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Any ℓ−states solutions of the Schrödinger equation interacting with Hellmann-generalized Morse potential model

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Abstract

The approximate analytical solutions of the radial Schrödinger equation have been obtained with a newly proposed potential called Hellmann-generalized Morse potential. The potential is a superposition of Hellmann potential and generalized Morse or Deng-Fan potential. The Hellmann-generalized Morse potential actually comprises of three different potentials which includes Yukawa potential, Coulomb potential and Deng-Fan potential. The aim of combining these potentials is to have a wide application. The energy eigenvalue and the corresponding wave function are calculated in a closed and compact form using the parametric Nikiforov-Uvarov method. The energy equation for some potentials such as Deng-Fan, Rosen Morse, Morse, Hellmann, Yukawa and Coulomb potentials have also been obtained by varying some potential parameters. Some numerical results have been computed. We have plotted the behavior of the energy eigenvalues with different potential parameters and also reported on the numerical result. Finally, we computed the variance and information energy for the Hellmann-generalized Morse potential.

Keywords: Wave equation; Eigensolutions; Variance; Information energy

1. Introduction

Wave functions and their corresponding eigenvalues give significant information in describing various quantum systems. Thus, the exact and approximate solutions of the Schrödinger wave equation have a lot of effects in the non-relativistic quantum mechanics [1−4]. With the experimental verification of the Schrödinger wave equation, researchers have devoted much interest in solving the radial Schrödinger equation to obtain bound state solutions for some physical potential models [5−10]. These potentials play important roles in different fields of studies in physics such as; plasma, solid state and atomic physics [11]. Some of these potentials include generalized Morse potential [12], Yukawa potential [13], Wood−Saxon potential [14], Hulthén potential [15], Eckart potential [16], Makorov potential [17], Hellmann potential [18]. These potentials have been studied with the Schrödinger equation by different authors using different methods [19−23], which includes asymptotic

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iteration method (AIM) [24–30], the supersymmetric shape invariance method [31–36], the Nikiforov-Uvarov (NU) method [1–8,21], the variational method [37]. For instance, Ita [38] solved the Schrödinger equation with the Hellmann potential and obtained the energy eigenvalues and the corresponding wave functions using both 1/N expansion method and the NU method. Kocak et al. [39] in their work solved the Schrödinger equation with the Hellmann potential using the asymptotic iteration method and obtained energy eigenvalues and the wave functions. Mesa et al. [40] in their study, obtained bound state spectrum of the Schrödinger equation with generalized Morse potential and Pöschl–Teller potential respectively. Arda and Sever [41] solved one-dimensional Schrödinger equation for the generalized Morse potential with the NU method and obtained the bound state solutions for the effective mass for some diatomic molecules. Similarly, Zhang et al. [12] solved any ℓ state solutions of the Schrödinger equation with the generalized Morse potential using the basic concept of the supersymmetric shape invariance formalism and the function analysis method. It is important to note that the solutions of a combination of two or more of these potentials give a significant result with diverse applications. For instance, Onate and Ojonubah [42] obtained the solutions of the radial Schrödinger equation with a combination of Coulomb potential, Yukawa potential and inversely quadratic Yukawa potential (class of Yukawa potentials) which has application in plasma physics, solid state physics and atomic physics. Awoga and Ikot [43] obtained the solutions of the Schrödinger equation with generalized inverted hyperbolic potential. Onate et al. [44,45], obtained analytical solutions of both Dirac equation and Klein Gordon equation with Hellmann–Frost–Musulin potential (combination of Hellmann potential and Frost-Musulin potential) and combined potential (combination of general Manning–Rosen potential, hyperbolical potential and Pöschl–Teller potentials). The Hellmann–Frost–Musulin potential has application in condense matter physics, atomic and molecular physics while the combined potential has application in high energy physics, nuclear physics, atomic and molecular physics. Since the combination of two or more potentials gives better result especially when a stabilizing potential like the Yukawa is added, stability of the nucleus is achieved and has been used extensively in nuclear physics [46–50]. For instance, Hamzavi et al. [47] applied the inversely quadratic Yukawa potential and a tensor interaction term to the solutions of the approximate spin and pseudospin symmetries of the Dirac. Their results show that by applying the tensor interaction term, the degeneracies between spin and pseudospin state doublets were removed. The addition of the Hellmann and the generalized Morse potential model to the non-relativistic regime is lacking in literature and so call for essential interest.

In this paper, we investigate the applicability of the radial Schrödinger equation with a combination of Hellmann potential and generalized Morse or Deng-Fan potential known as the Hellmann-generalized Morse potential. This potential in a real sense, is a combination of three potentials, i.e. Yukawa, Coulomb and generalized Morse/Deng-Fan. The Hellmann-generalized Morse potential is given as

$$V(r) = D_e\left[1 + \frac{-a + be^{-\alpha r}}{rD_e} - \frac{2(e^{\alpha r_e} - 1)}{e^{\alpha r} - 1} + \left(\frac{e^{\alpha r_e} - 1}{e^{\alpha r} - 1}\right)^2\right], \tag{1}$$

where D_e is the dissociation energy, r_e is the equilibrium bond length, a and$b$ are the potential strengths and α is the screening parameter. To obtain any ℓ–state solutions of the radial Schrödinger equation with potential (1) analytically, a suitable approximation scheme must be employed. In the present work, we considered the following approximation type suggested by Greene and Aldrich [51] and Dong et al. [52].

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1 - e^{-\alpha r})^2}, \tag{2}$$

which is valid for $\alpha << 1$.

