Bound State Solutions of Duffin-Kemmer-Petaiu Equation with Yukawa Potential

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The approximate analytic solutions of the Duffin-Kemmer-Petiau equation with Yukawa potential are obtained using SUSY QM by applying the Pekeris approximation type. The energy equation is obtained for each $J = 0$ and $J \neq 0$, and numerical results are obtained for $J \neq 0$ for various values of angular quantum number n, total angular momentum J and some parameters using super-symmetric quantum mechanics (SUSY QM).

1. Introduction

The Duffin-Kemmer-Petiau equation introduced by Duffin, Kemmer and Petiau [1] is a relativistic equation that describes spin-0 and spin-1 particles in the description of the Standard Model [2]. There has been a revival of interest in the DKP equation and its relevance to different problems in the area of particle and nuclear physics. An understanding of this equation reveals that the exact analytical solutions of the relativistic wave equations are important in the relativistic quantum mechanics since the wave contains all the necessary information to describe a quantum system and the investigation of particles on the basis of single equation in the relativistic regime [3]. However, there are only a few potentials for which the relativistic Dirac, Klein-Gordon and Duffin-Kemmer-Petiau equations can be analytically solved.

The Duffin-Kemmer-Petiau equation under a vector potential possessed the same mathematical structure as that of the Klein-Gordon equation [4]. With that similar mathematical structure, various authors have solved different problems of the DKP equation with certain potentials. Fainberg and Pimentel [5,6] clearly presented an equivalence between the Klein-Gordon and Duffin-Kemmer-Petiau theories of physical S-matrix elements in the case of charged scalar particles interacting in minimal way with quantized electromagnetic field. Hassanabadi et al. [7] presented the Duffin-Kemmer-Petiau equation under a scalar Coulomb interaction. Kasri and Chetouani [8] obtained the bound state energy eigen-values for the relativistic DKP Oscillator and DKP Coulomb potentials using the Exact quantization rule. Chargui et al. [9] solved Duffin-Kemmer-Petiau equation with a Pseudo-scalar linear plus Coulomb-like potential in two-dimensional space-time. Hamzavi and Ikhdair [10] solved the DKP equation for a vector Deformed Woods-Saxon potential for any J state using Nikiforov-Uvarov method. Yusuk et al. [11] presented an application of the relativistic Duffin-Kemmer-Petiau equation in the presence of a vector deformed Hulthén potential for spin-0 particles using Nikiforov-Uvarov method.

Since the Duffin-Kemmer-Petiau equation is being increasingly used to describe the interactions of the relativistic spin-0 and spin-1 bosons, it would be interesting to probe whether the DKP equation is amendable to the approximate solutions in the framework of SUSY QM.

The DKP formalization is discussed in Sec. 2 and the solution of DKP equation with Yukawa potential along with the numerical results are given in Sec. 3.

2. DKP Equation

The Duffin-Kemmer-Petiau Hamiltonian for scalar and vector interactions [7] is

$$[\beta.\vec{P}c + mc^2 + U_s + \beta^0 U_v^0]\psi(r) = \beta^0 E\psi(r) \quad (1)$$

Where,

$$\psi(r) = \begin{pmatrix} \psi_{upper} \\ i\psi_{lower} \end{pmatrix} \quad (2)$$

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the upper and lower components, respectively, are

$$\psi_{upper} \equiv \begin{pmatrix} \emptyset \\ \varphi \end{pmatrix}, \qquad \psi_{lower} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \qquad (3)$$

β^0 is the usual 5 x 5 matrix and U_s, U_v^0 represent the scalar and vector interactions. The equation in (3+0) dimensions [1,3] is written as

$$(mc^2 + U_s)\emptyset = (E - U_v^0)\varphi + \hbar c\vec{\nabla}.\vec{A} \qquad (4)$$

$$\vec{\nabla}\emptyset = (mc^2 + U_s)\vec{A}, \quad (mc^2 + U_s) = (E - U_v^0)\emptyset \qquad (5)$$

Where, $\vec{A} = (A_1, A_2, A_3)$. ψ in Eqn. (4) is a simultaneous eigen-function of J^2 and J_3, i.e.,

$$J^2 \begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix} = \begin{bmatrix} L^2 \psi_{upper} \\ (L+S)^2 \psi_{lower} \end{bmatrix} = J(J+1) \begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix} \qquad (6)$$

$$J_3 \begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix} = \begin{bmatrix} L_3 \psi_{upper} \\ (L_3 + S_3)\psi_{lower} \end{bmatrix} = M \begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix} \qquad (7)$$

and the general solution is considered as [7]

$$\psi_{J,M} = \begin{bmatrix} f_{n,J}(r) Y_{J,M}(\Omega) \\ g_{n,l}(r) Y_{J,M}(\Omega) \\ i\Sigma_L h_{n,J}(r) Y_{J,L,1}^M(\Omega) \end{bmatrix} \qquad (8)$$

