



Thermodynamic Properties And The Analytic Solutions Of The Relativistic Dirac Equation For A Particle Under The Improved Tietz Molecular Diatomic Potential Function

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ABSTRACT

Using the supersymmetric quantum mechanics (SUSY) approach, we obtained analytical approximate solutions to the Dirac equation with an improved Tietz diatomic potential function under spin and pseudospin symmetries. The energy eigenvalues equation and the corresponding wave spinor for the two symmetries have been explicitly obtained. The non-relativistic limit of the spin symmetry has been obtained. The solution of Morse, Deng-Fan, and Rosen-Morse were obtained by changing the numerical value of q . We have also obtained the thermodynamic properties of a system under the improved Tietz potential function.

Keywords

Solutions of wave equation; bound states; relativistic wave equations.

Pac's No: 03.65.Ge; 03.65.Pm; 02.30.Gp

1. INTRODUCTION

Solving Relativistic symmetries of the Dirac Hamiltonian had been discovered many years ago. These symmetries however, in the recent time have been recognized empirically in nuclear and hadronic spectroscopies [1]. The analytical solutions of the relativistic Dirac equation plays a crucial role in quantum mechanics because, they contain all the necessary information required to understand the quantum behaviour of the Dirac particles which describe a relativistic particle with $-1/2$ [2]. Many authors have investigated the Dirac equation under spin and pseudospin symmetries in the presence or absence of coulomb tensor interaction for most of the physical potential models such as Manning-Rosen potential [3, 4], Ekart potential [5], Morse potential [6], Pseudoharmonic potential [7, 8],

Coulomb potential [9, 10], double ring-shaped Kratzer and Oscillator potentials [11], Yukawa potential [12], Hulthén potential [13], Tietz potential [14], Hyperbolic potential [15] etc. The concept of pseudospin symmetry was first introduced by Hecht and Adler in 1969. The pseudospin symmetry was considered in the context of deformation, superdeformation [16], magnetic moment interpretation, identical band [17] and effective shell-model coupling scheme [18]. The pseudospin symmetry limit occurs when

$\Sigma(r) = V(r) + S(r) = \text{constant}$ [19, 20]. However, the spin symmetry is relevant to mesons [19]. Spin symmetry limit occurs when $\Delta(r) = V(r) - S(r) = \text{constant}$ [19, 20]. Despite the effort of various authors in both relativistic and non-relativistic wave equations, there is no report on the improved Tietz potential with thermodynamic properties yet.

Motivated by the interest in the interaction of fermionic particle within the thermodynamic properties under a system in the field of diatomic potentials, we attempt to study the k state

solutions for the fermionic massive spin $-1/2$ particles interacting with the improved Tietz

diatomic potential function. The improved Tietz diatomic potential function is given as [21, 22]

$$U_{IT}(r) = D_e \left[1 - \frac{e^{\alpha r_e + q}}{e^{\alpha r} + q} \right]^2. \quad (1)$$

This potential is related to other diatomic potential models such as the Morse potential (put $q = 0$) and Deng-Fan potential (put $q = -1$). In the next section, we briefly

discuss Dirac equation. In section 3 and 4, we present bound states solutions and thermodynamic properties of a particle under improved Tietz potential respectively. In section 5 and 6, we discuss results and concluding remark.

2. Dirac Equation

The Dirac equation for fermionic massive spin -1/2 particle moving in an attractive scalar potential $S(r)$ and repulsive vector potential $V(r)$ in a spherical coordinate, is given by [1, 4, 23]

$$[\vec{\alpha} \cdot \vec{p} + \beta(M + S(r)) - (E - V(r))] \psi(\vec{r}) = 0, \quad (2)$$

