invariance and solvable potential [20–25] and others. The major challenge faced by some authors using these methods is tedious calculations that are involved. Recently, Tezcan and Sever [26] derived the parametric form of Nikiforov-Uvarov method from the conventional.

Nikiforov-Uvarov method which is easier. Similarly, Falaye et al. [27] derived Formula method for bound state problem from factorization and asymptotic iteration methods. However, the condition for energy equation for Formula method is not very simple. It requires more mathematical skills to obtain the energy equation. The present work looks into the condition for energy equation when using Formula method. In this study, we intend to investigate the solutions of the relativistic Dirac equation with double tensor potential term newly proposed for the Trigonometric scarf potential.

This work is divided into three folds. In the first fold, we simplified the energy condition for the Formula method for special cases i.e. when $c_1 = c_2 = c_3 = 1$ and give a simple energy condition. In the second fold, we obtained the solution of Dirac equation under spin and pseudospin symmetries with a new tensor interaction proposed in this work. In the third fold, we calculated Fisher information in position space and in momentum space. The Trigonometric scarf potential is given as

$$V(r) = -\frac{V_0}{\sin^2(\alpha r)}, \tag{1}$$

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where V_0 is the potential depth. This potential has been studied by Suparmi et al. [28] as well as Falaye and Oyewumi [29].

Fisher information introduced in the concept of statistical estimation over the years has applications in other areas of study such as physics [30]. It determines the uncertainty relations of constant mass quantum system and measures the inhomogeneity of the system. It is equally used to predict the localization of a particle. In this study, the authors want to examine the effect of the strength of the interacting potential on both the Fisher information for position space and momentum space.

Formula method for bound state problem

Given a second order differential equation of the form [26,31]

$$\left(\frac{d^2}{dr^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\right)\psi(s)=0. \tag{2}$$

The condition for energy equation given by Falaye et al. [27] reads

$$k_0^2=\left[\frac{k_1^2-k_0^2-\left[\frac{1-2n}{2}-\frac{1}{2\alpha_3}\left(\alpha_2-\sqrt{(\alpha_3-\alpha_2)^2+4\xi_1}\right)\right]^2}{2\left[\frac{1-2n}{2}-\frac{1}{2\alpha_3}\left(\alpha_2-\sqrt{(\alpha_3-\alpha_2)^2+4\xi_1}\right)\right]}\right]^2, \tag{3}$$

where

$$k_1^2=\frac{(1-\alpha_1)+\sqrt{(1-\alpha_1)^2+4\xi_3}}{2}, \tag{4}$$

$$k_0^2=\frac{1}{2}+\frac{\alpha_1}{2}-\frac{\alpha_2}{2\alpha_3}+\sqrt{\left(\frac{1}{2}+\frac{\alpha_1}{2}-\frac{\alpha_2}{2\alpha_3}\right)^2+\frac{\xi_1}{\alpha_3^2}+\frac{\xi_2}{\alpha_3}+\xi_3}. \tag{5}$$

In order to make work easier, we elucidate Eq. (3) to have a simplified energy condition as follows

$$\begin{aligned}\sqrt{\xi_1}\left(1+\sqrt{1-4(\xi_1+\xi_2+\xi_3)}+2n\right)&=\xi_3-\xi_1-n(n+1)\\&-n\sqrt{1-4(\xi_1+\xi_2+\xi_3)}\\&-\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4(\xi_1+\xi_2+\xi_3)}\right)^2.\end{aligned} \tag{6}$$

Dirac equation with tensor coupling potential

The Dirac equation for fermionic massive spin-1/2 particles with an attractive scalar potential $S(r)$, vector potential $V(r)$, tensor potential $U(r)$ and relativistic E (in units $\hbar=c=1$) is given by [32–37]

$$[\vec{\alpha}.\vec{p}+\beta(M+S(r))-i\beta\vec{\alpha}.\mathbf{r}U(r)+V(r)-E]\psi(\vec{r})=0, \tag{7}$$

where M is the mass of the fermionic particle, $\vec{p}=i\vec{\nabla}$ is the three-dimensional momentum operator while $\vec{\alpha}$ and β are the usual 4×4 usual Dirac matrices. The eigenvalues of spin-orbit coupling operator are $\kappa=\left(j+\frac{1}{2}\right)>0$ for unaligned spin $j=l-\frac{1}{2}$ and $k=-\left(j+\frac{1}{2}\right)<0$ for aligned spin $j=l+\frac{1}{2}$. The spinor wave functions can be classified according to their angular momentum j the spin-orbit quantum number κ, and the radial quantum number n Hence, they can be written as follows