2. Parametric Nikiforov-Uvarov method

The Nikiforov-Uvarov method is based on the solution of a generalized second-order linear differential equation with special orthogonal function [53].

$$\psi_n''(s) + \frac{\overline{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\overline{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{3}$$

where $\sigma(s)$ and $\overline{\sigma}(s)$ are polynomials of at most second degree and $\overline{\tau}(s)$ is a first degree polynomial. The Schrödinger equation

$$\psi''(r) + [E - V(r)]\psi(r) = 0 \tag{4}$$

can be solved by this method. To make the application of this powerful method (Nikiforov-Uvarov method) simpler and straight forward without checking the validity of solution [54], we present a shortcut for the method. At first, we write the general form of the Schrödinger-like equation as [55–57].

$$\psi_n''(s)+\left(\frac{\theta_1-\theta_2 s}{s(1-\theta_3 s)}\right)\psi_n'(s)+\left(\frac{-\gamma_1 s^2+\gamma_2 s-\gamma_3}{s^2(1-\theta_3 s)^2}\right)\psi_n(s)=0 \tag{5}$$

Following the parametric Nikiforov-Uvarov method [53], we obtain the bound state energy condition as [55].

$$\theta_2 n+(2n+1)\left(\sqrt{\theta_9}+\theta_3\sqrt{\theta_8}\right)+n(n-1)\theta_3+\theta_7 = -2\theta_3\theta_8-2\sqrt{\theta_8\theta_9}+(2n+1)\theta_5, \tag{6}$$

with the wave function as

$$\psi(s)=N_{n,\ell}s^{\theta_{12}}(1-\theta_3 s)^{-\theta_{12}-\frac{\theta_{13}}{\theta_3}}P_n^{\left(\theta_{10}-1,\frac{\theta_{11}}{\theta_3}-\theta_{10}-1\right)}(1-2\theta_3 s). \tag{7}$$

The parameters in (6) and (7) are deduce as follows

$$\theta_4=\frac{1}{2}(1-\theta_1),\theta_5=\frac{1}{2}(\theta_2-2\theta_3),\theta_6=\theta_5^2+\gamma_1,$$
$$\theta_7=2\theta_4\theta_5-\gamma_2,\theta_8=\theta_4^2+\gamma_3,\theta_9=\theta_3\theta_7+\theta_3^2\theta_8+\theta_6$$
$$\theta_{10}=\theta_1+2\theta_4-2\sqrt{\theta_8},\theta_{11}=\theta_2-2\theta_5+2\left(\sqrt{\theta_9}-\theta_3\sqrt{\theta_8}\right),$$
$$\theta_{12}=\theta_4-\sqrt{\theta_8},\theta_{13}=\theta_5-\left(\sqrt{\theta_9}-\theta_3\sqrt{\theta_8}\right). \tag{8}$$

To see the accuracy of the approximation, we plotted the effective potential with three different values of the angular momentum in Fig. 2.

3. Solution to the Schrödinger equation

To obtain the solution of the Schrödinger-like equation given in (3) with potential (1), we write the radial Schrödinger equation of the form [42].

$$\frac{d^2R}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left\{E-V(r)-\frac{L\hbar^2}{2\mu r^2}\right\}\right]R(r)=0, \tag{9}$$

where$L=l(l+1)$, μ is the particle mass, E is the non-relativistic energy and $V(r)$ is the interacting potential. Substituting the Hellmann-generalized Morse potential (1) and approximation (2) into (9), we have

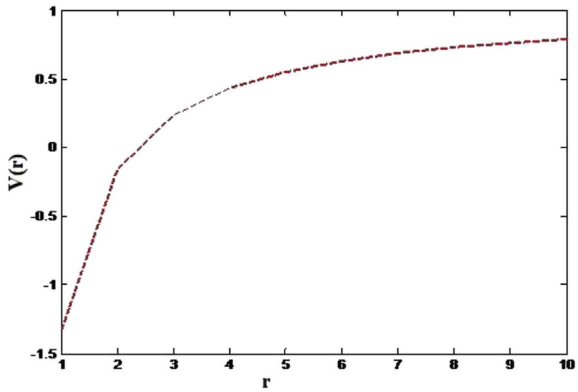

Fig. 1. Hellmann-generalized Morse potential with $D_e=0$, $a=2$, $b=1$, $r_e=1$ and $\alpha=0.2$.

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left\{E-D_e\left(1+\frac{\alpha}{1-e^{-\alpha r}}\left(\frac{-a+be^{-\alpha r}}{D_e}\right)-\frac{2c}{e^{\alpha r}-1}+\frac{c^2}{(e^{\alpha r}-1)^2}\right)-\frac{\alpha^2}{(1-e^{-\alpha r})^2}\frac{L\hbar^2}{2\mu}\right\}\right]R(r)=0, \tag{10}$$

where $c=e^{\alpha r_e}-1$. To use the parametric Nikiforov-Uvarov method, we defined a variable of the form $s=e^{-\alpha r}$ and substitute it into (10) to have

$$\left[\frac{d^2}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d}{ds}+\frac{1}{s^2(1-s^2)}\left\{-\wedge_1 s^2+\wedge_2 s-\wedge_3\right\}\right]R(s)=0, \tag{11}$$

where

$$\wedge_1=-\frac{2\mu E}{\alpha^2\hbar^2}+\frac{2\mu(D_e+2cD_e+D_e c^2-\alpha b)}{\alpha^2\hbar^2}, \tag{12}$$

$$\wedge_2=\frac{4\mu E}{\alpha^2\hbar^2}+\frac{2\mu(-2D_e-2cD_e+\alpha a+\alpha b)}{\alpha^2\hbar^2}, \tag{13}$$

$$\wedge_3=-\frac{2\mu E}{\alpha^2\hbar^2}+\frac{2\mu(D_e-\alpha a)}{\alpha^2\hbar^2}+L. \tag{14}$$

