Approximate Relativistic Dirac Equation of a Particle in the Field of Shifted Deng-Fan Potential

Atachegbe Clement Onate*
Physic Section, Edoja Community Secondary School, Uchuchu-Ibaji, Kogi State, Nigeria

We report here the analytic solution of the Dirac equation for Shifted Deng-Fan potential for any spin-orbit coupling term κ by using the SUSY QM approach within the framework of spin and pseudo-spin symmetries. The energy eigenvalues are obtained in a closed form by applying an approximation to spin-orbit coupling potential.

1. Introduction

In the relativistic regime, we deal with the Dirac and Klein-Gordon equations even though the number of physical potentials is limited [1]. In particle physics, the Dirac equation is a relativistic wave-equation, which provides a description of spin ½ elementary particles such as electrons consistent with both the principles of quantum mechanics and the theory of special relativity. This was the first theory to fully account for relativity in quantum mechanics. It explains the fine details of the hydrogen spectrum in a completely rigorous way [2]. The equation also implied that existence of a new form of matter, namely the antimatter, hitherto unsuspected and unobserved, predicted its experimental discovery. It equally provided a theoretical justification for the introduction of several component wave functions in Pauli's phenomenological theory of spin.

Within the present study, we intend to work on the spin and pseudo-spin symmetries under the Shifted Deng-Fan scalar and vector potential when a Coulomb tensor term is included. The non-relativistic bound state solution of the Shifted Deng-Fan potential have been studied by Oyewumi et al. [3] within the framework of asymptotic iteration method. The Shifted Den-Fan potential under investigation in this study is defined as

$$V(r)=D_1\left(1-\frac{b}{e^{\alpha r}-1}\right)^2-D_2, b=e^{\alpha r}-1 \tag{1}$$

The scheme of our work is as follows: In the next section, we present a review of the Dirac equation. In Sec. 3, we present the bound state energy for spin and pseudo-spin symmetry while in the last section our numerical results and concluding remark are presented.

2. Dirac Equation with Tensor Coupling

The Dirac equation with tensor coupling is given by [4,5,6]

$$\left[\vec{\alpha}.\vec{p}+\beta(M+S(r))-i\beta\vec{\alpha}.\hat{r}U(r)\right]\psi(r)$$
$$=\left[E-V(r)\right]\psi(r) \tag{2}$$

Where, $V(r), S(r)$ and $U(r)$ are vector, scalar and tensor potentials, respectively. Also E, M and $\vec{p}$ denote the relativistic energy, fermion mass and momentum operator, respectively. The matrices α and β are given as

$$\alpha=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{3}$$

Where, I is a 2 x 2 unitary matrix and the spin matrices are

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4}$$

The total angular momentum operator $\vec{J}$ and the spin-orbit coupling operator $k=(\vec{\sigma}.\vec{L}+1)$ with $\vec{L}$ being the orbital angular momentum of spherical nucleons, commute with the Dirac Hamiltonian. The eigenvalues of spin-orbit coupling operator $k=\left[j+\frac{1}{2}\right]>0$ and $k=-\left[j+\frac{1}{2}\right]<0$, for the

*onateca12@gmx.ux

unaligned spin $j=\ell-\frac{1}{2}$ and the aligned spin $j=\ell+\frac{1}{2}$, respectively. The set (H,K,J^2,J_2) is taken as the complete set of the conservative quantities. Therefore, we can write the spinors as [5,6]

$$\psi_{nk}(r)=\begin{pmatrix} f_{nk}(r) \\ g_{nk}(r) \end{pmatrix}=\begin{pmatrix} \frac{F_{nk}(r)}{r}Y_{jm}^{\ell}(\theta,\varphi) \\ i\frac{G_{nk}(r)}{r}Y_{jm}^{\bar{\ell}}(\theta,\varphi) \end{pmatrix} \tag{5}$$

Where, $f_{nk}(r)$ and $g_{nk}(r)$ are the upper and lower components of the Dirac spinors. $Y_{jm}^{\ell}(\theta,\varphi)$ and $Y_{jm}^{\bar{\ell}}(\theta,\varphi)$ denote the spin and pseudo-spin spherical harmonics, respectively, and m is the projection of the angular momentum on the Z-axis. By using the following relations