where $\vec{p} = -i\vec{\nabla}$ is the momentum operator, E denote the relativistic energy of the system, α and β are 4×4 usual Dirac matrices [1]. For a particle in a spherical field [4], the total angular momentum operator J and the spin-orbit matrix operator $k = (\sigma \cdot L + 1)$, where σ and L are the Pauli matrix and orbital angular momentum respectively, commute with the Dirac Hamiltonian. The eigenvalues of k are $k = -(j + 1/2)$ for the aligned spin ($s_{1/2}, p_{3/2}, etc$) and $k = (j + 1/2)$ for the unaligned spin ($p_{1/2}, d_{3/2}, etc$). The complete set of conservative quantities can be chosen as (H, K, J^2, J_z) [4]. The Dirac spinor is

$$\psi_{nk}(r) = \begin{pmatrix} f_{nk}(r) \\ g_{nk}(r) \end{pmatrix} = \begin{pmatrix} \frac{F_{nk}(r)}{r} Y_{jm}^l(\theta, \vartheta) \\ \frac{G_{nk}(r)}{r} Y_{jm}^{\bar{l}}(\theta, \vartheta) \end{pmatrix}, \quad (3)$$

where $F_{nk}(r)$ and $G_{nk}(r)$ are the radial wave functions of the upper and lower components respectively with $Y_{jm}^l(\theta, \vartheta)$ and $Y_{jm}^{\bar{l}}(\theta, \vartheta)$ for spin and pseudospin spherical harmonics coupled to the angular momentum on the Z -axis [1]. Now substitute Eq. (3) into Eq. (2), we recast the following differential equations

$$\left(\frac{d^2}{dr^2} + \frac{\kappa}{r^2} \right) F_{nk}(r) = (M + E_{nk} - V(r) + S(r)) G_{nk}(r), \quad (4)$$

$$\left(\frac{d^2}{dr^2} - \frac{\kappa}{r^2} \right) G_{nk}(r) = (M - E_{nk} + V(r) + S(r)) F_{nk}(r), \quad (5)$$

which later give

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - (M + E_{nk} - \Delta(r))(M - E_{nk} + \Sigma(r)) \right\} F_{nk}(r) + \left\{ \frac{d\Delta(r)}{dr} - \frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} + \frac{\kappa}{r} \right) \right\} F_{nk}(r) = 0 \quad (6)$$

for $\kappa(\kappa+1) = \ell(\ell+1)$, $r \in (0, \infty)$,

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - (M + E_{nk} - \Delta(r))(M - E_{nk} + \Sigma(r)) \right\} G_{nk}(r) - \left\{ \frac{d\Sigma(r)}{dr} - \left(\frac{d}{dr} - \frac{\kappa}{r} \right) \right\} G_{nk}(r) = 0 \quad (7)$$

for $\kappa(\kappa-1) = \bar{\ell}(\bar{\ell}+1)$, $r \in (0, \infty)$. It is noted that $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$.

3. Bound State Solutions

In this section, we obtained the energy eigenvalues and wave functions for both the spin and pseudospin symmetries.

3.1: The Spin Symmetry Limit:

Under the spin symmetry, $\frac{d\Delta(r)}{dr} = 0$, $\Delta(r) = C_s = \text{constant}$ [24, 25] with $\Sigma(r)$ as the Improved Tietz diatomic potential ($U_{IT}(r)$). Thus, we recast Eq. (6) in the form:

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - U_{IT}(r)(M + E_{nk} - C_s) \right\} F_{nk}(r) + \{ (M + E_{nk} - C_s)(M - E_{nk}) \} F_{nk}(r) = 0. \quad (8)$$

Eq. (8) can be solved analytically only for s-wave case due to

the spin-orbit coupling term $\frac{k(k+1)}{r^2}$. Therefore, we shall employ a suitable approximation scheme [22, 24] to the spin-orbit coupling term. For a short-range potential, the following formula:

$$\frac{1}{r^2} \approx \frac{1}{r_e^2} \left[C_0 + \frac{C_1}{e^{\alpha r} + q} + \frac{C_2}{(e^{\alpha r} + q)^2} \right], \quad (9)$$

is a good approximation to $\frac{1}{r^2}$. This approximation was used to study arbitrary l -state pseudo-Hulthén wave functions [22]. The parameters in Eq. (9) are obtain as follows:

$$C_0 = 1 + \frac{3(1 - \alpha r_e + 2q e^{-\alpha r_e} + q^2 e^{-2\alpha r_e})}{\alpha^2 r_e^2} - \frac{2q\alpha r_e e^{-\alpha r_e} + q^2 \alpha r_e e^{-2\alpha r_e}}{\alpha^2 r_e^2}, \quad (10)$$

$$C_1 = \frac{6(q\alpha r_e - 3q - e^{\alpha r_e} - 3q^2 e^{-\alpha r_e} - q^3 e^{-2\alpha r_e})}{\alpha^2 r_e^2} + \frac{2\alpha(2r_e e^{\alpha r_e} - q^3 e^{-2\alpha r_e})}{\alpha^2 r_e^2} \quad (11)$$

$$C_2 = \frac{6q(3q + 2e^{\alpha r_e} + 2q^2 e^{-\alpha r_e}) + 3(e^{2\alpha r_e} + 13q^4 e^{-2\alpha r_e})}{\alpha^2 r_e^2} + \frac{2q\alpha r_e (q^2 e^{-\alpha r_e} - e^{\alpha r_e}) + \alpha r_e q^4 e^{-2\alpha r_e} + \alpha r_e e^{2\alpha r_e}}{\alpha^2 r_e^2}. \quad (12)$$

Substituting potential (1) and approximation (9) into Eq. (8), we obtain the following:

$$\frac{d^2 F_{nk}(r)}{dr^2} = [V_{eff}^s(r) - E_{nk}^s] F_{nk}(r), \quad (13)$$

where we have used the following for simplicity

$$V_{eff(s)}(r) = \frac{\frac{k(k+1)C_1}{r^2} - 2bD_e(M + E_{nk} - C_s)}{e^{\alpha r} + q} + \frac{\frac{k(k+1)C_2}{r^2} + D_e b^2(M + E_{nk} - C_s)}{(e^{\alpha r} + q)^2}, \quad (14)$$

$$-E_{nk}^s = [(M - E_{nk}) + D_e](M + E_{nk} - C_s) + \frac{k(k+1)C_0}{r^2}, \quad b = e^{\alpha r} + q. \quad (15)$$

In other to obtain the solution of Eq. (13), we use the essential concept of the supersymmetric quantum mechanics and shape invariance formalism [4,25,26] which enable us to write the ground-state wave function for the upper component as

$$F_{0k}(r) = \exp\left(-\int Q(r) dr\right), \quad (16)$$

where $Q(r)$ is called the superpotential function in supersymmetric quantum mechanics and shape invariance approach. Inserting Eq. (16) into Eq. (13) results to the following equation

$$Q^2(r) - Q'(r) = V_{eff}(r) - E_{nk}^{s(1)}. \quad (17)$$

Since it is required that the superpotential function should be made compatible with the right-hand side of the nonlinear Riccati equation (17), we now propose a superpotential function $Q(r)$ of the form:

$$Q(r) = \beta - \frac{\beta_0}{e^{\alpha r} + q}. \quad (18)$$

where β and β_0 are two parametric constants to be determine later. In view of the superpotential of equation(18), we can now construct the two partner potentials $V_{\pm}(r)$ as follows

$$U_+(r) = Q^2(r) + \frac{dQ(r)}{dr} = \beta^2 - \frac{2\beta\beta_0}{e^{\alpha r} + q} + \frac{\beta_0(\beta_0 + \alpha)e^{\alpha r} + \beta_0^2(1 - e^{\alpha r})}{(e^{\alpha r} + q)^2}, \quad (19)$$

$$U_-(r) = Q^2(r) - \frac{dQ(r)}{dr} = \beta^2 - \frac{2\beta\beta_0}{e^{\alpha r} + q} + \frac{\beta_0(\beta_0 - \alpha)e^{\alpha r} + \beta_0^2(1 - e^{\alpha r})}{(e^{\alpha r} + q)^2}. \quad (20)$$