Comparing (11) with (5), we obtain

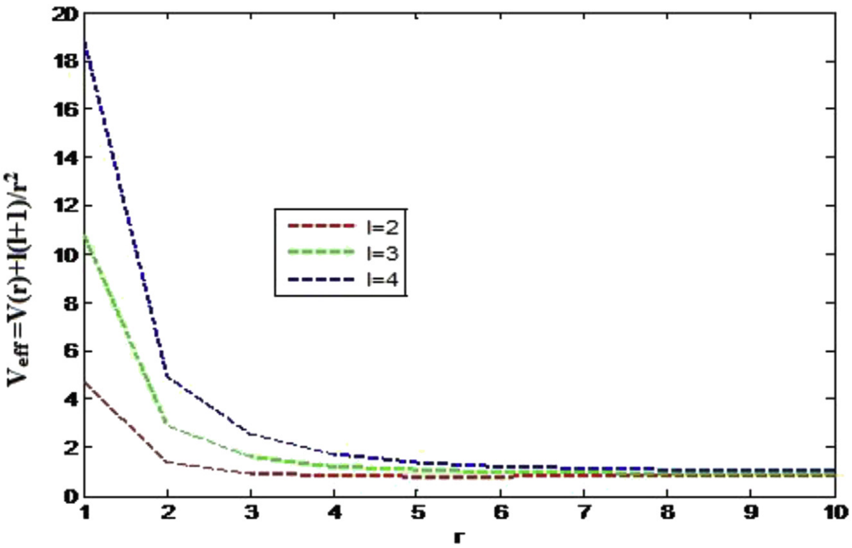

Fig. 2. Effective potential $V_{eff} = V(r) + \frac{\ell(\ell+1)}{r^2}$ for three different values of angular momentum with $D_e = 5$, $a = 2$, $b = 1$, $r_e = 1$ and $\alpha = 0.2$.

$$\left.\begin{array}{l}\theta_1=\theta_2=\theta_3=1,\theta_4=0,\theta_5=-\frac{1}{2},\theta_6=\frac{1}{4}+\wedge_1,\\ \theta_7=-\wedge_2,\theta_8=\wedge_3,\theta_9=\frac{1}{4}\left((1+2\ell)^2+\frac{8\mu D_e c^2}{\alpha^2\hbar^2}\right),\\ \theta_{10}=1+2\sqrt{\wedge_3},\theta_{11}=2\left(1+\sqrt{\wedge_3}+\sqrt{\theta_9}\right),\theta_{12}=\sqrt{\wedge_3},\\ \theta_{13}=-\frac{1}{2}-\sqrt{\wedge_3}-\sqrt{\theta_9}.\end{array}\right\}. \tag{15}$$

Now substituting the appropriate parametric constants in (15) into (6) and (7), we obtain the energy equation for the Hellmann-generalized Morse potential as

$$E_{n\ell} = D_e + \frac{L\alpha^2\hbar^2}{2\mu} - \alpha a - \frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu[2cD_e+\alpha(a-b)]}{\alpha^2\hbar^2} - n(n+1) - 2L - \left(\left(n+\frac{1}{2}\right)\sqrt{(1+2\ell)^2+\frac{8\mu D_e c^2}{\alpha^2\hbar^2}}+\frac{1}{2}\right)}{2n+1+\sqrt{(1+2\ell)^2+\frac{8\mu D_e c^2}{\alpha^2\hbar^2}}}\right]^2, \tag{16}$$

and the corresponding wave function as

$$R(s) = N_{n\ell}s^u(1-s)^v p_n^{(2u,2v-1)}(1-2s). \tag{17}$$

Using the normalization condition, we obtain the normalization constant as follows:

$$\int_0^\infty R_{n,\ell}(r)\times R_{n,\ell}(r)^* dr = 1, \tag{18}$$

$$-\frac{1}{\alpha}\int_1^0 |R_{n,\ell}(s)|^2\frac{ds}{s} = 1,\ \ s = e^{-\alpha r}, \tag{19}$$

$$\frac{1}{2\alpha}\int_{-1}^1 |R_{n,\ell}(y)|^2\frac{2}{1-y}dy = 1,\ \ y = 1-2s. \tag{20}$$

Substituting (17) into (20), we have

$$\frac{N_{n\ell}^2}{2\alpha}\int_{-1}^1\left(\frac{1-s}{2}\right)^{2u-1}\left(\frac{1+s}{2}\right)^{2v}\left[P_n^{(2u,2v-1)}(p)\right]^2 ds = 1, \tag{21}$$

where

$$v=\sqrt{(1+2\ell)^2-\frac{2\mu D_e c^2}{\hbar^2}}, \quad (22)$$

$$u=\frac{1}{2}+\frac{1}{2}\sqrt{\ell(\ell+1)+\frac{2\mu(D_e-E_{n\ell}-\alpha a)}{\alpha^2\hbar^2}}. \quad (23)$$

Comparing (21) with the integral of the form

$$\int_{-1}^{1}\left(\frac{1-p}{2}\right)^x\left(\frac{1+p}{2}\right)^y\left[P_n^{(2x,2y-1)}(p)\right]^2dp =\frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)}. \quad (24)$$

We have the normalization constant as

$$N_{n\ell}^2=\frac{n!2u\alpha\Gamma(2u+2v+n+1)}{\Gamma(2u+n+1)\Gamma(2v+n+1)}. \quad (25)$$