Where, spherical harmonics $Y_{J,M}(\Omega)$ are of the order J. $Y_{J,L,1}^M(\Omega)$ are normalized vector spherical harmonics, and f_{nJ}, g_{nJ} and h_{nJL} represent the radial wave functions. The equations above yield the coupled differential equation [12]

$$(E_{n,J} - U_v^0) F_{n,J}(r) = (mc^2 + U_s) G_{n,J}(r) \qquad (9)$$

$$\left[\frac{dF_{n,J}(r)}{dr} - \frac{J+1}{r} F_{n,J}(r)\right] = \frac{1}{\alpha J}(mc^2 + U_s) H_{1,n,J}(r) \qquad (10)$$

$$\left[\frac{dF_{n,J}(r)}{dr} + \frac{J}{r} F_{n,J}(r)\right] = \frac{1}{\alpha J}(mc^2 + U_s) H_{-1,n}(r) \qquad (11)$$

$$-\alpha J\left(\frac{dH_{1,n,J}(r)}{dr} + \frac{J+1}{r} H_{1,n,J}(r)\right) = \zeta\left[\frac{dH_{-1,n,J}(r)}{dr} - \frac{J}{r} H_{-1,n,J}(r)\right] = \frac{1}{\hbar c}\left[(mc^2 + U_s)F_{n,J}(r) - (E_{n,J} - U_v^0)G_{n,J}(r)\right] \qquad (12)$$

This gives the equation

$$\frac{d^2F_{n,J}(r)}{dr^2}\left[1 + \frac{\zeta^2 j}{\alpha^2 j}\right] - \frac{dF_{n,J}(r)}{dr}\left[\frac{U_s^{'}}{(m+U_s}\left(1 + \frac{\zeta^2 j}{\alpha^2 j}\right)\right] + F_{n,J}(r)\left\{-\frac{J(J+1)}{r^2}\left(1 + \frac{\zeta^2 j}{\alpha^2 j}\right) + \frac{U_s^{'}}{(m+U_s)}\left(\frac{J+1}{r} - \frac{\zeta^2 j}{\alpha^2 j}\frac{J}{r}\right) - \frac{1}{\alpha^2 j}\left[(m+U_s)^2 - (E_{n,l} - U_v^0)^2\right]\right\} = 0 \qquad (13)$$

Where, $\alpha j = \sqrt{\frac{J+1}{2J+1}}$, $f_{n,j}(r) = \frac{F(r)}{r}$, $g_{n,j}(r) = \frac{G(r)}{r}$, $h_{n,jJ\pm 1} = H_{\pm\frac{1}{r}}$ and $\zeta J = \sqrt{J/(2J+1)}$.

When U_s we recover the well-known formula [14,15]

$$\left[\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} + (E_{n,J} - U_v^0)^2 - m^2\right] F_{n,J}(r) = 0 \qquad (14)$$

3. DKP Equation with Yukawa Potential

Given the Yukawa potential [16,17] as

$$V_{YP}(r) = U_v^0 = -\frac{V_0 e^{-\alpha r}}{r} \qquad (15)$$

Due to the presence of the spin-orbit coupling term in Eqn. (14), the equation cannot be solved exactly. In order to obtain the approximate analytical solutions of Eqn. (14) with arbitrary J values, one must take an approximation to the centrifugal term. The approximation applied is given as [18,19]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)} \tag{16}$$

The approximation given in Eqn. (16) can only be a good approximation to the centrifugal term when the potential parameter α becomes small [18]. Now, substituting Eqns. (15) and (16) into Eqn. (14), we obtain Schrödinger-like equation

$$\frac{d^2 F_{n,J}(r)}{dr^2} = [V_{eff} - \overline{E}_{n,J}]F_{n,J}(r) \tag{17}$$

Where we have used the following substitutions

$$A = -4\alpha V_0 (E_{n,J} - \alpha V_0) \tag{18a}$$

$$B = 4\alpha^2 [J(J+1) - V_0^2] \tag{18b}$$

$$\overline{E}_{n,J} = E_{n,J}^2 - m^2 \tag{18c}$$

$$V_{eff} = \frac{Ae^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{Be^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \tag{18d}$$