$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B})=\vec{A}.\vec{B}+i\vec{\sigma}.(\vec{A}\times\vec{B}) \tag{6a}$$

$$(\vec{\sigma}.\vec{p})=\vec{\sigma}.\hat{r}\left[\hat{r}.\vec{p}+i\frac{\vec{\sigma}.\vec{L}}{r}\right] \tag{6b}$$

and relations

$$(\vec{\sigma}.\vec{L})Y_{jm}^{\ell}(\theta,\varphi)=(k-1)Y_{jm}^{\bar{\ell}}(\theta,\varphi) \tag{7a}$$

$$(\vec{\sigma}.\vec{L})Y_{jm}^{\ell}(\theta,\varphi)=-(k-1)Y_{jm}^{\ell}(\theta,\varphi) \tag{7b}$$

$$(\vec{\sigma}.\hat{r})Y_{jm}^{\bar{\ell}}(\theta,\varphi)=-Y_{jm}^{\ell}(\theta,\varphi) \tag{7c}$$

$$(\vec{\sigma}.\hat{r})Y_{jm}^{\ell}(\theta,\varphi)=-Y_{jm}^{\bar{\ell}}(\theta,\varphi) \tag{7d}$$

we have the following coupled differential equations [5,6,7]

$$\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)F_{nk}(r)=[M+E_{nk}+\Delta(r)]G_{nk}(r) \tag{8a}$$

$$\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)G_{nk}(r)=[M-E_{nk}+\Sigma(r)]F_{nk}(r) \tag{8b}$$

Where

$$\Sigma(r)=V(r)+S(r),\qquad \Delta(r)=V(r)-S(r) \tag{9}$$

From Eqns. (8a) and (8b), we have

$$\left(\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}+\frac{2k}{r}U(r)-\frac{dU(r)}{dr}-U^2(r)+\frac{\frac{d\Delta(r)}{dr}}{M+E_{nk}-\Delta(r)}\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)\right)F_{nk}(r)$$
$$=[M+E_{nk}-\Delta(r)][M-E_{nk}+\Sigma(r)]F_{nk}(r) \tag{10}$$

$$\left(\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}+\frac{2k}{r}U(r)+\frac{dU(r)}{dr}-U^2(r)+\frac{\frac{d\Sigma(r)}{dr}}{M-E_{nk}+\Sigma(r)}\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)\right)G_{nk}(r)$$
$$=[M+E_{nk}-\Delta(r)][M-E_{nk}+\Sigma(r)]G_{nk}(r) \tag{11}$$

Where

$$k(k-1)=\bar{\ell}(\bar{\ell}+1)\quad\text{and}\quad k(k+1)=\ell(\ell+1)$$

2.1. The spin symmetry limit

For spin symmetry to occur, $\frac{d\Delta(r)}{dr}=0$ and $\Delta(r)=C_s=\text{constant}$ [8-12]. The sum potential is then considered as the Shifted Deng-Fan potential which is given as

$$\Sigma(r) = D_1\left(1-\frac{b}{e^{\alpha r}-1}\right)^2 - D_2 \tag{12}$$

$$U(r) = -\frac{H}{r}, \quad r \ge R_c \tag{13}$$

$$H = \frac{z_a z_b e^2}{4\pi\varepsilon_0} \tag{14}$$

Where, R_c is the Coulomb radius, z_a and z_b denote the charges of the projection particle *a* and the target nucleus *b*. The coefficient of the inverse square term is obtained by considering

$$\beta = k(k+1) + 2kH + H + H^2 = (k+H)(K+H+1) \tag{15}$$

Substituting Eqn. (12) and approximation in [13]

$$\frac{1}{r^2} \approx \alpha^2\left(C_0 + \frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right) \tag{16}$$

into Eqn. (10), we have a second-order differential equation of the form

$$\left(\frac{d^2}{dr^2} - \frac{D_1 b^2 (M+E_{s,nk}-C_s)e^{-2\alpha r}}{(1-e^{-\alpha r})^2} + \frac{2D_1^b (M+E_{s,nk}-C_s)e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{\alpha^2 \beta e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right)F_{nk}(r)$$
$$= [(M-E_{s,nk}) - D_2 + D_1][M+E_{s,nk}-C_s]F_{nk}(r) + \alpha^2 \beta C_0 F_{nk}(r) \tag{17}$$

Where, $k = \ell$ and $k = -\ell - 1$ for $k < 0$ and $k > 0$, respectively.