Now, comparing Eq. (17) with Eq. (13), we deduce the values of the two parametric constants and establish the relationship between the constants and other variables:

for constants :

$$\beta^2 = -E_{nk}^{s(1)}, \quad (21)$$

$$2\beta\beta_0 = 2bD_e(M + E_{nk} - C_s) - \frac{k(k+1)C_1}{r_e^2}, \quad (22)$$

$$\beta_0^2 = D_e b^2(M + E_{nk} - C_s) \beta_0 + \frac{k(k+1)C_2}{r_e^2}, \quad (23)$$

$$\alpha\beta_0 = 0. \quad (24)$$

In other to obtain the value of each of these parameters and establish the relationship between them as stated before, it is interested to note that the interest of the author lies only in the bound state solutions which demand that the radial part of the wave function F_{nk} must satisfy the boundary conditions

$$\frac{F_{nk}(r)}{r} = \begin{cases} 0, & r \rightarrow \infty \\ \infty, & r \rightarrow 0 \end{cases}. \quad (25)$$

Considering the regularity conditions and their consequences (i.e. $\beta > 0, \beta_0 > 0$) and the simplification of equations (21),

(22), (23) and (24), the value of β and β_0 are obtain as

$$\beta_0 = \frac{\alpha}{2} \left[1 \pm \sqrt{1 + \frac{4}{\alpha^2 r_e^2} [k(k+1)C_2 + D_e b^2 r_e^2 (M + E_{nk} - C_s)]} \right], \quad (26)$$

$$\beta = \frac{\frac{k(k+1)(C_2 - C_1)}{r_e^2} + bD_e(2+b)(M + E_{nk} - C_s) - \beta_0^2}{2\beta_0}. \quad (27)$$

From the two partner potentials, we can obtain the following relationship

$$U_+(r, a_0) = U_-(r, a_1) + R(a_1), \quad (28)$$

where a_1 is a new set of parameter uniquely determine from an old set of parameter a_0 and a_0 is a function of a_1 , i.e. $a_1 = f(a_0) = a_0 - \alpha$. The residual term $R(a_1)$ is independent of the variable r . However, Eq. (28) shows that the partner potentials are shape invariant. By using the shape invariant approach [27], we can determine exactly, the energy eigenvalues equation of the shape invariant potential $U_-(r)$ after obtaining the following relationship

$$R(a_1) = \left[\frac{(2\beta\beta_0 + a_0^2) - a_0^2}{2a_0} \right] - \left[\frac{(2\beta\beta_0 + a_1^2) - a_1^2}{2a_1} \right], \quad (29)$$

$$R(a_2) = \left[\frac{(2\beta\beta_0 + a_1^2) - a_1^2}{2a_1} \right] - \left[\frac{(2\beta\beta_0 + a_2^2) - a_2^2}{2a_2} \right], \quad (30)$$

$$R(a_n) = \left[\frac{(2\beta\beta_0 + a_{n-1}^2) - a_{n-1}^2}{2a_{n-1}} \right] - \left[\frac{(2\beta\beta_0 + a_n^2) - a_n^2}{2a_n} \right]. \quad (31)$$

Following the formalism of shape invariance approach, the energy levels of the system can be determine as

$$E_{nk}^s = E_{nk}^{s(-)} + E_{0k}^s = \sum_{i=1}^n R(a_i) + E_{0k}^s$$

$$= \left[\frac{(2\beta\beta_0 + a_n^2) - a_n^2}{2a_n} \right]^2 \quad (32)$$

This gives energy equation for the spin symmetry as

$$-E_{s,nk}^{s2} + M^2 + A_\gamma + \frac{k(k+1)C_0}{r_e^2} =$$

$$\left[\frac{k(k+1)(C_2 - C_1) + A_\beta - \left(\frac{\alpha}{2}\right)[1 - 2n - A_\beta]}{\alpha(1 - 2n - A_\beta)} \right]^2 \quad (33a)$$