We employ the basic concept of the super-symmetric quantum mechanics method and shape invariance approach to solve Eqn. (17) [20]. The ground state function can then be written in the form

$$F_{0,J}(r) = N \exp(-\textstyle\int W(r)) \tag{19}$$

Where, N is the normalization constant and $W(r)$ is called the super-potential in the super-symmetric quantum mechanics [20]. In other to make the right-hand side of Eqn. (17) compatible with the left-hand side that enable us proceed to the next step and obtain the partner potential, we propose the super-symmetric super-potential of the form

$$W(r) = P - \frac{Q}{1-e^{-2\alpha r}} \tag{20}$$

With the comparison of the two sides of Eqn. (17), which is the Riccati equation with a solution of the form given as super-symmetric super-potential of Eqn. (20), expressed in the form

$$W^2(r) \pm W'(r) = V_{eff} - \overline{E}_{n,J} \tag{21}$$

We obtain now the following equations

$$Q = \alpha \pm \sqrt{\alpha^2 + 4\alpha^2[J(J+1) - V_0^2]} \tag{22}$$

$$P = \frac{-4\alpha V_0 (E_{n,J} - \alpha V_0) - Q^2}{2Q} \tag{23}$$

and

$$P^2 = -\overline{E}_{n,J} \tag{24}$$

We can now find the Hamiltonian of Eqn. (17) as it is related to the super-potential via [21,22]

$$V_{eff}^{\pm}(r) = W^2(r) \pm W'(r) \tag{25}$$

If the shape invariance condition exists, all desired results are directly obtained. The latter exists if [23]

$$V_{eff}^{+}(a_0, r) = V_{eff}^{-}(a_1, r) + R(a_1) \tag{26}$$

Where, a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping F : $a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ does not include the variable r. Provided that the above is satisfied, everything desired is given via the following relations [24-26]

$$H_s = -\frac{\partial^2}{\partial r^2} + V_{eff}^{-}(a_s, r) + E_s \tag{27a}$$

$$H_s \phi_{n-s}^{-}(a_s, r) = E_n \phi_{n-s}^{-}(a_s, r) \qquad n \ge s \tag{27b}$$

$$\phi_{n-s}^{-}(a_s, r) = \frac{A_s^{\dagger}}{\sqrt{E_n - E_s}} \phi_{n-(s+1)}^{-}(a_{s+1}, r) \tag{27c}$$

$$A_s^{\dagger} = -\frac{\partial}{\partial r} + W(a_s, r) \tag{27d}$$

$$E_n = \Sigma_{s=1}^{n} R(a_s) \tag{27e}$$

Then, the super-symmetric partner potentials given in Eqn. (25) are fully written by substituting Eqn. (20) into Eqn. (25)

$$V_{eff}^{+}(r)=P^2+\frac{Q^2}{(1-e^{-2\alpha r})^2}-\frac{2PQ}{1-e^{2\alpha r}}+\frac{2\alpha Qe^{-2\alpha r}}{(1-e^{-2\alpha r})^2}$$

$$=P^2+\frac{Q(Q-2P)}{1-e^{-2\alpha r}}+\frac{Q(Q+2\alpha)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad (28)$$

$$V_{eff}^{-}(r)=P^2+\frac{Q^2}{(1-e^{-2\alpha r})^2}-\frac{2PQ}{1-e^{2\alpha r}}-\frac{2\alpha Qe^{-2\alpha r}}{(1-e^{-2\alpha r})^2}$$

$$=P^2+\frac{Q(Q-2P)}{1-e^{-2\alpha r}}+\frac{Q(Q+2\alpha)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad (29)$$

If these partner potentials satisfy the relationship

$$V_{eff}^{+}(r,a_0)=V_{eff}^{-}(r,a_1)+R(a_1) \quad (30)$$

Where, $a_0=Q$ and the Hamiltonian is shape-invariant. We can easily say $a_1=f(a_0)=a_0-2\alpha$. In simpler words, the relation in Eqn. (30) means that these potentials are the same apart from a constant [20]. We now generalize, $a_n=a_0-2\alpha n$, where in our study, $Q=a_0$. In terms of the parameters of the problem, it can easily be written

$$R(a_1)=\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_0^2}{2a_0}\right)^2-\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_1^2}{2a_1}\right)^2$$

We have now the shape-invariance condition via $Q\rightarrow Q-2\alpha$. According to [27-29], the energy of the system is obtained via the summation $\Sigma_{k=1}^{n}R(a_k)$. Therefore, it can first be deduced

$$R(a_2)=\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_1^2}{2a_1}\right)^2-\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_2^2}{2a_2}\right)^2$$

$$R(a_3)=\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_2^2}{2a_2}\right)^2-\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_3^2}{2a_3}\right)^2$$

$$R(a_4)=\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_3^2}{2a_3}\right)^2-\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_4^2}{2a_4}\right)^2$$

$$\vdots$$

$$R(a_n)=\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_{n-1}^2}{2a_{n-1}}\right)^2-\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_n^2}{2a_n}\right)^2 \quad (31)$$

Which, determines the spectrum as

$$\overline{\overline{E}}_{0J}=0 \quad (32)$$

$$\overline{E}_{nJ}^{-}=\Sigma_{k=1}^{n}R(a_k)=\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_0^2}{2a_0}\right)^2-\left(\frac{-4\alpha V_0(E_{nJ}-\alpha V_0)-a_n^2}{2a_n}\right)^2 \quad (33)$$

