Dirac and Schrödinger Equations in Presence of Actual and Exponential Inverted Generalized Hyperbolic Potential

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Abstract

Using two different methods, we approximately solved the Dirac equation and the Schrödinger equation with an inverted generalized hyperbolic potential in actual and exponential form. By varying the values of parameters a, b, c and d, we obtained energy eigenvalues for some well-known potential models. We have shown the behavior of ground state energy with α for some potential models. A comparison of the numerical results shows an excellent agreement between the energy eigenvalues of the hyperbolical potential and the inverted generalized hyperbolic potential.

1. Introduction

The analytical solutions of both the relativistic and non-relativistic wave equations for some known physical potential models have been a great line of interest in several research areas in quantum mechanics due to their importance in quantum systems [1]. However, only a few of these potentials can explicitly be solved for all n and ℓ quantum numbers [2] for some special cases of interaction due to the presence of the inverse square term. The presence of the centrifugal term requires the use of approximation schemes. The most frequently used are the Pakeris approximation scheme [3] and the Greene-Aldrich approximation type [4]. Considering the effect of the relativistic wave equation with several potential fields, there are various works on the Dirac equation and the Klein-Gordon equation using approximation schemes. The Klein-Gordon equation is the equation of motion of a quantum scalar or pseudo-scalar field whose quanta are spin-less particles (pion) [5]. It describes the quantum amplitude for finding a point particle in various places. On the account of the Dirac equation, it describes the elementary spin - ½ particles such as electron consistent with both the particles of quantum mechanics and the theory of special relativity [6]. Dirac equation provides a theoretical justification for the interaction of several component wave functions in Pauli's phenomenological theory of spin. In the case of bound state solutions in this study, the Dirac equation is viewed under spin and pseudospin symmetry. The reliability of the pseudospin symmetry was analyzed by Marcos et al. [7] and they found that the nuclear surface strongly increases the effect of the pseudospin-orbital potential. They also pointed out that the pseudospin symmetry cannot be justified by the smallness of the potential $\sum(r)$ but by the strong compensation of different contributions to the single-particle energy of a nucleon in the Dirac equation [8]. Page et al. [9] however pointed out that the condition of the difference between the scalar potential and vector potential $\Delta(r)$ results in the spin symmetry which is relevant to mesons.

The aim of this paper is to investigate the amendability of the inverted generalized hyperbolical potential with both the relativistic and non-relativistic wave equations in its actual form and its exponential form. Thus, in this study, we investigate the energy spectrum of both the Dirac equation and the Schrödinger equation in the framework of parametric Nikiforov-Uvarov method and the supersymmetric approach, respectively, with the actual and exponential inverted generalized hyperbolic potential. The inverted generalized hyperbolic potential in its actual form under consideration is given by [10]

$$V_{IGH}(r) = -aV_0\coth(\alpha r) + bV_1\coth^2(\alpha r) - cV_2\,\mathrm{csch}^2(\alpha r) + d, \tag{1}$$

where, a, b, c and d are real constants, V_0, V_1 and V_2 are potential depth. By varying the magnitude of the real constants, we can easily obtain some known potential models as we shall see later. This potential is closely related to the real hyperbolical potential, asymmetric and symmetric Rosen-Morse potential. However, the potential (1) can be studied explicitly using the following approximation scheme

$$\frac{1}{r^2} = \alpha^2\,\mathrm{csc}\,h^2(\alpha r), \tag{2}$$

which is a good approximation to the centrifugal term $\frac{1}{r^2}$ and it is valid for $\alpha \ll 1$. The potential (Eqn. (1)) in its exponential form is

$$V_{IGH}(r) = d + bV_1 - \frac{aV_0\left(1+e^{-2\alpha r}\right)}{1-e^{-2\alpha r}} + \frac{4\left(bV_1 - cV_2\right)e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}, \tag{3}$$

and the approximation (Eqn. (2)) in the form

$$\frac{1}{r^2} \approx \frac{4\alpha^2}{\left(1-e^{-2\alpha r}\right)^2}, \tag{4}$$

where we have used the following transformation

$$\sinh(\alpha r) = \frac{e^{\alpha r} - e^{-\alpha r}}{2}, \tag{5}$$

$$\cosh(\alpha r) = \frac{e^{\alpha r} + e^{-\alpha r}}{2}. \tag{6}$$

The scheme of our work is as follows. In section 2, we briefly review the parametric Nikiforov-Uvarov method. In section 3, we obtained the bound state solutions for Dirac and Schrödinger equations. In section 4, we discussed the work and result while in the final section, we state the conclusion.