3.1. Special cases

i. The potential (1) can be reduced to other useful potentials by varying the potential parameters which also reduce the energy equation (16). When the parameter $a=b=0$, the potential (1) turns to generalized Morse or Deng-Fan potential

$$V(r)=D_e\left[1+-\frac{2(e^{\alpha r_e}-1)}{e^{\alpha r}-1}+\left(\frac{e^{\alpha r_e}-1}{e^{\alpha r}-1}\right)^2\right], \quad (26)$$

and the energy equation becomes

$$E_{n\ell}=D_e+\frac{L\alpha^2\hbar^2}{2\mu}-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{4\mu c D_e}{\alpha^2\hbar^2}-n(n+1)-2L-\left(\left(n+\frac{1}{2}\right)\sqrt{(1+2\ell)^2+\frac{8\mu D_e c^2}{\alpha^2\hbar^2}}+\frac{1}{2}\right)}{2n+1+\sqrt{(1+2\ell)^2+\frac{8\mu D_e c^2}{\alpha^2\hbar^2}}}\right]^2. \quad (27)$$

ii. When the parameter $D_e=0$, the potential (1) reduces to Hellmann potential

$$V(r)=\frac{-a+be^{-\alpha r}}{r}, \quad (28)$$

and the energy equation (16) becomes

$$E_{n\ell}=\frac{L\alpha^2\hbar^2}{2\mu}-\alpha a-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu(a-b)}{\alpha\hbar^2}-(n+\ell+1)^2-L}{2(n+\ell+1)}\right]^2. \quad (29)$$

When the parameters $b=D_e=0$ and the screening parameter approaches zero, potential (1) becomes Coulomb potential

$$V(r)=\frac{-a}{r}, \quad (30)$$

and the energy equation (16) becomes the energy levels for pure Coulomb potential.

$$E_{n\ell}=\frac{L\alpha^2\hbar^2}{2\mu}-\alpha a-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu a}{\alpha\hbar^2}-(n+\ell+1)^2-L}{2(n+\ell+1)}\right]^2. \quad (31)$$

iii. When $a=D_e=0$, potential (1) becomes Yukawa potential

$$V(r)=\frac{be^{-\alpha r}}{r}, \quad (32)$$

and the energy equation (16) becomes

$$E_{n\ell}=\frac{L\alpha^2\hbar^2}{2\mu}-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{-2\mu b}{\alpha\hbar^2}-(n+\ell+1)^2-L}{2(n+\ell+1)}\right]^2. \quad (33)$$

When the parameter $a=b=0$ and $c=e^{\alpha r_e}+1$, the potential (1) turns to Rosen Morse potential.

Similarly, when the parameter $a=b=0$ and $c=e^{\alpha r_e}$, the potential (1) turns to Morse potential.

4. Variance under Hellmann-generalized Morse potential

The variance $V(\rho)$ is define as

$$V(\rho)=\langle r^2\rangle_\rho-\langle r\rangle_\rho^2=\int\left(r-\langle r\rangle_\rho\right)^2\rho(r)dr=\int_0^\infty r^2\rho(r)dr-\int_0^\infty r\rho(r)dr, \quad (34)$$

and the normalization condition is obtained with (35)

$$N_{n\ell}e^{-\alpha ru}\left(1-e^{-\alpha r}\right)^v p_n^{(2u,2v-1)}\left(1-2e^{-\alpha r}\right) = N_{n\ell}e^{-\alpha ru}\left(1-e^{-\alpha r}\right)^v {}_2F_1\left(-n,n+2(v+u),2u+1,e^{-\alpha r}\right). \quad (35)$$

$${}_2F_1(a,b,d,1)=\frac{\Gamma(d)\Gamma(d+a-b)}{\Gamma(d+a)\Gamma(d-b)}. \quad (36)$$

$$\int_0^\infty z^2e^{-pz}dz=\frac{n!}{p^{n+1}},\ p>0,\ n=1,2,3,---. \quad (37)$$

The integral (34) can be evaluated using (25), and the definitions (35), (36) and (37) in (34) leading to (38)

$$V(\rho)=\frac{1,562.5(0.0392)^{2v}n!\frac{\Gamma(2u+2v+n+1)}{\Gamma(2u+n+1)\Gamma(2v+n+1)}\left(1-0.0064u^2\right)\left[\frac{\Gamma(2u+1)\Gamma(1-2n-2v)}{\Gamma(2u+1-n)\Gamma(1-n-2v)}\right]^2}{\left(\frac{1}{2}+\frac{1}{2}\sqrt{\ell(\ell+1)+\frac{50\mu(D_e-E_{n\ell}-0.2a)}{\hbar^2}}.\right)^2}. \quad (38)$$

4.1. Information energy

The information energy $E(\rho)$ is given by

$$E(\rho)=4\pi\int_0^\infty r^2\rho^2(r)dr. \quad (39)$$

Using (25), (35), (36) and (37) in (39), we have

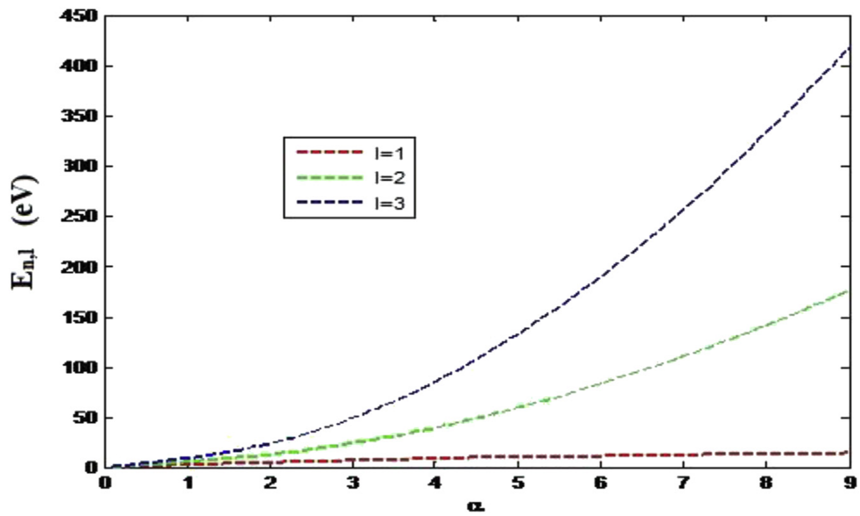

Fig. 3. Energy spectra of the Hellmann-generalized Morse potential against the screening parameter with $D_e = 5$, $a = 2$, $b = 1$, $r_e = 1$ and $\hbar = \mu = 1$.