$$\psi_{n\kappa}(\vec{r})=\begin{pmatrix}f_{n\kappa}(\vec{r})\\g_{n\kappa}(\vec{r})\end{pmatrix}=\frac{1}{r}\begin{pmatrix}F_{n\kappa}(r)Y_{jm}^{\ell}(\theta,\varphi)\\iG_{n\kappa}(r)Y_{jm}^{\tilde{\ell}}(\theta,\varphi)\end{pmatrix}, \tag{8}$$

where $f_{nk}(\vec{r})$ is the upper component and $g_{nk}(\vec{r})$ is the lower component of the Dirac spinors. $Y_{jm}^{\ell}(\theta,\varphi)$ and $Y_{jm}^{\tilde{\ell}}(\theta,\varphi)$ are spin and pseudospin spherical harmonics, respectively, and m is the projection of the angular momentum on the z-axis. Substituting Eq. (8) into Eq. (7) and making use of the following equations

$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B})=\vec{A}.\vec{B}+i\vec{\sigma}.(\vec{A}\times\vec{B}), \tag{9}$$

$$(\vec{\sigma}.\vec{P})=\vec{\sigma}.\mathbf{r}\left(\mathbf{r}.\vec{P}+i\frac{\vec{\sigma}.\vec{L}}{r}\right), \tag{10}$$

the two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{nk}(r)$ and $G_{nk}(r)$ are obtain as

$$\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)F_{nk}(r)=(M+E_{nk}-V(r)+S(r))G_{nk}(r), \tag{11}$$

$$\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)G_{nk}(r)=(M-E_{nk}+V(r)+S(r))F_{nk}(r), \tag{12}$$

where

$$\Delta(r)=V(r)-S(r) \tag{13}$$

and

$$\sum(r)=V(r)+S(r). \tag{14}$$

Eliminating $G_{nk}(r)$ and $F_{nk}(r)$ from Eqs. (11) and (12) respectively, we have

$$\begin{aligned}&\left(\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}+\frac{2kU(r)}{r}-\frac{dU(r)}{dr}-U^2(r)\right.\\&\left.+\frac{\frac{d\Delta(r)}{dr}}{M+E_{nk}-\Delta(r)}\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)\right)F_{nk}(r)\\&=[(M+E_{nk}-\Delta(r))(M-E_{nk}+\sum(r))]F_{nk}(r),\end{aligned} \tag{15}$$

$$\begin{aligned}&\left(\frac{d^2}{dr^2}-\frac{k(k-1)}{r^2}+\frac{2kU(r)}{r}+\frac{dU(r)}{dr}-U^2(r)\right.\\&\left.+\frac{\frac{d\sum(r)}{dr}}{M-E_{nk}+\sum(r)}\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)\right)G_{nk}(r)\\&=[(M+E_{nk}-\Delta(r))(M-E_{nk}+\sum(r))]G_{nk}(r).\end{aligned} \tag{16}$$

Spin symmetry limit

The spin symmetry limit occurs when $\frac{d\Delta(r)}{dr}=0$ and $\sum(r)=V(r)$. Hence, Eq. (15) becomes

$$\begin{aligned}&\left[\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}+\frac{2kU(r)}{r}-\frac{dU(r)}{dr}-U^2(r)\right.\\&\left.-\aleph_0\left[\sum(r)+(M-E_{nk})\right]\right]F_{nk}(r)=0,\end{aligned} \tag{17}$$

where we have defined the following for mathematical simplicity:

$$\Delta(r)=C_s, \tag{18}$$

$$\aleph_0=M+E_{nk}-C_s, \tag{19}$$

Pseudospin symmetry limit

The pseudospin symmetry limit occurs when $\frac{d\sum(r)}{dr}=0$ and $\Delta(r)=V(r)$. Thus, Eq. (16) becomes to

$$\begin{aligned}&\left[\frac{d^2}{dr^2}-\frac{k(k-1)}{r^2}+\frac{2kU(r)}{r}+\frac{dU(r)}{dr}-U^2(r)\right.\\&\left.-\aleph_1((M+E_{nk})+\Delta(r))\right]G_{nk}(r)=0,\end{aligned} \tag{20}$$

where

$$\aleph_1=M-E_{nk}+C_p, \tag{21}$$

$$\sum(r)=C_p. \tag{22}$$

Eigensolutions of the relativistic Dirac equation

In this section, we obtain the solutions of the Dirac equations for both the spin and pseudospin symmetries under tensor interaction. Here, we proposed a Coulomb-Hulthén like tensor potential as

$$U(r) = -\frac{H_C}{r} - \frac{H_H e^{-\delta r}}{1 - e^{-\delta r}}. \tag{23}$$