2. Parametric Nikiforov-Uvarov method

The method Nikiforov-Uvarov (NU) method is based upon reducing the second-order linear differential equation to a hyper-geometric type equation [11]. By introducing an appropriate coordinate transformation $s = s(x)$, one can write an equation of the form

$$\psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{7}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials of degree two at most, and $\bar{\tau}(s)$ is a polynomial of degree one at most. To use parametric Nikiforov-Uvarov method, Tezcan and Sever [12-15], we transformed Eq. (7) into the following form

$$\psi''(r) + \left[\frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\right]\psi'(s) + \left[\frac{-As^2 + Bs - C}{\left(s(1-s)\right)^2}\right]\psi(s) = 0. \tag{8}$$

According to the parametric Nikiforov-Uvarov method, the condition for eigenvalues and eigen-functions are [12-15]

$$n\alpha_2 - (2n+1)\alpha_5 + \left[n(n-1) + 2\alpha_8\right]\alpha_3 = -\sqrt{4\alpha_8\alpha_9} - (2n+1)\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right). \tag{9}$$

and

$$\psi_{n,\ell}(s) = N_{n,\ell}\, s^{\alpha_{12}}\left(1-\alpha_3 s\right)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} \times P_n^{\left(\alpha_{10}-1,\, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1\right)}\left(1 - 2\alpha_3 s\right) \tag{10}$$

respectively, where

$$\left.\begin{aligned}
&\alpha_4 = \frac{1-\alpha_1}{2}, \alpha_5 = \frac{\alpha_2 - 2\alpha_3}{2}, \alpha_6 = \alpha_5^2 + A,\\
&\alpha_7 = 2\alpha_4\alpha_5 - B, \alpha_8 = \alpha_4^2 + C,\\
&\alpha_9 = \alpha_3\left(\alpha_7 + \alpha_3\alpha_8\right) + \alpha_6,\\
&\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8},\\
&\alpha_{11} = \alpha_2 - 2\alpha_5 + 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right),\\
&\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \alpha_{13} = \alpha_5 - \left(\sqrt{\alpha_9} + \alpha_5\sqrt{\alpha_8}\right)
\end{aligned}\right\}. \tag{11}$$

3. Dirac equation with exponential inverted generalized hyperbolic potential

The Dirac equation for a spin-1/2 particle with mass M moving in the field of a repulsive vector and attractive scalar potentials $(V(r)$ and $S(r))$ in relativistic units $(c=\hbar=1)$ is given as

$$\left[\vec{\alpha}.\vec{p}+\beta\left(M+S(r)\right)-\left(E-V(r)\right)\right]\psi(\vec{r})=0, \tag{12}$$

where $\vec{p}=-i\vec{\nabla}$ is the momentum operator, E denote the relativistic energy of the system, α and β are 4×4 usual Dirac matrices. The spin and pseudospin symmetry for the Dirac equation are given by

$$\left[\frac{d^2}{dr^2}-(M+E_{n,\mathrm{k}}-\Delta(r))(M-E_{n,\mathrm{k}}+\Sigma(r))\right]F_{n,\ell}(r)$$

$$=\left[\frac{k(k+1)}{r^2}-\frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr}+\frac{k}{r}\right)}{M+E_{n,\mathrm{k}}-\Delta(r)}\right]F_{n,\mathrm{k}}(r), \tag{13}$$

$$\left[\frac{d^2}{dr^2}-(M+E_{n,\mathrm{k}}-\Delta(r))(M-E_{n,\mathrm{k}}+\Sigma(r))\right]G_{n,\ell}(r)$$

$$=\left[\frac{k(k-1)}{r^2}-\frac{\frac{d\sum(r)}{dr}\left(\frac{d}{dr}-\frac{k}{r}\right)}{M-E_{n,\mathrm{k}}+\sum(r)}\right]G_{n,\mathrm{k}}(r) \tag{14}$$

where

$\Delta(r)=V(r)-S(r)$ and $\sum(r)=V(r)+S(r)$

respectively,

3.1. Spin symmetry.

The spin symmetry limit in the Dirac equation occurs when $\frac{d\Delta(r)}{dr}=0,\ \Delta(r)=C_s$ and $\sum(r)=V(r)$