2.2. The pseudo-spin symmetry limit

For pseudo-spin symmetry, the condition $\frac{d\Sigma(r)}{dr} = 0$ and $\Sigma(r) = C_{ps} = \text{constant}$ holds. Here, we consider the difference potential as the Shifted Deng-Fan potential, which is given as

$$\Delta(r) = D_1\left(1-\frac{b}{e^{\alpha r}-1}\right)^2 - D_2 \tag{18}$$

and

$$\delta = k(k-1) + 2kH - H + H^2 = (k+H)(k+H-1) \tag{19}$$

When, $D_2 = 0$, the potential (Eqn. (18)), turns into Deng-Fan potential. Substituting Eqns. (16), (18) and (19) into Eqn. (11) yields

$$\left(\frac{d^2}{dr^2} - \frac{\alpha^2\delta - D_1 b^2 (M-E_{ps,nk}+C_{ps})e^{-\alpha r}}{(1-e^{-\alpha r})^2} - \frac{2bD_1 (M-E_{ps,nk}+C_{ps})e^{-\alpha r}}{1-e^{-\alpha r}}\right)G_{nk}(r)$$
$$= [\delta C_0 \alpha^2 + (D_2 - D_1)(M - E_{ps,nk} + C_{ps}) + (M + E_{ps,nk})(M - E_{ps,nk} + C_{ps})]G_{nk}(r) \tag{20}$$

Where, $k = -\bar{\ell}$ and $k = \bar{\ell} + 1$ for $k < 0$ and $k > 0$, respectively.

3. Bound State Solutions

Here, we report the approximate solutions of the Dirac equation (spin and pseudo-spin) with the Shifted Deng-Fan potential in the presence of tensor interaction.

3.1. Approximate solution of the spin symmetry

In this symmetry limit, the Schrödinger-like equation takes the following form

$$\frac{d^2 F_{nk}(r)}{dr^2} = [V_{eff} - E_{s,nk}]F_{nk}(r) \tag{21}$$

Where we have used the following notations for mathematical simplicity

$$V_{eff} = \frac{V_1 e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{V_2 e^{-\alpha r}}{(1-e^{-\alpha r})^2} \quad (22)$$

$$V_1 = -D_1 b(E_{s,nk} + M - C_s)(b+2),$$

$$V_2 = \alpha^2 \beta + D_1 b^2 (E_{s,nk} + M - C_s) \quad (23)$$

$$E_{s,nk} = \alpha^2 C_0 \beta + (D_1 - D_2)(E_{s,nk} + M - C_s)$$

$$+(E_{s,nk} - M)(E_{s,nk} + M - C_s) \quad (24)$$

The equation we are dealing with is a Riccati equation of the form

$$W^2(r) - W(r) = V_{eff} - E_{s,nk} \quad (25)$$

Where, we search the super-potential as a solution of the form

$$W(r) = \frac{A + (B-A)e^{-\alpha r}}{1-e^{-\alpha r}} \quad (26)$$

Now, using the basic concept of the super-symmetric quantum mechanics formalism and the shape invariance technique [14-17] to solve Eqn. (25), where the ground state function for the upper component $F_{nk}(r)$ is written in the form

$$F_{0,-1}(r) = \exp\left(-\int W(r) dr\right) \quad (27)$$

we obtain the following relations

$$A^2 = -E_{s,nk} \quad (28)$$

$$B = \frac{-\alpha \pm \alpha\sqrt{1 + 4\beta + 4D_1 b^2 (E_{s,nk} + M - C_s)\alpha^{-2}}}{2} \quad (29)$$

$$A = \frac{B^2 - bD_2(E_{s,nk} + M - C_0)(2+b)}{2B} \quad (30)$$

In this work, the super-symmetric partner potentials $U_{\pm}(r) = W^2(r) \pm W'(r)$ are given items of the super-potential of Eqn. (26) as

$$U_+(r) = \left(\frac{B^2 - bD_1(E_{s,nk} + M - C_s)(2+b)}{2B}\right)^2$$

$$+ \frac{B(2A-B)E^{-\alpha r}}{1-e^{-\alpha r}} + \frac{B(B-\alpha)e^{-\alpha r}}{(1-e^{-\alpha r})^2} \quad (31)$$

$$U_-(r) = \left(\frac{B^2 - bD_1(E_{s,nk} + M - C_0)(2+b)}{2B}\right)^2$$

$$+ \frac{B(2A-B)e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{B(B+\alpha)}{(1-e^{-\alpha r})^2} \quad (32)$$