where H_C is the strength of the Coulomb-like tensor and H_H is the strength of the Hulthén-like tensor. In other for the proposed tensor potential to match up with the Trigonometric scarf potential, we substitute δ for 2α and adopt the following approximation scheme:

$$\frac{1}{r^2} \approx \frac{\delta^2}{(1 - e^{-\delta r})^2}. \tag{24}$$

Solutions of the spin symmetry limit

To obtain the solutions of the spin symmetry limit in the presence of tensor interaction, we substitute our potential (1), tensor potential (23) and approximation (24) into Eq. (17) to have

$$\frac{d^2F_{n\kappa}(y)}{dy^2} + \frac{(1-y)}{y(1-y)}\frac{dF_{n\kappa}(y)}{dy} + \frac{-\xi_1 y^2 + \xi_2 y - \xi_3}{y^2(1-y)^2}F_{n\kappa}(y) = 0, \tag{25}$$

where

$$\xi_1 = (\kappa + H_C)(\kappa + H_C + 1) + \frac{H_H}{\delta}\left(2\kappa + 2H_H + \frac{H_H}{\delta}\right) + \frac{\aleph_0(M - E_{n\kappa})}{\delta^2}, \tag{26}$$

$$\xi_2 = \frac{2\aleph_0[2V_0 + M - E_{n\kappa}]}{\delta^2} - \frac{H_H}{\delta}, \tag{27}$$

$$\xi_3 = \frac{\aleph_0(M - E_{n\kappa})}{\delta^2}, \tag{28}$$

Substituting Eqs. (26), (27) and (28) into Eq. (6), we have energy equation for the spin symmetry as

$$\frac{\aleph_0(M - E_{nkp})}{\delta^2} = \left[\frac{n(n+1) + \frac{\delta^2 + 2H_H\delta^{-1} - 8\aleph_0 V_0}{2\delta^2} + \left(n + \frac{1}{2}\right)\sqrt{1 + 4(\xi_1 + H_H\delta^{-1} - 4\aleph_0 V_0\delta^{-2})}}{1 + 2n + \sqrt{1 + 4(\xi_1 + H_H\delta^{-1} - 4\aleph_0 V_0\delta^{-2})}}\right]^2. \tag{29}$$

The upper component of the wave function is given by

$$F_{n\kappa}(r) = N_{n\kappa} y^{\sqrt{\xi_3}}(1-y)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\xi_1-\xi_3+\frac{H_H\delta^{-1}-4\aleph_1 V_0}{\delta^2}}} P_n^{\left(2\sqrt{\lambda_3},\, 2\sqrt{\frac{1}{4}+\xi_1-\xi_3+\frac{H_H\delta^{-1}-4\aleph_1 V_0}{\delta^2}}\right)}(1-2y), \tag{30}$$

and the lower component is obtain as

$$G_{n\kappa s}(r) = \frac{1}{M + E_{n\kappa s} - C_s}\left(\frac{d}{dr} + \frac{k}{r} - U(r)\right)F_{n\kappa s}(r). \tag{31}$$

Solutions of the pseudospin symmetry limit

To avoid repetition of previous algebra, we follow the same procedures as explained in the spin symmetry limit to obtain the negative energy eigenvalues for the pseudospin symmetry as

$$\frac{\aleph_2(M + E_{n\kappa p})}{\delta^2} = \left[\frac{n(n+1) + \frac{\delta^2 - 2H_H\delta^{-1} + 8\aleph_1 V_0}{2\delta^2} + \left(n + \frac{1}{2}\right)\sqrt{1 + 4(\lambda_1 - 2(\kappa + H_C) - H_H\delta^{-1} + 4\aleph_2 V_0\delta^{-2})}}{1 + 2n + \sqrt{1 + 4(\lambda_1 - 2(\kappa + H_C) - H_H\delta^{-1} + 4\aleph_2 V_0\delta^{-2})}}\right]^2. \tag{32}$$

where

$$\lambda_1 = (\kappa + H_C)(\kappa + H_C - 1) + \frac{H_H}{\delta}\left(2\kappa + 2H_H - \frac{H_H}{\delta}\right) + \frac{\aleph_1(M + E_{n\kappa})}{\delta^2}, \tag{32b}$$