$$E(\rho)=\frac{981.875(0.0392)^{4v}(n!)^2\left(\frac{\Gamma(2u+2v+n+1)}{\Gamma(2u+n+1)\Gamma(2v+n+1)}\right)^2\left(\frac{\Gamma(2u+1)\Gamma(1-2n-2v)}{\Gamma(2u+1-n)\Gamma(1-n-2v)}\right)^4}{\frac{1}{2}+\frac{1}{2}\sqrt{\ell(\ell+1)+\frac{50\mu(D_e-E_{n\ell}-0.2a)}{\hbar^2}}}. \quad (40)$$

In Figs. 1 and 2, we plotted the Hellmann-generalized Morse potential and effective potential respectively. In Fig. 3, we graphically show the variation of Hellmann-generalized Morse potential with the screening parameter. It can be seen that the energy eigenvalues increases as the screening parameter increases. In Figs. 4–7, we plotted energy for Hellmann-generalized Morse, Rosen Morse, Morse and Deng-Fan respectively with the reduced Planck constant. In Table 1, we reported the energy eigenvalues (in units $\mu=\hbar=1$) of the Hellmann-generalized potential for various orbital and angular momentum n and ℓ for different values of the potential strength ($a=b=1$, $a=2$, $b=1$ and $a=1$, $b=2$) with $\alpha=0.2$, $D_e=5$ and $r_e=1$. In Table 2, we obtained the energy eigenvalues for Deng-Fan potential, (i.e. $a=b=0$), Rosen Morse potential ($a=b=0$ and $c=e^{\alpha r_e}+1$) and Morse potential ($a=b=0$ and $c=e^{\alpha r_e}$) with $\alpha=0.2$, $D_e=5$ and $r_e=10$. These potentials are physically alike. The numerical results obtained in Table 2 showed an agreement between the potentials. In Table 3, we numerically reported the energy eigenvalues of the Hellmann potential ($D_e=0$), Yukawa potential ($D_e=a=0$) and Coulomb potential ($D_e=b=0$). It can be seen that the numerical values are in agreement.

5. Conclusion

In this paper, we obtained solutions of the Schrödinger equation with a combination of three different potentials (Coulomb potential, Yukawa potential and generalized Morse of Deng-Fan potential) which we called Hellmann-generalized Morse potential using parametric Nikiforov-Uvarov method. The eigensolutions are obtained and numerical results are presented for Hellmann-generalized Morse potential and other useful potentials like the Deng-Fan, Rosen

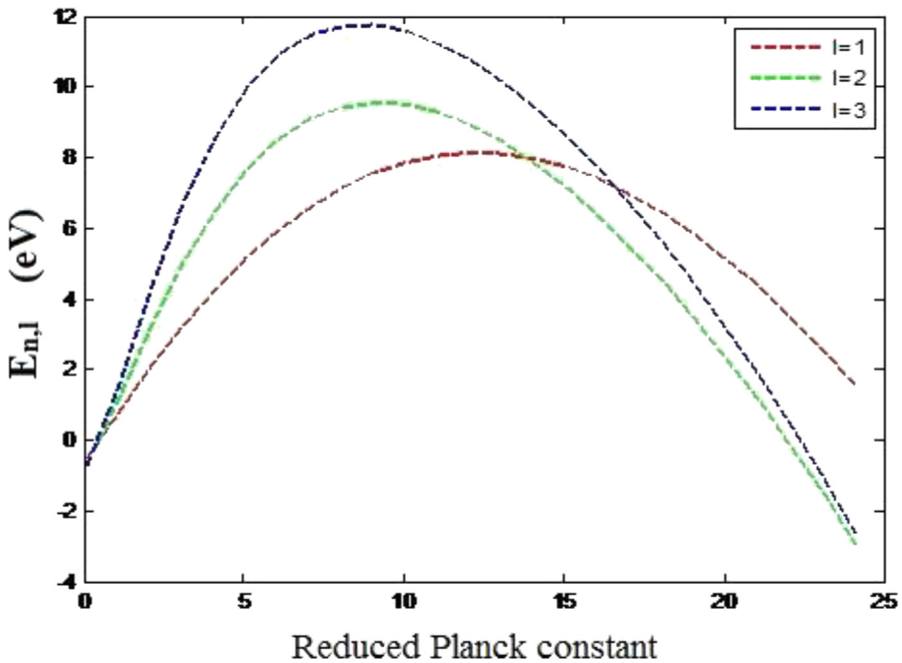

Fig. 4. Energy spectra of the Hellmann-generalized Morse potential against the reduced Planck constant($\hbar$) with $D_e=5$, $a=2$, $b=1$, $r_e=1$ and $\mu=1$.