[7, 9]. Substituting the potential (Eqn. (3)) and approximation (Eqn. (4)) into (Eqn. (13)) and by introducing a transformation $z=e^{-2\alpha r}$, the second-order differential equation (13) becomes

$$\frac{d^2F_{n,\mathrm{k}}(r)}{dz^2}+\frac{1-z}{z(1-z)}\frac{dF_{n,\mathrm{k}}(r)}{dz}+T \tag{15}$$

$$T=\frac{1}{\left(z(1-z)\right)^2}\left[Az^2+Bz+C\right]F_{n,\mathrm{k}}(r) \tag{15a}$$

where

$$A=\frac{\beta\left(d+bV_1+aV_0+M-E_{nk,s}\right)}{4\alpha^2} \tag{16}$$

$$B=\frac{\beta\left(d+2cV_2+M-E_{nk,s}-bV_1\right)}{2\alpha^2}+k(k+1) \tag{17}$$

$$C=\frac{\beta\left(d+bV_1+M-E_{nk,s}-aV_0\right)}{4\alpha^2}, \tag{18}$$

$$\beta=M+E_{nk}-C_s. \tag{19}$$

Now, to test the validity of approximation in Eqn. (2), we define the following function

$$f(r)=\alpha^2k(k+1)\,\mathrm{csch}^2(\alpha r). \tag{20}$$

Comparing Eq. (15) with Eq. (8), we deduce the following

$$\left.\begin{aligned}
&\alpha_1=\alpha_2=\alpha_3=1,\alpha_4=0,\\
&\alpha_5=-\frac{1}{2},\alpha_6=\frac{1}{4}+A,\alpha_7=-A,\\
&\alpha_9=\frac{1}{4}+\frac{\beta\left(bV_1-cV_2\right)}{\alpha^2}+k\left(k+1\right),\\
&\alpha_{10}=1+2\sqrt{\zeta_3},\alpha_8=A,\\
&\alpha_{11}=1+\alpha_{10}+2\sqrt{\alpha_9},\\
&\alpha_{12}=\sqrt{\zeta_3},\alpha_{13}=-\frac{1}{2}\left(1+\sqrt{\zeta_3}\right)-\sqrt{\alpha_9}
\end{aligned}\right\}. \tag{21}$$

Substituting Eqn. (21) in Eqns. (9) and (10), the positive energy and the upper component wave function for the spin symmetry in the Dirac equation as

$$4\alpha^2\left(\frac{n(n+1)+k(k+1)+\dfrac{\beta(2\eta-aV_0)}{2\alpha^2}+\dfrac{\aleph_1}{2}}{2n+1+\sqrt{(1+2k)^2+\dfrac{4\eta\beta}{\alpha^2}}}\right)^2.$$

$$=E_{nk,s}^2-M^2+C_s\left(M-E_{nk,s}\right)+\aleph_0. \qquad (22)$$

$$\aleph_0=\beta\left(aV_0-d-bV_1\right)+4k(k+1)\alpha^2 \qquad (22a)$$

$$\aleph_1=\left[1+(2n+1)\sqrt{(1+2k)^2+\frac{4\eta\beta}{\alpha^2}}\right]. \qquad (22b)$$

The upper component of the wave function is obtained as

$$F_{n,k}(z)=N_{n,k}z^{\sqrt{\zeta_3}}\left(1-z\right)^{-\frac{1}{2}\left(1+\sqrt{\zeta_3}\right)-\sqrt{\alpha_9}} \times P_n^{\left(2\sqrt{\zeta_3},2\sqrt{\alpha_9}+1\right)}\left(1-2z\right). \qquad (23)$$

where we have used the following for mathematical simplicity $\eta=bV_1-cV_2$.

3.2 Pseudospin symmetry limit

The pseudospin symmetry limit occurs when $\frac{d\sum(r)}{dr}=0,\ \sum(r)=C_{ps}$ and $\Delta(r)=V(r)=$ potential in Eqn. (3) [7, 9]. With the potential (Eqn. (3)), the approximation (Eqn. (4)) and the transformation $z=e^{-2\alpha r}$, Eqn. (14) becomes

$$\frac{d^2G_{n,k}(r)}{dz^2}+\frac{1-z}{z(1-z)}\frac{dG_{n,k}(r)}{dz}+T_1=0, \qquad (24)$$

where,

$$A_1=-\frac{\beta_1\left(d+bV_1+aV_0+M-E_{nk,ps}\right)}{4\alpha^2} \qquad (25)$$

$$B_1=-\frac{\beta_1\left(2cV_2+d-M+E_{nk,ps}-bV_1\right)}{2\alpha^2}+k(k-1),$$

(26)

$$C_1=\frac{\beta_1\left(-d-bV_1-M+E_{nk,ps}+aV_0\right)}{4\alpha^2}, \qquad (27)$$

$$\beta_1=M-E_{nk,ps}+C_{ps} \qquad (28)$$

$$T_1=\frac{1}{\left(z(1-z)\right)^2}\left[A_1z^2+B_1z+C_1\right]G_{n,k}(r). \qquad (29)$$