The lower component of the Dirac spinor is obtained as

$$G_{n\kappa}(r) = N_{n\kappa} y^{\sqrt{\lambda_3}}(1-y)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\lambda_1-\lambda_3+\frac{4\aleph_1 V_0 - H_H\delta^{-1}}{\delta^2}}} P_n^{\left(2\sqrt{\lambda_3},\, 2\sqrt{\frac{1}{4}+\lambda_1-\lambda_3+\frac{4\aleph_1 V_0 - H_H\delta^{-1}}{\delta^2}}\right)}(1-2y), \tag{33}$$

and the upper component is also obtained as

$$F_{n\kappa p}(r) = \frac{1}{M - E_{n\kappa p} + C_p}\left(\frac{d}{dr} - \frac{k}{r} + U(r)\right)G_{n\kappa p}(r), \tag{34a}$$

$$\lambda_3 = \frac{\aleph_1(M + E_{n\kappa})}{\delta^2}. \tag{34b}$$

Non-relativistic limit

In this section, we obtain the non-relativistic limit of the spin symmetry which is usually equal to the solution of the Schrödinger equation. Using the following transformation: $M + E_{n\kappa} = \frac{2\mu}{\hbar^2}$, $M - E_{n\kappa} = -E_{n\ell}$, $C_P = H = 0$, $\kappa \to \ell$, Eq. (32) becomes

$$E_{n\ell} = -\frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\frac{2\mu V_0}{\alpha^2\hbar^2} - \ell(\ell+1) - n\left(n + 1 + \frac{1}{2n}\right) - \left(n + \frac{1}{2}\right)\sqrt{(1+2\ell)^2 - \frac{8\mu V_0}{\alpha^2\hbar^2}}}{2n + 1 + \sqrt{(1+2\ell)^2 - \frac{8\mu V_0}{\alpha^2\hbar^2}}}\right]^2. \tag{35}$$

with the corresponding wave function as

$$U_{n\ell}(r) = N_{n\ell} y^{\sqrt{-\frac{\mu E_{n\ell}}{2\alpha^2\hbar^2}}}(1-y)^{\frac{1}{2}+\frac{1}{2}\sqrt{(1+2\ell)^2 - \frac{8\mu V_0}{\alpha^2\hbar^2}}} P_n^{\left(\sqrt{-\frac{2\mu E_{n\ell}}{\alpha^2\hbar^2}},\, \sqrt{(1+2\ell)^2 - \frac{8\mu V_0}{\alpha^2\hbar^2}}\right)}(1-2y). \tag{36}$$

Fisher information and trigonometric scarf potential

The Fisher information for a physical system is given as [38–44]

$$I(\rho) = \int_0^\infty \frac{1}{\rho(r)}\left(\frac{d\rho(r)}{dr}\right)^2 dr, \tag{37}$$

where $\rho(r) = U_{n\ell}^2(r)$. In the integral method, the position space has integral limits of 0 to 1 while momentum space has integral limits of -1 to $+1$. Substituting the value of $\rho(r)$ into equation (37), we have

$$I(\rho) = N_{n\ell}^2[\aleph_{F1} + \aleph_{F2}], \tag{38}$$

where

$$\aleph_{F1} = \lambda^2\int_0^1 s^{\varepsilon+1}(1-s)^{\lambda-2}\Re ds + (\varepsilon+1)^2\int_0^1 s^{\varepsilon-1}(1-s)^{\lambda}\Re ds + \frac{16}{(2s-1)}\int_0^1 s^{\varepsilon+1}(1-s)^{\lambda-1}\Re ds, \tag{39}$$

$$\aleph_{F2} = \lambda(\varepsilon+1)\int_0^1 s^{\varepsilon}(1-s)^{\lambda}\Re ds + \frac{4\lambda}{(2s-1)}\int_0^1 s^{\varepsilon+1}(1-s)^{\lambda-1}\Re ds + \frac{16}{(2s-1)}\int_0^1 s^{\varepsilon}(1-s)^{\lambda}\Re ds, \tag{40}$$

$y = e^{-2\alpha r}$, $s = 1 - y$ and $\Re = (P_n^{(\lambda,\varepsilon)}(2s-1))^2$.

Using integral of the form

$$\int_0^1 x^t(1-x)^z(P_n^{(t,z)}(2x-1))^2 dx = \frac{2^{t+z+1}\Gamma(z+n+1)\Gamma(t+n+1)}{(2n+t+z)\Gamma(n+1)\Gamma(t+z+n+1)}, \tag{41}$$