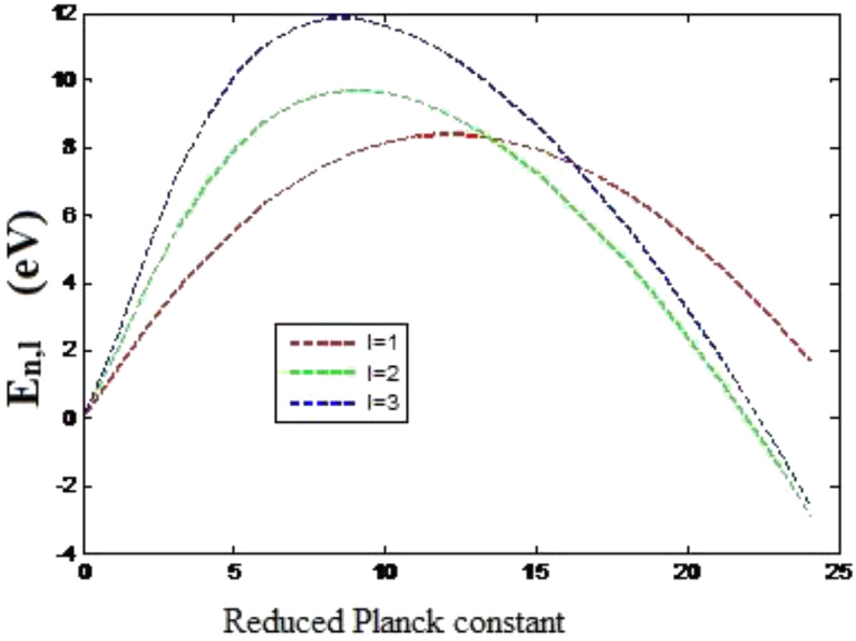

Fig. 5. Energy spectra of the Rosen Morse potential against the reduced Planck constant($\hbar$) with $D_e = 5$, $a = b = 0$, $c = e^{\alpha r_e} + 1$, $r_e = 1$ and $\mu = 1$.

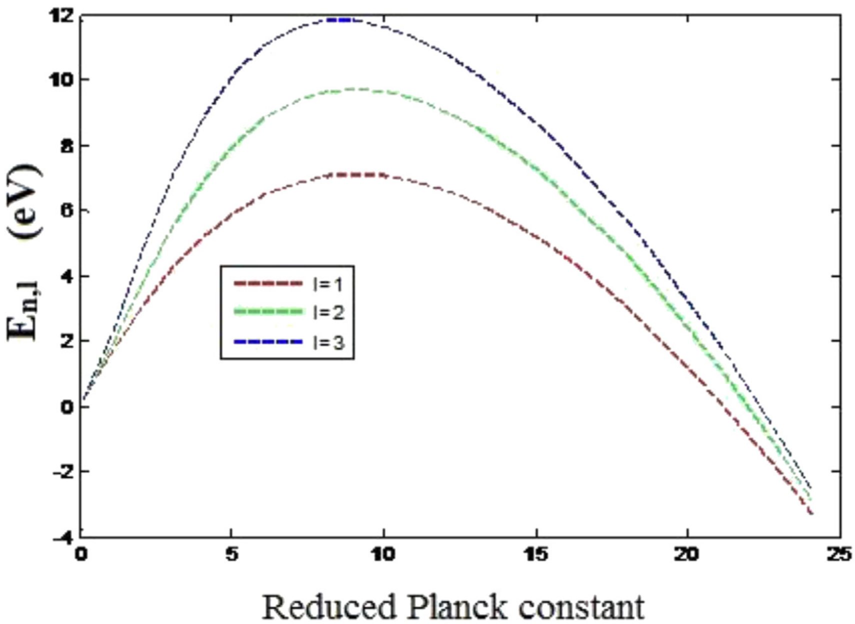

Fig. 6. Energy spectra of the Morse potential against the reduced Planck constant($\hbar$) with $D_e = 5$, $a = b = 0$, $c = e^{\alpha r_e}$, $r_e = 1$ and $\mu = 1$.

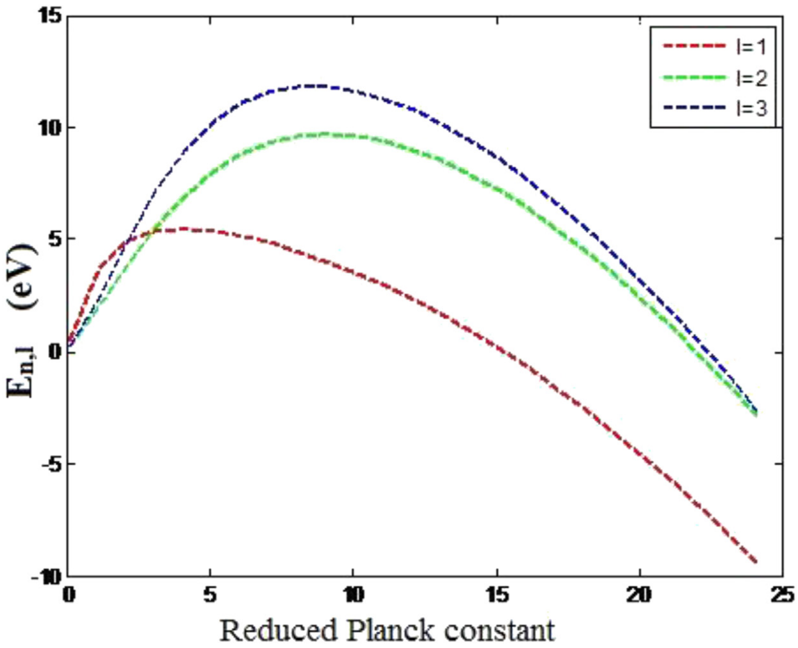

Fig. 7. Energy spectra of the Deng Fan potential against the reduced Planck constant($\hbar$) with $D_e = 5$, $a = b = 0$, $r_e = 1$ and $\mu = 1$.