To avoid repetition of algebra, a first inspection for the relationship between the present set of parameters $(\beta_1 set)$ and the previous (βset) enable us to know that the negative solution for Eqn. (22) can be obtained by using the parameter map

$$F_{nk}\leftrightarrow G_{nk},$$

With these transformations and following the previous work we obtain the relativistic energy spectrum for the pseudospin symmetry as

$$E_{nk,ps}^2-M^2+C_{ps}\left(E_{nk,ps}-M\right)+4k(k-1)\alpha^2=$$

$$\left(2\alpha\left(\frac{n\left(n+1+\dfrac{1}{2n}\right)-\aleph_a+\left(n+\dfrac{1}{2}\right)\aleph_b}{1+2n+\aleph_b}\right)\right)^2, \qquad (30)$$

$$\aleph_a=2\lambda(\lambda-1)+\frac{\beta_1\left(aV_0-2\eta\right)}{2\alpha^2}, \qquad (31)$$

$$\aleph_b=\sqrt{(1-2k)^2-\frac{4\eta\beta_1}{\alpha^2}}. \qquad (32)$$

The lower component wave function is obtain as

$$G_{n,k}(z)=N_{n,k}z^{\sqrt{\zeta_{31}}}\left(1-z\right)^{-\Im}P_n^{\Re}\left(1-2z\right). \qquad (33)$$

$$\Re = 2\left(\sqrt{\zeta_{31}}, \sqrt{\alpha_9} + \frac{1}{2}\right), \quad (34)$$

$$\Im = \frac{1}{2}\left(1 + \sqrt{\zeta_{31}}\right) - \sqrt{\alpha_9}. \quad (35)$$

3.3 Schrödinger equation with actual inverted generalized hyperbolic potential via supersymmetric approach.

Given the 3-dimensional Schrödinger equation of the form [16, 17]

$$\left(-\frac{\hbar^2}{2\mu}\left[\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+ \\ \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+ \\ \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\end{array}\right]+V(r)-E\right)\psi(\boldsymbol{r})=0, \quad (36)$$

and setting the wave function $\psi(\boldsymbol{r}) = \frac{R_{nl}(r)Y_{ml}(\theta,\phi)}{r}$, we obtain the radial part of the Schrödinger equation by the separation of variables as

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{n,\ell}-V(r)\right)-\frac{\ell(\ell+1)}{r^2}\right]R_{n,\ell}(r)=0. \quad (37)$$

Substituting potential in Eqn. (1) and the approximation in Eqn. (2) into Eqn. (37), we obtained a differential equation of the form

$$\frac{d^2R_{n,\mathrm{k}}(r)}{dr^2}=\left[\varepsilon_{T_1}\operatorname{csch}^2(\alpha r)-\varepsilon_{T_2}\coth(\alpha r)+\varepsilon_{T_3}\right]R_{n,\mathrm{k}}(r), \quad (38)$$

where

$$\left.\begin{array}{l}\varepsilon_{T_1}=\varepsilon_0+\varepsilon\varepsilon_1, \varepsilon_{T_2}=aV_0\varepsilon, \varepsilon=\frac{2\mu}{\hbar^2} \\ \varepsilon_{T_3}=\varepsilon(\varepsilon_2+\varepsilon_n), \varepsilon_0=\ell(\ell+1)\alpha^2, \\ \varepsilon_1=bV_1-cV_2, \varepsilon_2=bV_1+\mathrm{d}, \varepsilon_n=E_{nk},\end{array}\right\}. \quad (39)$$

For bound state, the possible solution to the Riccati equation (38) satisfying the above condition is the super-potential function of the form [18-21]

$$W(r)=\varepsilon_4-\varepsilon_3\coth(\alpha r). \quad (40)$$

In order to make the left hand side of Eq. (40) compatible to the right hand side, the following condition must be fulfilled:

$$W^2(r)-W'(r)=\varepsilon_{T_1}\operatorname{csc}h^2(\alpha r)-\varepsilon_{T_2}\coth(\alpha r)+\varepsilon_{T_3}. \quad (41)$$