we have Fisher information in the position space as

$$I(\rho) = \left[\begin{array}{l} \frac{\alpha\lambda^2(2n+\lambda+\varepsilon+1)\Gamma(\lambda+n-1)\Gamma(\lambda+\varepsilon+n+2)}{2(2n+\varepsilon+\lambda-1)\Gamma(\lambda+\varepsilon+n)\Gamma(\lambda+n+1)} \\ \quad + \frac{\alpha(\varepsilon+1)^2(2n+\lambda+\varepsilon+1)\Gamma(\varepsilon+n)\Gamma(\lambda+\varepsilon+n+2)}{2(2n+\varepsilon+\lambda-1)\Gamma(\lambda+\varepsilon+n)\Gamma(\varepsilon+n+1)} \\ + \frac{\alpha(\varepsilon+1)(2n+\lambda+\varepsilon+1)\Gamma(\varepsilon+n+1)\Gamma(\lambda+\varepsilon+2)}{(2n+\varepsilon+\lambda)\Gamma(\lambda+\varepsilon+1)\Gamma(\varepsilon+n+2)} \\ \quad + \frac{4\alpha\lambda(2n+\lambda+\varepsilon+1)\Gamma(\lambda+n)\Gamma(\lambda+\varepsilon+n+2)}{(2s-1)(2n+\varepsilon+\lambda)\Gamma(\lambda+\varepsilon+n+1)\Gamma(\lambda+n+1)} \\ + \frac{4\alpha(\varepsilon+1)(2n+\lambda+\varepsilon+1)\Gamma(\varepsilon+n+1)\Gamma(\lambda+\varepsilon+n+2)}{(2s-1)(2n+\varepsilon+\lambda)\Gamma(\lambda+\varepsilon+n+1)\Gamma(\varepsilon+n+2)} + \frac{32\alpha(2n+\lambda+\varepsilon+1)}{(2s-1)(2n+\varepsilon+\lambda+2)} \end{array}\right]. \tag{42}$$

To obtain the Fisher information in the momentum space, we make a transformation of variable as $z = 1 - 2y$, then, Eq. (37) turns to

$$I(\gamma) = \frac{1}{4\alpha}\int_{-1}^{1}\frac{2}{\gamma(z)(1-z)}\left(\frac{d\gamma(z)}{dz}\right)^2 dz, \tag{43}$$

where we have replaced ρ with γ for easy differentiation of position space from momentum space. Substituting the value of the probability density into Eq. (43) gives

$$I(\gamma) = N_{n\ell}^2[\aleph_{F3} + \aleph_{F4}], \tag{44}$$

where

$$\aleph_{F3} = \left(\frac{\lambda^2}{4}\int_{-1}^{1}\left(\frac{1-z}{2}\right)^{\lambda-2}\left(\frac{1-+z}{2}\right)^{\varepsilon+1} + \frac{(\varepsilon+1)^2}{4}\int_{-1}^{1}\left(\frac{1-z}{2}\right)^{\lambda}\left(\frac{1+z}{2}\right)^{\varepsilon-1} + \frac{4}{z^2}\int_{-1}^{1}\left(\frac{1-z}{2}\right)^{\lambda}\left(\frac{1+z}{2}\right)^{\varepsilon+1}\right)\mathbb{R}dz \tag{45}$$

$$\aleph_{F4} = \left(-\frac{\lambda(\varepsilon+1)}{4}\int_{-1}^{1}\left(\frac{1-z}{2}\right)^{\lambda-1}\left(\frac{1+z}{2}\right)^{\varepsilon+1} + \frac{\lambda}{z}\int_{-1}^{1}\left(\frac{1-z}{2}\right)^{\lambda-1}\left(\frac{1+z}{2}\right)^{\varepsilon+1} - \frac{(\varepsilon+1)}{z}\int_{-1}^{1}\left(\frac{1-z}{2}\right)^{\lambda}\left(\frac{1+z}{2}\right)^{\varepsilon}\right)\mathbb{R}dz, \tag{46}$$

$z = 1 - 2y$ and $\mathbb{R} = (P_n^{(\lambda,\varepsilon)}(z))^2$.

Using integral of the form

$$\int_{-1}^{1}\left(\frac{1-x}{2}\right)^a\left(\frac{1+x}{2}\right)^b(P_n^{(\lambda,\varepsilon)}(x))^2 dz = \frac{2\Gamma(a+n+1)\Gamma(b+n+1)}{n!\Gamma(a+b+2n+1)\Gamma(a+b+n+1)}, \tag{57}$$

we have the Fisher information in momentum space as

Table 1
Bound state energy eigenvalues in unit of fm^{-1} of the Trigonometric scarf potential for the spin symmetry with $M = 10\,fm^{-1}$, $C_s = 5\,fm^{-1}$, $V_0 = -5\,fm^{-1}$ and $\delta = 0.75$.