Morse, Morse, Hellmann, Yukawa and Coulomb. It is observed that when the Yukawa potential strength in potential (1) equal to the dissociation energy and is equal to zero as the screening parameter approaches zero, the energy levels approaches the familiar pure Coulomb potential energy levels. We have also computed the variance and information energy under the Hellmann-generalized Morse potential. Our results find applications where the component potentials are useful.

Table 1
Energy spectra for Hellmann-generalized Morse potential with $D_e = 5$, r_e=1, α=0.2 and $\mu = \hbar = 1$.

n	ℓ	$E_{n\ell}$ $a = b = 1$	$E_{n\ell}$ $a = 1, b = 2$	$E_{n\ell}$ $a = 2, b = 1$
0	1	0.3283679	0.4122219	0.4372743
	3	0.7303891	0.8124575	0.4484402
	5	1.4394427	1.5171254	1.1609905
1	1	1.1281301	1.2020894	0.8534265
	3	1.5058288	1.5777102	1.2332089
	5	2.1711869	2.2394080	1.9022374
2	1	1.8266046	1.8914471	1.5610569
	3	2.1816981	2.2446049	1.9180913
	5	2.8075646	2.8670601	2.5473785
3	1	2.4331146	2.4895386	2.1760214
	3	2.7675157	2.8221329	2.5122341
	5	3.3572067	3.4086383	3.1051194
4	1	2.9558225	3.0044568	2.7065523
	3	3.2712255	3.3181701	3.0236496
	5	3.8276804	3.8716445	3.5830929
5	1	3.4018994	3.4433118	3.1598821
	3	3.6998101	3.7396393	3.4593800
	5	4.2256426	4.2626783	3.9880135

Table 2
Energy spectra for Deng-Fan, Rosen Morse and Morse potentials with $D_e = 5$, $r_e = 10$, $\alpha = 0.2$ and $\mu = \hbar = 1$.

n	ℓ	$E_{n\ell}$ for Deng-Fan $c = e^{\alpha r_e} - 1$	$E_{n\ell}$ for Rosen Morse $c = e^{\alpha r_e} + 1$	$E_{n\ell}$ for Morse $c = e^{\alpha r_e}$
0	1	0.4118458	0.3972941	0.4035778
	3	0.6761791	0.6456389	0.6587868
	5	1.1506835	1.0919517	1.1172266
1	1	1.0814890	1.0490430	1.0631015
	3	1.3403530	1.2935065	1.3137654
	5	1.8050517	1.7328494	1.7640483
2	1	1.6923995	1.6468213	1.6666170
	3	1.9459609	1.8874933	1.9128553
	5	2.4011521	2.3200271	2.3551954
3	1	2.2464333	2.1919159	2.2156472
	3	2.4948514	2.4288831	2.4575748
	5	2.9408204	2.8547634	2.8921782
4	1	2.7453587	2.6855672	2.7116524
	3	2.9887860	2.9189135	2.9493799
	5	3.4258060	3.3382908	3.3764452
5	1	3.1908614	3.1289708	3.1560336
	3	3.4294440	3.3587773	3.3896677
	5	3.8577764	3.7717972	3.8093859

Table 3
Energy spectra for Hellmann, Yukawa and Coulomb potentials with $\alpha = 0.001$ and $\mu = \hbar = 1$.

n	ℓ	$-E_{n\ell}$ for Hellmann $a = 2, b = -1$	$-E_{n\ell}$ for Yukawa $a = 0, b = -3$	$-E_{n\ell}$ For Coulomb $a = 3, b = 0$
0	1	1.1247501	1.1227501	1.1257501
	3	0.2806251	0.2786251	0.2816251
	5	0.1242501	0.1222501	0.1252501
1	1	0.5001674	0.4981674	0.5011674
	3	0.1797808	0.1777808	0.1807808
	5	0.0914929	0.0894193	0.0924193
2	1	0.2815640	0.2795640	0.2825640
	3	0.1250020	0.1230020	0.1260020
	5	0.0701116	0.0681116	0.0711116
3	1	0.1803826	0.1783826	0.1813826
	3	0.0917288	0.0899729	0.0929729
	5	0.0555040	0.0535040	0.0565040
4	1	0.1254207	0.1234207	0.1264207
	3	0.0753653	0.0685365	0.0715365
	5	0.0450561	0.0430561	0.0460561
5	1	0.0922811	0.0902811	0.0932811
	3	0.0558407	0.0538407	0.0568407
	5	0.0373268	0.0353267	0.0383267