Then, the two constants in Eqn. (36) are deduce as follows

$$\varepsilon_{T_3}=\varepsilon_4(\varepsilon_4)+\varepsilon_3(\varepsilon_3), \quad (42)$$

$$\varepsilon_4=\frac{\varepsilon_{T_2}}{2\varepsilon_3}, \quad (43)$$

$$\varepsilon_3=\alpha\left(\frac{1\pm\sqrt{(1+2\ell)^2+4\alpha^{-2}\left(\varepsilon_{T_1}-\varepsilon_0\right)}}{2}\right), \quad (44)$$

where $\varepsilon_4>0$. We find the Hamiltonian of Eqn. (37) as it is related to the super-potential function via [22-26]

$$V_{\pm}(r)=W^2(r)\pm\frac{dW(r)}{dr}, \quad (45)$$

and the ground state wave function $U_{0,\ell}(r)$ is simply calculated from

$$U_{0,\ell}(r)=\exp\left(-\int W(r)dr\right). \quad (46)$$

The two partner potential are given as

$$V_{+}(r)=\varepsilon_4^2+\varepsilon_3^2+\varepsilon_3(\varepsilon_3+\alpha)\operatorname{csc}h^2(\alpha r)-\square, \quad (47)$$

$$V_{-}(r)=\varepsilon_4^2+\varepsilon_3^2+\varepsilon_3(\varepsilon_3-\alpha)\operatorname{csc}h^2(\alpha r)-\square. \quad (48)$$

$$\square=2\varepsilon_3\varepsilon_4\coth(\alpha r). \quad (49)$$

If the shape invariance condition is satisfied and all desirable results are obtained, then

$$V_{+}(a_0,r)=V_{-}(a_1,r)+R(a_1), \quad (50)$$

where a_1 is a new set of parameters uniquely determined from an old set a_0 via mapping of the form: $a_1 \rightarrow a_0-\alpha$ and subsequently $a_n \rightarrow a_0-n\alpha$ where, $\varepsilon_3=a_0$ as the residual term $R(a_1)$ is independent of the variable r. Substituting for the parameters in Eqn. (42) one gets the energy equation as

$$E_{n\ell}=\chi-\frac{\alpha^2\hbar^2E_{FN}}{2\mu}-\frac{\alpha^2\hbar^2E_{SN}}{2\mu}, \quad (51)$$

$$E_{FN}=\left(\frac{1+2n+\chi_0}{2}\right)^2, \quad (52)$$

$$E_{SN}=\frac{\left(\frac{2\mu aV_0}{\hbar^2}\right)^2}{\alpha^2\left(1+2n+\chi_0\right)^2}, \quad (53)$$

$$\chi=bV_1+d, \quad (54)$$

$$\chi_0=\sqrt{\left(1+2\ell\right)^2+\frac{8\mu\left(bV_1-cV_2\right)}{\alpha^2\hbar^2}}. \quad (55)$$

Special cases:

putting $d=b=0, a=1, c=-1$, the potential (1) reduces to generalized version of the Eckart potential

$$E_{n\ell}=-\frac{\alpha^2\hbar^2}{2\mu}\left[\left(\frac{\chi_{S1}}{2}\right)^2+\left(\frac{\frac{2\mu V_0}{\hbar^2}}{\alpha^2\chi_{S1}}\right)^2\right] \quad (56)$$

$$\chi_{S1}=1+2n+\sqrt{\left(1+2\ell\right)^2+\frac{8\mu V_2}{\alpha^2\hbar^2}} \quad (57)$$

Putting $d=c=0, a=b=-1,$ the potential (Eqn.(1)) turns to asymmetric trigonometric Rosen-Morse potential

$$V_{IGH}(r)=V_0\coth(\alpha\,\mathrm{r})-V_1\coth^2(\alpha r), \quad (58)$$

and Eqn. (51) turns to

$$E_{n\ell}=V_0-V_1-\frac{\alpha^2\hbar^2}{2\mu}\left[\left(\frac{\chi_{S2}}{2}\right)^2+\left(\frac{\frac{-2\mu V_0}{\hbar^2}}{\alpha^2\chi_{S2}}\right)^2\right] \quad (59)$$

$$\chi_{S2}=1+2n+\sqrt{\left(1+2\ell\right)^2-\frac{8\mu V_1}{\alpha^2\hbar^2}}. \quad (60)$$