Considering Eqn. (29), we obtain

$$\overline{E}_{nJ}=\overline{E}_{nJ}^{-}+\overline{E}_{0J}=-\left(\frac{4\alpha V_0(E_{nJ}-\alpha V_0)+a_n^2}{2a_n}\right)^2 \quad (34)$$

Which by means of Eqn. (18c), is obtained as

$$E_{nJ}^2 - M^2 = -\left[\frac{2V_0(E_{nJ} - \alpha V_0) + \frac{\alpha}{2}\left[2n - 1 + \sqrt{(1+2J)^2 - (2V_0)^2}\right]^2}{2n - 1 + \sqrt{(1+2J)^2 - (2V_0)^2}}\right] \quad (35)$$

In order to obtain the non-normalized wave function via standard function analysis, we define a variable of the form $x = \exp(-2\alpha r)$. Substitute this transformation into Eqn. (17), we then have a second-order differential equation

$$\frac{d^2F_{nJ}(x)}{dx^2} + \frac{1}{x}\frac{dF_{nJ}(x)}{dx} + \frac{A + Bx + Cx^2}{(x(1-x))^2}F_{nJ}(x) = 0 \quad (36)$$

Where

$$A = \frac{E_{nJ}^2 - M^2}{4\alpha^2}$$

$$B = \frac{E_{nJ}V_0}{\alpha} - J(J+1) - \frac{E_{nJ}^2 + M^2}{2\alpha^2}$$

$$C = \frac{E_{nJ}^2 - M^2}{4\alpha^2} - \frac{E_{nJ}V_0}{\alpha} + V_0^2 \quad (37)$$

From the transformation, the following equation can be written

$$F_{nJ}(x) = x^{\eta}(1-x)^{\delta}U_{nJ}(x) \quad (38)$$

Where

$$\eta = \left(\frac{E_{nJ}^2 - M^2}{4\alpha^2}\right) \quad \text{and}$$

$$\delta = 1 + (1 + 4J(J+1) - 4V_0^2)^{\frac{1}{2}} \quad (39)$$

Now, let $\varepsilon = -\eta$, then Eqn. (36) becomes a new second-order homogeneous linear differential equation of the form

$$U''(x) + U'(x)\left[\frac{(2\varepsilon+1) - x(2\varepsilon+\delta+1)}{x(1-x)}\right] - U(x)\left[\frac{(2\varepsilon+\delta)^2 + C}{x(1-x)}\right] \quad (40)$$

Consequently, the total radial wave function is obtain as

$$F_{nJ}(x) = N_{n,J}x^{\eta}(1-x)^{\delta}2U_{nJ}(-n, n + (2\varepsilon + 2\delta), 2\varepsilon + 1, x) \quad (41)$$

Table 1: Bound state eigen-value E_{nJ} (MeV) with $m = 938$ (MeV), $\alpha = 0.65$ and $V_0 = 10$.

n	J	$E_{n,J}$	$E_{n,J}$
0	0	-160.3625151	182.1014974
	1	-156.9975596	178.4538035
	2	-150.5979486	171.5084237
	3	-141.7535645	161.8916920
1	0	-160.6068168	181.8456200
	1	-157.2323107	178.2079104
	2	-150.8148186	171.2812398
	3	-141.9463564	161.6897366
2	0	-425.0692502	440.7660604
	1	-418.6791465	434.3277452
	2	-406.2715828	421.8161346
	3	-388.5583503	403.9302076
3	0	572.4071480	583.3390995
	1	-566.8584775	577.8365280
	2	-555.8898835	566.9518625
	3	-539.7682615	550.9379055

Table 2: Bound state eigen-value E_{nJ} (MeV) with $V_0 = 5$ and $m = 938$ (MeV).

α	$E_{0,1}$	$E_{1,1}$	$E_{2,1}$	$E_{2,2}$
0.10	-103.4388645 104.4473920	-103.4508153 104.4354331	-296.7999051 297.6740534	-296.2177896 297.0920518
0.20	-116.9209014 118.9909369	-116.9503847 118.9613830	-330.8604638 332.5858970	-328.3384378 330.0626740
0.30	-135.1895476 138.4339998	-135.2455087 138.3777851	-373.0995402 375.6518430	-366.8999680 369.4492323
0.40	-160.5692332 165.1761845	-160.6665870 165.0781844	-424.9447519 428.2720094	-412.8870142 416.2168957
0.50	-197.0640905 203.3215618	-197.2289212 203.1553580	-487.4438450 491.4300992	-467.0172860 471.0389424
0.60	-251.9231435 260.2471186	-252.2035845 259.9640956	-560.4738965 564.8964205	-529.3157690 533.8669365

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