l	k	n	(l, j)	$H_C = 0$, $H_H = 0$.	$H_C = 1$, $H_H = 1$	$H_C = 1$, $H_H = 0.5$	$H_H = 1$ $H_C = 0.5$
0	−1	0	$0S_{1/2}$	4.7237672	4.7289360	4.7200571	4.7090993
0	−1	1	$1S_{1/2}$	4.1285665	4.1314516	4.1234540	4.1107862
0	−1	2	$2S_{1/2}$	3.4488016	3.4485993	3.4417282	3.4266059
0	−1	3	$3S_{1/2}$	2.6464611	2.6415479	2.6363001	2.6172088
1	−2	0	$0P_{3/2}$	4.7308929	4.6908329	4.7009399	4.6741910
1	−2	1	$1P_{3/2}$	4.1359835	4.0917485	4.1035415	4.0743967
1	−2	2	$2P_{3/2}$	3.4566804	3.4063290	3.4205407	3.3878340
1	−2	3	$3P_{3/2}$	2.6551561	2.5947338	2.6128631	2.5742037
2	−3	0	$0d_{5/2}$	4.7450523	4.6592250	4.6888789	4.6459801
2	−3	1	$1d_{5/2}$	4.1507171	4.0587867	4.0909757	4.0449676
2	−3	2	$2d_{5/2}$	3.4723312	3.3711835	3.4071632	3.3564352
2	−3	3	$3d_{5/2}$	2.6724112	2.5556962	2.5980442	2.5392819
3	−4	0	$0f_{7/2}$	4.7660727	4.6344984	4.6840296	4.6248187
3	−4	1	$1f_{7/2}$	4.1725805	4.0329867	4.0859226	4.0228810
3	−4	2	$2f_{7/2}$	3.4955368	3.3436398	3.4017804	3.3328430
3	−4	3	$3f_{7/2}$	2.6979620	2.5250249	2.5920792	2.5129842
1	1	0	$0P_{1/2}$	4.7308929	4.8228052	4.7784390	4.7972846
1	1	1	$1P_{1/2}$	4.1359835	4.2291289	4.1842109	4.2025899
1	1	2	$2P_{1/2}$	3.4566804	3.5523226	3.5062703	3.5241772
1	1	3	$3P_{1/2}$	2.6551561	2.7558218	2.7074765	2.7248934
2	2	0	$0d_{3/2}$	4.7450523	4.8775708	4.8170068	4.8495903
2	2	1	$1d_{3/2}$	4.1507171	4.2860376	4.2243102	4.2569682
2	2	2	$2d_{3/2}$	3.4723312	3.6125844	3.5487915	3.5818162
2	2	3	$3d_{3/2}$	2.6724112	2.8218579	2.7541961	2.7881734
3	3	0	$0f_{5/2}$	4.7660727	4.9368628	4.8612758	4.9066846
3	3	1	$1f_{5/2}$	4.1725805	4.3475829	4.2703000	4.3162653
3	3	2	$2f_{5/2}$	3.4955368	3.6776344	3.5974844	3.6445495
3	3	3	$3f_{5/2}$	2.6979620	2.8928704	2.8075407	2.8567866

$$I(\gamma) = \left[\begin{array}{l} \frac{\alpha\lambda^2\Gamma(\lambda+n-1)\Gamma(\lambda+\varepsilon+2n+2)\Gamma(\lambda+\varepsilon+n+2)}{\Gamma(2n+\varepsilon+\lambda)\Gamma(\lambda+\varepsilon+n)\Gamma(\lambda+n+1)} \\ \quad + \frac{\alpha(\varepsilon+1)^2\Gamma(n+\varepsilon)\Gamma(\varepsilon+\lambda+2n+2)\Gamma(\lambda+\varepsilon+n+2)}{\Gamma(2n+\varepsilon+\lambda)\Gamma(\lambda+\varepsilon+n)\Gamma(\varepsilon+n+2)} \\ - \frac{\alpha\lambda(\varepsilon+1)\Gamma(\lambda+n)\Gamma(\varepsilon+\lambda+2n+2)\Gamma(\lambda+\varepsilon+n+2)}{\Gamma(n+1+\lambda)\Gamma(\lambda+\varepsilon+2n+1)\Gamma(\varepsilon+\lambda+n+1)} \\ \quad + \frac{2\alpha\lambda\Gamma(n+\lambda)\Gamma(\lambda+\varepsilon+2n+2)\Gamma(\lambda+\varepsilon+n+2)}{z\Gamma(2n+\varepsilon+\lambda+1)\Gamma(\lambda+\varepsilon+n+1)\Gamma(\lambda+n+1)} \\ - \frac{4\alpha(\varepsilon+1)\Gamma(n+\varepsilon+1)\Gamma(\varepsilon+\lambda+2n+2)\Gamma(\lambda+\varepsilon+n+2)}{z\Gamma(2n+\varepsilon+\lambda+1)\Gamma(\lambda+\varepsilon+n+1)\Gamma(\varepsilon+n+2)} + \frac{16\alpha}{z^2} \end{array}\right]. \tag{58}$$