Putting $a_0 = B$ we deduced that the two partner potentials $U_+(r)$ and $U_-(r)$ satisfy the following relationship

$$U_+(r, a_0) = U_-(r, a_0) + R(a_1) \quad (33)$$

Where, a_1 is a function of a_0 and expressed mathematically as $a_1 = f(a_0) = B + \alpha$, and the remainder $R(a_1)$ is independent of the variable *r*. Eqn. (33) shows that the two partner potentials in Eqns. (31) and (32) are shape invariant. The energy spectra of the potential $U_-(r)$ can be determined by using the shape invariance approach. The energy spectra of the potential are given by

$$E_{s,nk} = \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + R(a_3) + \ldots$$

$$+R(a_{n-1}) + R(a_n)$$

$$\left(\frac{a_0^2 - bD_1(E_{s,nk} M - C_s)(2+b)}{2a_0}\right)^2$$

$$-\left(\frac{a_1^2 - bD_1(E_{s,nk} M - C_s)(2+b)}{2a_1}\right)^2 + \cdots$$

$$+\left(\frac{a_{n-1}^2 - bD_1(E_{s,nk} + M - C_s)(2+b)}{2a_{n-1}}\right)^2$$

$$-\left(\frac{a_n^2 - bD_1(E_{s,nk} + M - C_s)(2+b)}{2a_n}\right)^2 \quad (34)$$

Where, the quantum number n = 0, 1, 2, 3, …. When, $a_n = B + \alpha n$, then the energy spectra of the spin symmetry is obtained by substituting the values of $E_{s,nk}$ and a_n into Eqn. (31). We then obtain

$$E_{s,nk}^2 - M^2 + C_s(M - E_{s,nk}) - \alpha^2 C_0 (k+H)(k+H+1) + (D_2 - D_1)(E_{s,nk} + M - C_s)$$
$$= \left(\frac{bD_1(E_{s,nk} + M - C_s)(2+b) - \frac{\alpha^2}{4}\left(1 + 2n + \sqrt{1 + 4(k+H)(k+H+1) + 4D_1 b^2 (E_{s,nk} + M - C_s)\alpha^{-2}}\right)^2}{\alpha(1 + 2n + \sqrt{1 + 4(k+H)(k+H+1) + 4D_1 b^2 (E_{s,nk} + M - C_s)\alpha^{-2}})} \right)^2 \tag{35}$$

Non-relativistic limit

The non-relativistic Schrödinger equation is spineless while the Dirac equation is for spin-1/2 particles. However, a relationship exists between these two important equations [18]. The approximate non-relativistic solutions can be obtained from the solutions of the relativistic one [18,19]. According to Sun [18], when the energies of the potentials $S(r)$ and $V(r)$ are small compared to the rest mass mc^2, then, the non-relativistic energy can be determined by using the following transformations: $E_{n,k}M \to \frac{2\mu}{\hbar^2}$, $M - E_{n,k} \to -E_{n,\ell}$ and $k \to \ell$. With, $C_s = 0$, the relativistic energy equation of Eqn. (35) turns into

$$E_{n\ell} = D_1 - D_2 + \frac{\hbar^2\alpha^2}{2\mu} C_0 \ell(\ell+1) - \frac{\hbar^2}{2\mu}\left(\frac{\frac{2\mu D_1 b(2+b)}{\hbar^2} - \frac{\alpha^2}{4}\left(1 + 2n + \sqrt{(1+2\ell)^2 + \frac{8\mu D_1 b^2}{\hbar^2\alpha^2}}\right)^2}{\alpha\left(1 + 2n + \sqrt{(1+2\ell)^2 + \frac{8\mu D_1 b^2}{\hbar^2\alpha^2}}\right)} \right)^2 \tag{36}$$

To test the accuracy of the energy equation given above, we put $D_1 = D$, and $D_2 = 0$ then Eqn. (36) becomes

$$E_{n\ell} = D + \frac{\hbar^2\alpha^2}{2\mu} C_0 \ell(\ell+1) - \frac{\hbar^2}{2\mu}\left(\frac{\frac{2\mu D b(2+b)}{\hbar^2} - \frac{\alpha^2}{4}\left(1 + 2n + \sqrt{(1+2\ell)^2 + \frac{8\mu D b^2}{\hbar^2\alpha^2}}\right)^2}{\alpha\left(1 + 2n + \sqrt{(1+2\ell)^2 + \frac{8\mu D b^2}{\hbar^2\alpha^2}}\right)} \right)^2 \tag{37}$$

This equation is equal to Eqn. (25) of Ref. [13] of the non-relativistic limit with Deng-Fan potential in the framework of SUSY QM.