Taking $c=0,$ the potential (Eqn. (1)) becomes the hyperbolical potential and the energy equation (Eqn. (51)) reduces to

$$V_{GVE}(r)=d-aV_0\coth(\alpha\,\mathrm{r})+bV_1\coth^2(\alpha r), \quad (61)$$

$$E_{n\ell}=bV_1+d-\frac{\alpha^2\hbar^2}{2\mu}\left[\left(\frac{\chi_{S3}}{2}\right)^2+\left(\frac{\frac{2\mu aV_0}{\hbar^2}}{\alpha^2\chi_{S3}{}^2}\right)^2\right], \quad (62)$$

$$\chi_{S3}=1+2n+\sqrt{\frac{8\mu bV_1}{\alpha^2\hbar^2}+\left(1+2\ell\right)^2}. \quad (63)$$

Putting $a=c=d=0,$ the potential (1) reduces to symmetric trigonometric Rosen-Morse potential

$$V_{GVE}(r)=bV_1\coth^2(\alpha r), \quad (64)$$

and the energy equation (51) reduces to

$$E_{n\ell}=bV_1-\frac{\hbar^2\alpha^2}{2\mu}\left(\frac{1+2n+\sqrt{\frac{8\mu bV_1}{\alpha^2\hbar^2}+(1+2\ell)^2}}{2}\right)^2 . \quad (65)$$

Putting $b=c=d=0, a=\left(\frac{1}{2}\right), \alpha=\left(\frac{1}{2x}\right)$,

potential (Eqn. (1)) becomes standard Hulthėn potential

$$V_{SH}(r)=-\frac{V_0}{2}\coth\left(\frac{r}{2x}\right), \quad (66)$$

and energy equation (51) reduces to

$$E_{n\ell}=-\frac{V_0}{2}-\frac{\hbar^2}{8\mu}\left[\left(\frac{(1+n+\ell)}{x}\right)^2+\left(\frac{\frac{-2\mu xV_0}{\hbar^2}}{1+n+\ell}\right)^2\right]. \quad (67)$$

4. Results and discussion

The energy equation of some potential models are obtained by varying the values of the real constants in potential (Eqn. (1)).

Generalized version of the Eckart potential: This is obtained when $b=d=0$

Asymmetric trigonometric Rosen-Morse potential: This is obtained when $c=d=0$.

Hyperbolical potential: This is obtained when $c=0$.

Symmetric trigonometric Rosen-Morse potential: This is obtained when $a=c=d=0$.

Standard Hulthėn potential: This is obtained when $b=c=d=0,\ a=\frac{1}{2}$ and $\alpha=\frac{1}{2x}$.

In Tables 1 and 2, we numerically reported the energy eigenvalues of the inverted generalized hyperbolic potential for spin and pseudospin symmetry respectively. In Table 3, we presented numerical results obtained from the two methods (supersymmetric method and Nikiforov-Uvarov method) with $V_0=10,\ a=0.2,\ V_1=1.1,$ $V_2=c=1,\ d=10$ and $b=1$. In Tables 4 and 5, we compared the present results with previous results for hyperbolical and Eckart potentials, respectively. It can be seen that our results agreed with the previous results.

5. Conclusion

In this work, we obtained the solutions of Dirac and Schrödinger equations with actual and exponential inverted generalized hyperbolic potential via Nikiforov-Uvarov method and the supersymmetric approach, respectively. We also obtained the solution of some known potential models by varying the numerical values of the real constants in the inverted generalized hyperbolic potential model such as generalized version of the Eckart potential, asymmetric trigonometric Rosen-Morse potential, standard Hulthėn potential, hyperbolical potential, and symmetric trigonometric Rosen-Morse potential. To test the accuracy of our results, we found the non-relativistic limit of the spin symmetry and computed the numerical results and then compared the results obtained from the non-relativistic Schrödinger equation. It is observed from the table that the result from the two methods are in excellent agreement. It is also seen that the numerical results of the hyperbolical potential and the inverted generalized hyperbolic potential are equal. The results of the Eckart potential is in good agreement with previous results. This shows that our methods and results are efficient, effective and accurate.