Discussion

The energy of the spin and pseudospin symmetry limits are calculated in Eqs. (36) and (39) respectively. The numerical results for the two symmetries are reported in Tables 1 and 2. It can be seen in Table 1 that in the absence of tensor interaction ($H_C = H_H = 0$), there are energy degeneracies between states such as $0P_{3/2} = 0P_{1/2}$, $1P_{3/2} = 1P_{1/2}$, $2P_{3/2} = 2P_{1/2}$, $3P_{3/2} = 3P_{1/2}$, $0d_{5/2} = 0d_{3/2}$, $1d_{5/2} = 1d_{3/2}$, $2d_{5/2} = 2d_{3/2}$, $3d_{5/2} = 3d_{3/2}$, $0f_{7/2} = 0f_{5/2}$, $1f_{7/2} = 1f_{5/2}$, $2f_{7/2} = 2f_{5/2}$, $3f_{7/2} = 3f_{5/2}$. However, when tensor potential is introduced, ($H_H \neq 0$, $H_C \neq 0$), the degeneracy doublets are removed completely. It is observed in the same table that the energy obtained for the spin symmetry are higher when the value of the strength of the Coulomb tensor term is higher than the value of the strength of the Hulthén tensor term (i.e. $H_C > H_H$) compared to when the strength of the Hulthén tensor term is higher than the strength of the Coulomb tensor term (i.e. $H_C < H_H$). In Table 2, we numerically presented the energy eigenvalues for the pseudospin symmetry for both the presence and absence of tensor interaction. In the absence of tensor interaction, the following degeneracies occurred: $1S_{1/2} = 1d_{3/2}$, $1P_{3/2} = 0f_{5/2}$, $1d_{5/}$

Table 2
Bound state energy eigenvalues in unit of fm^{-1} of the Trigonometric scarf potential for the pseudospin symmetry with $M = 10\,fm^{-1}$, $C_s = 5\,fm^{-1}$, $V_0 = 5\,fm^{-1}$ and $\delta = 0.75$.

$\tilde{l}$	n	k	(l, j)	$H_C = 0$, $H_H = 0$.	$H_C = 1$, $H_H = 1$	$H_C = 1$, $H_H = 0.5$	$H_H = 1$ $H_C = 0.5$
1	1	−1	$1S_{1/2}$	−4.4142785	−4.4334658	−4.4190772	−4.4255554
2	1	−2	$1P_{3/2}$	−4.4211946	−4.4184821	−4.4132980	−4.4122546
3	1	−3	$1d_{5/2}$	−4.4315252	−4.4068808	−4.4109816	−4.4023674
4	1	−4	$1f_{7/2}$	−4.4452178	−4.3987204	−4.4121402	−4.3959442
1	1	−1	$2S_{1/2}$	−3.9930077	−4.0128835	−3.9981892	−4.0050363
2	1	−2	$2P_{3/2}$	−3.9998696	−3.9980199	−3.9924560	−3.9918428
3	1	−3	$2d_{5/2}$	−4.0101200	−3.9865125	−3.9901581	−3.9820361
4	1	−4	$2f_{7/2}$	−4.0237076	−3.9784190	−3.9913074	−3.9756657
1	1	2	$0d_{3/2}$	−4.4142785	−4.4978860	−4.4568135	−4.4851932
2	1	3	$0f_{5/2}$	−4.4211946	−4.5255059	−4.4759832	−4.5113291
3	1	4	$0g_{7/2}$	−4.4315252	−4.5559967	−4.4983051	−4.5404006
4	1	5	$0h_{9/2}$	−4.4452178	−4.5892231	−4.5236743	−4.5722767
1	1	2	$1d_{3/2}$	−3.9930077	−4.0768099	−4.0356319	−4.0642115
2	1	3	$1f_{5/2}$	−3.9998696	−4.1042283	−4.0546566	−4.0901539
3	1	4	$1g_{7/2}$	−4.0101200	−4.1345045	−4.0768143	−4.1190172
4	1	5	$1h_{9/2}$	−4.0237076	−4.1675054	−4.1020013	−4.1506728

Table 3
Energy spectra for the trigonometric scarf type potential for various n and ℓ with $\alpha = 0.05$, $\hbar = 1$ and $\mu = 1/2$.