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References

[1] A.N. Ikot, L.E. Akpabio, Int. Rev. Phys. 4 (2010) 224.
[2] A.N. Ikot, E.J. Uwah, I.O. Akpan, L.E. Akpabio, IJPAP 6 (2010) 511.
[3] A.N. Ikot, E.J. Uwah, L.E. Akpabio, I.O. Akpan, Afr. J. Math. Phys. 8 (2010) 43.
[4] A.D. Antia, A.N. Ikot, L.E. Akpabio, Eur. J. Sci. Res. 46 (2010) 107.
[5] S.M. Ikhdair, R. Sever, Int. J. Mod. Phys. A 21 (2006) 6465.
[6] A.N. Ikot, L.E. Akpabio, Appl. Phys. Res. 2 (2010) 202.
[7] B.I. Ita, Nig. J. Phys. 20 (2008) 221.
[8] S.M. Ikhdair, R. Sever, Open Phys. 5 (4) (2007) 516–527.
[9] M. Hamzavi, S.M. Ikhdair, B.I. Ita, Phys. Scri 85 (2012) 045007.
[10] B.I. Ita, C.O. Ehi-Eromosele, A. Edobor-Osoh, A.I. Ikeuba, in: AIP Conference Proceedings, vol. 169, 2014, p. 360.
[11] E. Maghsoodi, E. Hassanabadi, O. Aydogdu, Phys. Scr. 86 (2012) 015005.
[12] L.H. Zhang, X.P. Li, C.S. Jia, Int. J. Quantum Chem. 111 (2011) 1870.
[13] C.A. Onate, Afr. Rev. Phys. 8 (2013) 0046.
[14] B. Gonul, K. Koksal, Phys. Scr. 76 (2007) 565.
[15] S. Meyur, S. Debnath, Mod. Phy. Lett. A 23 (2008) 2077.
[16] W.C. Qiang, J.Y. Wu, S.H. Dong, Phys. Scr. 79 (2009) 065011.
[17] A.A. Makarov, J.A. Smorodinsky, Kh Valiev, P. Winternitz, Nuov. Cim. A 52 (1967) 1061.
[18] R.K. Amlan, J. Math. Chem. 44 (2008) 260.
[19] O. Bayrak, I. Boztosun, H. Ciffci, Int. J. Quantum Chem. 107 (2007) 504.
[20] C.S. Jia, L. Yun, Y. Sun, Phys. Lett. A 305 (2002) 231.
[21] C. Berkdemir, A. Berkedemir, R. Sever, Phys. Rev. C 72 (2005) 027001.
[22] M. Simsek, H. Egrifes, J. Phys. A Math. Gen. 37 (2004) 4379.
[23] A.R. Akbarieh, H. Motavalli, Mod. Phys. Lett. A 23 (2008) 3008.
[24] O. Bayrak, I. Boztosun, Phys. Scr. 76 (2007) 92.
[25] A. Soylu, O. Bayrak, I. Boztosun, J. Phys. A Math. Theor. 41 (2008) 065308.
[26] O. Bayrak, I. Boztosun, J. Phys. A Math. Theor. 40 (2007) 11119.
[27] A. Soylu, O. Bayrak, I. Boztosun, J. Math. Phys. 48 (2007) 082302.
[28] K.J. Oyewunmi, B.J. Falaye, C.A. Onate, O.J. Oluwadare, W.A. Yahya, J. Mol. Phys. 112 (2014) 127.
[29] K.J. Oyewunmi, B.J. Falaye, C.A. Onate, O.J. Oluwadare, W.A. Yahya, Int. J. Mod. Phys. E 23 (2014) 1450005.
[30] S.H. Behere, Indian J. Phys. 55B (1982) 272.
[31] G.F. Wei, S.H. Dong, Phys. Lett. B 686 (2010) 288.
[32] H. Hassanabadi, S. Zarrinkamar, H. Rahimov, Commun. Theor. Phys. 56 (2011) 423.
[33] C.S. Jia, T. Chen, S. He, Phys. Lett. A 377 (2013) 682.
[34] H. Hassanabadi, E. Maghsoodi, S. Zarrinkamar, H. Rahimov, Can. J. Phys. 90 (2012) 633.
[35] H. Hassanabadi, L.L. Lu, S. Zarrinkamar, G. Wu, H. Rahimov, Acta Phys. Pol. A 122 (2012) 111.
[36] C. Onate, Afr. Rev. Phys. 9 (2015) 0033.
[37] B. Batiha, Aust. J. Basic Appl. Sci. 3 (2009) 3826.
[38] B.I. Ita, Ultra Sci. 22 (2009) 573.
[39] G. Kocak, O. Bayrak, I. Boztosun, J. Theor. Comp. Chem. 6 (2007) 893.
[40] A.D. Mesa, C. Quesne, Y.F. Smirnov, J. Phys. A 31 (1998) 321.
[41] A. Arda, R. Sever, Comm. Theor. Phys. 56 (2011) 51.
[42] C.A. Onate, J.O. Ojonubah, J. Theor. Appl. Phys. 10 (2016) 21.
[43] O.A. Awoga, A.N. Ikot, Pramana 79 (2012) 345.
[44] C.A. Onate, M.C. Onyeaju, A.N. Ikot, Ann. Phys. 375 (2016) 239.
[45] C.A. Onate, M.C. Onyeaju, A.N. Ikot, J.O. Ojonubah, Chin. J. Phys. 54 (2016) 820.
[46] S.M. Ikhdair, Phys. Scr. 88 (2013) 065007.
[47] M. Hamzavi, S.M. Ikhdair, B.I. Ita, Phys. Scr. 85 (2012) 045009.
[48] A.N. Ikot, S. Zarrinkamar, B.H. Yazarloo, H. Hassanabadi, Chin. Phys. B 23 (2014) 100306.
[49] H.P. Obong, A.N. Ikot, I.O. Owate, J. Korean Phys. Soc. 66 (2015) 867.
[50] A.N. Ikot, H. Hassanabadi, T.M. Abbey, Commun. Theor. Phys. 64 (2015) 637.
[51] R.L. Greene, C. Aldrich, Phys. Rev. A14 (1976) 2363.
[52] S.H. Dong, W.C. Qiang, G.H. Sun, V.R. Bezerra, J. Phys. A40 (2007) 10535.
[53] A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics, Birkhauser,Verlag, Basel, 1988.
[54] M. Hamzavi, K.E. Thylwe, A.A. Rajabi, Commun. Theor. Phys. 60 (2013) 1.
[55] C. Tezcan, R. Sever, Int. J. Theor. Phys. 48 (2009) 337.
[56] M.G. Miranda, G.H. Sun, S.H. Dong, Int. J. Mod. Phys. E19 (2010) 123.
[57] M.C. Zhang, G.H. Sun, S.H. Dong, Phys. Lett. A 374 (2010) 704.