$$A_\alpha = \sqrt{1 + \frac{4}{\alpha^2 r_e^2} [k(k+1)C_2 + D_e b^2 r_e^2 (M + E_{nk} - C_s)]} \quad (33b)$$

$$A_\beta = bD_e (2 + b)(M + E_{nk} - C_s), \quad (33c)$$

$$A_\gamma = C_s (E_{nk} - M) + D_e (M + E_{nk} - C_s). \quad (33d)$$

In other to compute the corresponding wave function, we define a variable of the form: $s = e^{-2\alpha r}$ and substitute it into Eq. (13) to have

$$\frac{d^2 F_{nk}(s)}{ds^2} + \frac{1-s}{s(1-s)} \frac{dF_{nk}(s)}{ds} + \left[\frac{As^2 + Bs + C}{(s(1-s))^2} \right] F_{nk}(s) = 0 \quad (34)$$

Comparing Eq. (34) with equation of the form

$$\frac{d^2 F_{nk}(z)}{dz^2} + \frac{\alpha_1 - \alpha_2 z}{s(1 - \alpha_3 z)} \frac{dF_{nk}(z)}{dz} + \left[\frac{Az^2 + Bz + C}{(z(1 - \alpha_3 z))^2} \right] F_{nk}(z) = 0 \quad (35)$$

the wave function is given by

$$F(z) = N_n z^\epsilon (1 - \alpha_3 z)^\zeta {}_2F_1 \left(-n, n + 2(\epsilon + \zeta) + \frac{\alpha}{\alpha_3} - 1; 2\epsilon + \alpha_1 \alpha_3 z \right) \quad (36)$$

where

$$\epsilon = \frac{(1 - \alpha_1) + \sqrt{(1 - \alpha_1)^2 - 4C}}{2},$$

$$\zeta = \frac{1}{2} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2\alpha_3} + \sqrt{\left(\frac{1}{2} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2\alpha_3}\right)^2 - \left(\frac{A}{\alpha_3 \alpha_3} + \frac{B}{\alpha_3} + C\right)},$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 1,$$

$$A = \frac{(M + E_{nk} - C_s)[D_e + (M - E_{nk})]}{\alpha^2} - \frac{k(k+1)C_0}{\alpha^2 r_e^2}, \quad (37)$$

$$B = \frac{2(M + E_{nk} - C_s)[D_e(b-1) - (M - E_{nk})]}{\alpha^2} - \frac{k(k+1)(2C_0 + C_1)}{\alpha^2 r_e^2}, \quad (38)$$

$$C = \frac{(M + E_{nk} - C_s)[D_e(b-1) - (M - E_{nk})]}{\alpha^2} - \frac{k(k+1)(C_0 + C_1 + C_2)}{\alpha^2 r_e^2}. \quad (39)$$

The complete wave function is written as

$$F(s) = N_n s^\epsilon (1-s)^\zeta {}_2F_1(-n, n + 2(\epsilon + \zeta); 2\epsilon + 1, s), \quad (40)$$

3.2: The Pseudospin Symmetry Limit:

Under the pseudospin symmetry, $\frac{d\Sigma(r)}{dr} = 0$, $\Sigma(r) = C_s = \text{constant}$ [28] with $\Delta(r)$ is taking as the improved Tietz diatomic potential ($U_s(r)$). Thus, we recast Eq. (7) in the form:

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + U_{II}(r)(M - E_{nk} + C_{ps}) \right\} G_{nk}(r) - (M - E_{nk} + C_{ps})(M + E_{nk}) G_{nk}(r) = 0. \quad (41)$$

Substituting potential (1) and approximation (9) into Eq. (41), we obtain the following:

$$\frac{d^2 G_{nk}(r)}{dr^2} = [V_{eff}^{ps}(r) - E_{nk}^{ps}] G_{nk}(r), \quad (42)$$

where

$$V_{eff}^{ps}(r) = \frac{\frac{k(k-1)C_1}{r_e^2} + 2bD_e(M - E_{nk} + C_{ps})}{e^{\alpha r} + q} + \frac{\frac{k(k-1)C_2}{r_e^2} - D_e b^2 (M - E_{nk} + C_{ps})}{(e^{\alpha r} + q)^2}, \quad (43)$$

$$-E_{nk}^{ps} = \frac{k(k-1)C_0}{r_e^2} + [(M + E_{nk}) - D_e](M - E_{nk} + C_{ps}), \quad (44)$$