Table 1: Bound states for the spin symmetry limit in units of fm^{-1} for $a=4,\ b=3,\ c=2,\ d=1,$ $M=1fm^{-1}\ C_s=5\,fm^{-1}, V_0=5, V_1=1, V_2=2, k=\ell,$ $\alpha=0.10$.

l	$n,\ \kappa$	(l,j)	$E_{s,n,\kappa}$
1	0, -2,1	$0p_{3/2}$, $0p_{1/2}$	4.001774786
2	0, -3,2	$0d_{5/2}$, $0d_{3/2}$	4.004018067
3	0, -4,3	$0f_{7/2}$, $0f_{5/2}$	4.007162883
4	0, -5,4	$0g_{9/2}$, $0g_{7/2}$	4.011210254
1	1, -2,1	$1p_{3/2}$, $1p_{1/2}$	4.004032636
2	1, -3,2	$1d_{5/2}$, $1d_{3/2}$	4.007014355
3	1, -4,3	$1f_{7/2}$, $1f_{5/2}$	4.011048661
4	1, -5,4	$1g_{9/2}$, $1g_{7/2}$	4.015983144

Table 2: Bound states for the pseudospin symmetry limit in units of fm^{-1} for $a=4,\ \ b=3,\ \ c=2,$ $d=1,\ M=1\,fm^{-1}, C_{ps}=-5\,fm^{-1}, V_0=5,$ $V_1=1, V_2=2,\ k=\ell$ and $\alpha=0.10$

$\tilde{l}$	n, κ	(l,j)	$E_{ps,n,\kappa}$
1	1, -1,2	$1s_{1/2}$, $0d_{3/2}$	-3.996129126
2	1, -2,3	$1p_{3/2}$, $0f_{5/2}$	-3.992989951
3	1, -3,4	$1d_{5/2}$, $0g_{7/2}$	-3.988962108
4	1, -4,5	$1f_{7/2}$, $0h_{9/2}$	-3.984039485
1	2, -1,2	$2s_{1/2}$, $1d_{3/2}$	-3.993338541
2	2, -2,3	$2p_{3/2}$, $1f_{5/2}$	-3.989250745
3	2, -3,4	$2d_{5/2}$, $1g_{7/2}$	-3.984316427
4	2, -4,5	$2f_{7/2}$, $1h_{9/2}$	-3.978501386

Table 3. Comparison between energy eigenvalues obtained from three different methods

state	α	$E_{n,\ell}^{IGH}$ (NU)	$E_{n,\ell}^{IGH}$ (SUSY)
2p	0.10	2.615564041	2.615564041
	0.15	3.898299549	3.898299549
	0.20	4.990620192	4.990620192
3p	0.10	4.732230991	4.732230991
	0.15	6.038288185	6.038288185
	0.20	6.903939477	6.903939477
3d	0.10	3.617467056	3.617467056
	0.15	5.272626079	5.272626079
	0.20	6.436844393	6.436844393
4p	0.10	5.999694634	5.999694634
	0.15	7.108121333	7.108121333
	0.20	7.706342651	7.706342651
4d	0.10	5.321768060	5.321768060
	0.15	6.714411071	6.714411071
	0.20	7.506720486	7.506720486
4f	0.10	4.670610995	4.670610995
	0.15	6.387083657	6.387083657
	0.20	7.357818179	7.357818179

Table 4. Comparison of energy eigenvalues for the Inverted Generalized Hyperbolical potential $E_{n,\ell}^{IGH}$ and the Hyperbolical Potential $E_{n,\ell}^{H}$

state	α	$E_{n,\ell}^{IGH}$	$E_{n,\ell}^{H}$ [2]
2p	0.10	2.61556	2.61556
	0.15	3.89823	3.89823
	0.20	4.99062	4.99062
3p	0.10	4.73223	4.73223
	0.15	6.03829	6.03829
	0.20	6.90394	6.90394
3d	0.10	3.61747	3.61747
	0.15	5.27263	5.27263
	0.20	6.43684	6.43684
4p	0.10	5.99969	5.99969
	0.15	7.10812	7.10812
	0.20	7.70634	7.70634
4d	0.10	5.32177	5.32177
	0.15	6.71441	6.71441
	0.20	7.50672	7.50672
4f	0.10	4.67061	4.67061
	0.15	6.38708	6.38708
	0.20	7.35782	7.35782
5p	0.10	6.80027	6.80027
5d	0.10	6.36810	6.36810
5f	0.10	5.96159	5.96159
5g	0.10	5.59631	5.59631
6p	0.10	7.32099	7.32099
6d	0.10	7.03872	7.03872
6f	0.10	6.77575	6.77575
6g	0.10	6.54204	6.54204