n	ℓ	$E_{n\ell}$ ($V_0 = -1$)	$E_{n\ell}$ ($V_0 = -2$)
0	0	−1.048767109	−2.069472480
1	0	−1.156296868	−2.218415177
	1	−1.161671283	−2.223680136
2	0	−1.273839218	−2.377364563
	1	−1.279463195	−2.382806146
	2	−1.290706514	−2.393687665
3	0	−1.401431750	−2.546341664
	1	−1.407305081	−2.551959812
	2	−1.419045257	−2.563193802
	3	−1.436639552	−2.580039062
4	0	−1.539124761	−2.725376334
	1	−1.545247057	−2.731170927
	2	−1.557483327	−2.742757150
	3	−1.575817232	−2.760129132
	4	−1.600225000	−2.783278201
5	0	−1.686975210	−2.914504310
	1	−1.693345908	−2.920475165
	2	−1.706077156	−2.932413257
	3	−1.725149032	−2.950311419
	4	−1.750532544	−2.974159062
	5	−1.782190633	−3.003942369

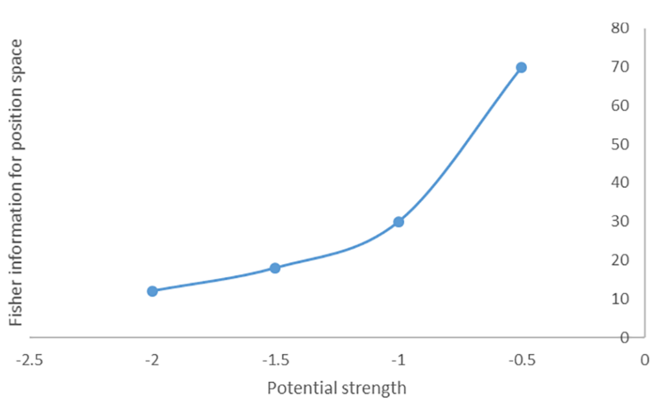

Fig. 1. Fisher information in position space against the potential strength.

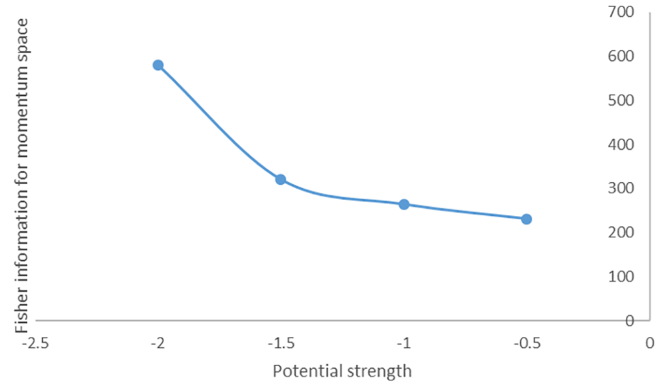

Fig. 2. Fisher information in momentum space against the potential strength.

$_2 = 0\,g_{7/2}$, $1f_{7/2} = 0\,h_{9/2}$, $2S_{1/2} = 1d_{3/2}$, $2P_{3/2} = 1f_{5/2}$, $2d_{5/2} = 1\,g_{7/2}$, $2f_{7/2} = 1\,h_{9/2}$. These degeneracies disappeared when tensor potential interacted with the system. In Table 3, we presented eigenvalues for the non-relativistic Schrödinger equation with Trigonometric scarf type potential for two values of the potential strength for various n and ℓ. It is observed that the energy decreases as both n and ℓ increases respectively. Similarly, the energy decreases as the potential strength decreases. In Figs. 1 and 2, we plotted Fisher information for position space and momentum space respectively against the potential strength. In the position space, Fisher information increases as the potential strength increases leading to an increase in the concentration of the system and a decrease in uncertainty. Hence, there is high degree of accuracy for predicting the localization of a particle in the system. In the momentum space, Fisher information decreases as the potential strength increases which indicates that the uncertainty is higher leading to higher disorderliness and low accuracy for predicting the localization of a particle in the system.

Conclusion

In this paper, the eigensolutions of the three-dimensional Trigonometric scarf potential is obtained using parametric Nikiforov-Uvarov method for both Dirac equation and Schrödinger equation. In the Dirac equation, the inclusion of the combined tensor potential removes the energy degeneracies between states in both the spin and pseudospin symmetries. This indicates that the Coulomb-Hulthén tensor term introduced in this work is better than the Coulomb tensor popularly used. The condition for energy equation for the Formula method has been simplified for a situation when $\alpha_1 = \alpha_2 = \alpha_3 = 1$. This makes it easier to obtain the equation using parametric Nikiforov-Uvarov parameters. The resulting wave function in the non-relativistic Schrödinger equation is squared to obtain probability density which was later used to calculate the Fisher information in position and momentum spaces. The Fisher information in position and momentum space were calculated using integral method. Our results obey Heisenberg uncertainty principle.

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