The negative energy solution of Eq. (6) can directly be obtained via the spin symmetry solution through the mapping [28, 29, 30, 31]

$$F_{nk}(r) \leftrightarrow G_{nk}(r), E_{nk} \rightarrow -E_{nk},$$

$$U_{II}(r) \rightarrow -U_{II}(r), -C_s \rightarrow C_{ps}, k \rightarrow k-1, \quad (45)$$

Following the previous procedures, the energy eigenvalue equation for the pseudospin symmetry under the improved Tietz diatomic potential is obtain as

$$-E_{ps,nk}^2 + M^2 - B_\gamma + \frac{k(k-1)C_0}{r_e^2} = \left[\frac{k(k-1)(C_2 - C_1) + B_\beta - \left(\frac{\alpha}{2}\right)[1 - 2n - B_\beta]}{\alpha(1 - 2n - B_\beta)} \right]^2 \quad (46)$$

$$B_\alpha = \sqrt{1 + \frac{4}{\alpha^2 r_e^2} [k(k-1)C_2 - D_e b^2 r_e^2 (M - E_{nk} + C_{ps})]},$$

$$B_\beta = bD_e (2 + b)(M - E_{nk} + C_{ps}),$$

$$B_\gamma = C_{ps} (E_{nk} - M) - D_e (M - E_{nk} + C_{ps}).$$

The corresponding wave function is

$$G(s) = N_n s^\delta (1-s)^{\lambda} {}_2P_1(-n, n+2(\eta+\lambda); 2\eta+1, s), \quad (47)$$

3.3: Non-Relativistic Limit:

Here we obtain the non-relativistic limit of the energy equation of the spin symmetry. To obtain this, we make the

following transformation $M - E_{s,nk} \rightarrow -E_{n\ell}$, $M + E_{s,nk} \rightarrow \frac{2\mu}{\hbar^2}$,

$C_s = 0$ and $k \rightarrow \ell$, with these, Eq. (33) turns to

$$E_{n,\ell} = D_e + \frac{\ell(\ell+1)C_0\hbar^2}{2\mu} \frac{\alpha^2\hbar^2}{2\mu} \left[\frac{B_2 + B_1 - \frac{1}{4}(1-2n-A)^2}{1-2n-A} \right]^2. \quad (48)$$

$$A_1 = \sqrt{1 + \frac{4\ell(\ell-1)C_2}{\alpha^2 r_e^2} + \frac{8\mu D_e b^2}{\alpha^2 \hbar^2}},$$

$$B_1 = \frac{2\mu b D_e (b+2)}{\alpha^2 \hbar^2},$$

$$B_2 = \frac{\ell(\ell+1)(C_2 - C_1)}{\alpha^2 r_e^2}.$$

4. Thermodynamic properties and the improved Tietz potential

In other to calculate the thermodynamic properties of a system within the improved Tietz diatomic potential model, we first calculate the vibrational partition function of the system. To begin, we first re-write the energy equation (48) in the form:

$$E_{n,\ell} = D_e + \frac{\ell(\ell+1)C_0\hbar^2}{2\mu} - A_4, \quad (49)$$

where

$$A_4 = \frac{\alpha^2 \hbar^2}{2\mu} \left[\frac{-\frac{1}{4}(\delta-2n)^2}{\delta-n} \right]^2.$$

$$\square = \frac{\ell(\ell+1)(C_2 - C_1)}{\alpha^2 r_e^2} + \frac{2\mu b D_e (b+2)}{\alpha^2 \hbar^2},$$

$$\delta = 1 - \sqrt{1 + \frac{4\ell(\ell-1)C_2}{\alpha^2 r_e^2} + \frac{8\mu D_e b^2}{\alpha^2 \hbar^2}} \text{ and } [\delta] \text{ means the largest}$$

integer inferior to δ . Now, the partition function of the system is calculated by

$$Z(\beta) = \sum_{n=0}^{\delta} e^{-\beta E_{n\ell}}, \quad \beta = \frac{1}{kT}. \quad (50)$$