Table 5. Comparison of energy eigenvalues for the Eckart Potential

state	α^{-1}	present	[27]	[28]
2p	0.025	0.099994	0.101594	0.100888
	0.050	0.096722	0.098298	0.098050
	0.075	0.079890	0.088588	0.088880
3p	0.025	0.038631	0.040311	0.040178
	0.050	0.031674	0.032396	0.032454
	0.075	0.023575	0.023773	0.023998
3d	0.025	0.040766	0.041479	0.041519
	0.050	0.031368	0.033211	0.032811
	0.075	0.022569	0.022964	0.024150
4p	0.025	0.016712	0.018547	0.018514
	0.050	0.009978	0.010855	0.010908
	0.075	0.004521	0.004792	0.004874
4d	0.025	0.017716	0.018977	0.019076
	0.050	0.010421	0.010686	0.011042
	0.075	0.004364	0.004505	0.004924
4p	0.025	0.017331	0.018946	0.019331
	0.050	0.010099	0.010219	0.011102
	0.075	0.003697	0.003992	0.004946

References

[1] C.S. Jia, T. Chen and S. He: Phys. Lett. A (2013) doi.org/10.1016/j.physleta.2013.101.016

[2] C.A. Onate, K.J. Oyewumi and B.J. Falaye: Few-Body **55** (2014) 61

[3] C.L. Pekeris: Phys. Rev. **45** (1934) 98

[4] R.L. Greene and C. Aldrich: Phys. Rev. A **14** (1978) 3363

[5] C.A. Onate: M. Sc. Thesis (2013) *Supersymmetric Shape Invariance Approach and its Application, University of Ilorin, Nigeria.*

[6] A. Khare and U. Sukhatme: J. Phys. A: Math. Gen. **21,** (1988) 501.

[7] S. Marcos, M. LŐpez-Quelle, R. Niembro, L.N. Savushkia and P. Bernardos: Eur. Phys. J. A **17** (2003) 173.

[8] C.S. Jia, P. Guo, Y.F. Diao, L.Z. Yi and X.J. Xie: Eur. Phys. J. A. **34** (2007) 41

[9] P.R. Page, T. Goldman and J.N. Ginocchio: Phys.Rev. Lett. **86** (2001) 204

[10] O.A. Awoga and A.N. Ikot: Pramana J. Phys. **79** (2012) 345

[11] A.F. Nikiforov and B.V. Uvarov: Special Functions of Mathematical Physics, Birkhäuser, Basel (1988)

[12] C. Tezcan and R. Sever. Int. J. Theor. Phys. **48** (2009) 337.

[13] C.A. Onate and J.O.A. Idiodi. Chin. J. Phys. **53** (2015) 120001

[14] S. M. Ikhdair and R. Sever. J. Mol. Structure: THEOCHEM **809** (2007) 103

[15] A.N. Ikot, H.P. Obong, I.O. Owate, M.C. Onyeaju, and H. Hassanabadi. High Energy Physics, 2015, (2015) 632603, http://dx.doi.org/10.1155/2015/632603

[16] S.M. Ikhdair and R. Sever: J. Math. Phys. **52** (2011) 122108

[17] B.J. Falaye, K.J. Oyewumi, T.T. Ibrahim, M.A. Punyasena and C.A. Onate: Can. J. Phys. **91** (2013) 98

[18] F. Cooper et al.: Phys. Rep. **251** (1995) 267

[19] G. Junker: *Supersymmetric Method in Quantum and Statistical Physics* Springer Berlin (1996)

[20] H. Hassanabadi, E. Maghsoodi and S. Zarrinkamar: Eur. Phys. J. Plus **127** (2012) 31

[21] C.A. Onate, K.J. Oyewumi and B.J. Falaye: Afr. Rev. Phys. **8** (2013) 0020

[22] H. Hassanabadi, E. Maghsoodi, S. Zarrinkamar and H. Rahimov: **53** (2012) 022104

[23] C.A. Onate: SOP TTPH **1** (2014) 118

[24] C.S. Jia, P.Guo, Y.F. Diao, L.Z. Yi and X.J. Xie: Eur. Phys. J. A. **34** (2007) 41

[25] H. Hassanabadi, S. Zarrinkamar and H. Rahimov: Commun. Theor. Phys. **56** (2011) 423

[26] C.A. Onate and J.O. Ojonubah. J. Theor. Appl. Phys. **10** (2016) 21

[27] F. Taşkin and G. Koçak. Chin. Phys. B **19** (2010) 090314

[28] S.H. Dong, W.C. Qiang, G.H. Sun and V.B. Bezerra. J. Phys. A: Math. Theor. **40** (2007) 10535

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