Substituting Eq. (49) into Eq. (50), we have

$$Z(\beta) = e^{-\beta \left(D_e - \frac{\ell(\ell+1)\delta^2\hbar^2}{2\mu} \right)} \sum_{n=0}^{\delta} e^{-\beta \left[\frac{-\frac{1}{4}(\delta-2n)^2}{\delta-n} \right]^2}, \quad \lambda = \frac{\gamma}{\sqrt{\beta}},$$

$$\gamma = \frac{\sqrt{2\mu}}{\hbar\delta}. \quad (51)$$

In the classical limit at high temperature T, for large δ and small β , $e^{-\beta \left(D_e - \frac{\ell(\ell+1)C_0\hbar^2}{2\mu} \right)} \approx 1$, the sum can be replace by the integral to have the partition function as

$$Z(\beta) = \lambda \int_0^{\delta} e^{-z^2} dz = \frac{\sqrt{\pi} \gamma \text{Erfi} \left(\frac{\delta}{\gamma} \sqrt{\beta} \right)}{\beta}, \quad z = \frac{-\frac{1}{4}(\delta-2n)^2}{\delta-2n}. \quad (52)$$

Having obtained the partition function in Eq. (52), it becomes less cumbersome to calculate the thermodynamic properties.

I: The Vibrational Mean Energy U:

$$U(\beta) = -\frac{\partial}{\partial \beta} \ln Z(\beta) = \frac{1}{\beta} \left[1 - \frac{\tau}{\text{DawsonF}(\tau)} \right] = \frac{-\tau}{\sqrt{\pi} \text{Erfi}(\tau)} \left[\frac{e^\tau}{\beta} - \frac{\sqrt{\pi} \text{Erfi}(\tau)}{2\beta\tau} \right], \quad \tau = \tau(\beta) = \frac{\delta}{\gamma} \sqrt{\beta}. \quad (53)$$

II: The Vibrational Specific Heat C:

$$C(\beta) = \frac{\partial}{\partial T} U(\beta) = k\beta^2 \frac{\partial}{\partial \beta} U(\beta) = \frac{1}{2} k \left[1 - \frac{\tau [\tau e^{\tau^2} + \sqrt{\pi}(1-\tau^2)\text{Erfi}(\tau)]}{2e^{\tau^2} \text{DawsonF}(\tau)^2} \right], \quad (54)$$

When $\beta \square 1$, $C(\beta) = 0$.

III: The Vibrational Mean Free Energy F:

$$F(\beta) = -kT \ln Z(\beta) = -\frac{1}{\beta} \ln \left(\frac{\sqrt{\pi} \gamma \text{Erfi}(\tau)}{\sqrt{\beta}} \right). \quad (55)$$

IV: The Vibrational Entropy S:

$$S(\beta) = k \ln Z(\beta) + kT \frac{\partial}{\partial \beta} \ln Z(\beta) = S(\beta) = k \ln Z(\beta) - k\beta \frac{\partial}{\partial \beta} \ln Z(\beta) = \frac{1}{2} k \left[1 - \frac{\tau}{\text{DawsonF}(\delta)} + \log \left(\frac{\pi}{2} \right) \right] + \frac{1}{2} k \left[2 \log \left(\frac{\gamma \text{Erfi}(\tau)}{\frac{1}{2} \sqrt{\beta}} \right) \right]. \quad (66)$$

5. Results and Discussion

In Tables 1 and 2, we numerically reported energy eigenvalue of the Improved Tietz potential for four different values of q ($q = 2, 3, 4$ and 5) for $\ell = 0$ and $\ell = 1$ respectively. It is seen that the energy for $\ell = 0$ in Table 1 are higher than

their counterpart for $\ell = 1$ in Table 2. The energy increases as q and n increase for both E_n and $E_{n\ell}$. By putting $q = 0, -1$ and 1 , our potential (1) turns to Morse, Deng-Fan and Rosen-Morse potentials respectively as $b = e^{\alpha r_e}$, $b = e^{\alpha r_e} - 1$ and $b = e^{\alpha r_e} + 1$ for the Morse, Deng-Fan and Rosen-Morse potentials. In Table 3, we present the

numerical results for the energy eigenvalues of the Morse, Deng-Fan and Rosen-Morse potentials. It is seen from Table 3 that as α increases, the energy decreases in all the states.

Table 1: Energy spectral E_n of the Improved Tietz diatomic potential with $\alpha = 0.25$, $D=5$ and $r_e = 10$ for three (3) different values of q .

n	$q = 2$	$q = 3$	$q = 4$
0	4.6604906	4.6606105	4.6607156
1	4.7133853	4.7132757	4.7131796
2	4.7615313	4.7612507	4.7610044
3	4.8050113	4.8046116	4.8042603
4	4.8439059	4.8434326	4.8430163
5	4.8782932	4.8777861	4.8773395
6	4.9082494	4.9077426	4.9072956
7	4.9338487	4.9333708	4.9329487
8	4.9551633	4.9547377	4.9543612
9	4.9722633	4.9719087	4.9715943
10	4.9852172	4.9849475	4.9847077

Table 2: Energy spectral $E_{n\ell}$ of the Improved Tietz diatomic potential with $\alpha = 0.25$, $D=5$ and $r_e = 10$ for four (4) different values of q .

n	$q = 2$	$q = 3$	$q = 4$
0	4.6548085	4.6557587	4.6573358
1	4.7076234	4.7083489	4.7097291
2	4.7556916	4.7562508	4.7574849
3	4.7990958	4.7995405	4.8006736
4	4.8379164	4.8382921	4.8393640
5	4.8722315	4.8725777	4.8736231
6	4.9021173	4.9024678	4.9035165
7	4.9276476	4.9280311	4.9291082
8	4.9488947	4.9493344	4.9504606
9	4.9659285	4.9664430	4.9676347
10	4.9788175	4.9794205	4.9806901

Table 3: Energy spectral $E_{n\ell}$ of the Morse, Deng-Fan and Rosen-Morse diatomic potentials with $D = 1$ and $r_e = 10$

state	α	Morse ($q = 0$)	Deng-Fan ($q = -1$)	Rosen Morse ($q = 1$)
2p	0.10	0.98921	0.97203	1.00918
	0.15	0.96829	0.96498	0.97222
	0.20	0.94657	0.94581	0.94748
3p	0.10	0.99111	0.97416	1.01097

	0.15	0.97405	0.97101	0.97781
	0.20	0.95928	0.95874	0.96003
3d	0.10	0.98958	0.98625	1.04934
	0.15	0.95587	0.93823	0.96744
	0.20	0.93303	0.93100	0.93558
4p	0.10	0.99278	0.98596	1.01256
	0.15	0.97894	0.97605	0.98260
	0.20	0.96974	0.96930	0.97041
4d	0.10	0.99142	0.96018	1.05112
	0.15	0.96148	0.95207	0.97293
	0.20	0.94548	0.94363	0.94790
4f	0.10	0.99003	0.98722	1.10949
	0.15	0.93711	0.91798	0.96017
	0.20	0.91260	0.90865	0.91760

6. Conclusion

In this work, we obtained the energy eigenvalues and the corresponding wave functions for the Improved Tietz diatomic potential model using a suitable approximation type via supersymmetric approach. We have also obtained the solutions of other potential model which are closely related to the Improved Tietz potential by changing the numerical value of q . The rotational vibrational partition function and the thermodynamic properties were equally obtained. Our results find applications in different areas of physics.

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