In order to obtain the un-normalized wave function, we apply the function analysis method. Now defining a new variable of the form $s = \exp(-\alpha r)$ and substitute it into Eqn. (17), we obtain a second-order differential equation in the following form

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} + \frac{As^2 + Bs + C}{s^2(1-s)^2} \right) F_{nk}(s) = 0 \tag{38}$$

Where,

$$A=\frac{(D_2-D_1+E_{s,nk}-M-D_1b(2+b))(M+E_{s,nk}-C_s)}{\alpha^2}-C_0(k+H)(k+H+1) \tag{39a}$$

$$B=\frac{(D_1b(2+b)-D_1b^2+2(D_1-D_2)+2(M-E_{s,nk}))(M+E_{s,nk}-C_s)}{\alpha^2}-(k+H+1)(k+H) \tag{39b}$$

$$C=\frac{(D_2-D_1-M+E_{s,nk})(M+E_{s,nk}-C_s)}{\alpha^2}-C_0(k+H)(k+H+1) \tag{39c}$$

Now, if we analyze the asymptotic behavior of Eqn. (38) at origin and at infinity, it can be tested that, when $r\to 0(s\to 1)$ and $r\to\infty(s\to)$ Eqn. (38) has a solution $F_{s,nk}(s)=s^{\bar{\lambda}}(1-s)^{\lambda}$, where

$$\bar{\lambda}=\left[(D_1b(2+b)+D_1-D_2+M-E_{s,nk})(M+E_{s,nk}-C_s)+C_0(k+H)(k+H+1)\right]^{\frac{1}{2}} \tag{40}$$

$$\lambda=\frac{1}{4}+\left(\frac{1+4(k+H)(k+H+1)+4D_1b^2(M+E_{s,nk}-C_s)}{16}\right)^{\frac{1}{2}} \tag{41}$$

Then, the components of wave function are obtained as follows:

$$F_{s,nk}(r)=(e^{-\alpha r})^{\bar{\lambda}}(1-e^{-\alpha r})^{\lambda}\,2F_1(-n,n+2(\bar{\lambda}+\lambda);2\bar{\lambda}+1,e^{-\alpha r}) \tag{42}$$

$$G_{s,nk}(r)=\frac{1}{M+E_{s,nk}-C_s}\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)F_{s,nk}(r) \tag{43}$$

3.2. Approximate solutions of the pseudo-spin symmetry

The Schrödinger-like equation takes the form

$$\frac{d^2G_{nk}(r)}{dr^2}=[V_{eff}-E_{ps,nk}]G_{nk}(r) \tag{44}$$

Where

$$V_{eff}=\frac{(\alpha^2\delta-D_1b^2(M-E_{p,nk}+C_p)e^{-\alpha r}}{(1-e^{-\alpha r})^2}+\frac{D_1b(M-E_{p,nk}+C_p)e^{-\alpha r}}{1-e^{-\alpha r}} \tag{45}$$

$$E_{p,nk}=\alpha^2C_0\delta-[(D_1-D_2)-(M+E_{p,nk})](M-E_{p,nk}+C_p) \tag{46}$$

We have decided to use the same variables so as to avoid repetition of algebra. It is clear that Eqn. (44) is similar to Eqn. (18) and thus substituting for the values of $E_{p,nk}$ and V_{eff} in Eqn. (44) we obtain the relativistic energy spectra for the pseudo-spin symmetry as

$$E_{p,nk}^2 - M^2 - C_p(M + E_{p,nk}) - \alpha^2 C_0(k+H)(k+H-1) + (D_1 - D_2)(M - E_{p,nk} + C_p)$$

$$= -\left(\frac{D_1 b(M - E_{p,nk} + C_p)(2+b) - \frac{\alpha^2}{4}\left(1 + 2n + \sqrt{1 + 4(k+H)(k+H-1) - \frac{4D_1 b^2(M - E_{p,nk} + C_p)}{\alpha^2}}\right)^2}{\alpha\left(1 + 2n + \sqrt{1 + 4(k+H)(k+H-1) - 4D_1 b^2(M - E_{p,nk} + C_p)\alpha^{-2}}\right)} \right)^2 \tag{47}$$

Taking the following parameters as

$$\lambda\!\!\!^{-} = \left[(D_1 b(2+b) + D_2 - D_1 + M + E_{p,nk})(M - E_{p,nk} + C_p) + C_0(k+H)(k+H-1)\right]^{\frac{1}{2}} \tag{48}$$

$$\lambda = \frac{1}{4} + \left[\frac{1 + 4(k+H)(k+H-1) - 4D_1 b^2(M - E_{p,nk} + C_p)}{16}\right]^{\frac{1}{2}} \tag{49}$$

Then, the upper and lower components of the wave function are given as

$$G_{p,nk}(r) = (e^{-\alpha r})^{\lambda\!\!\!^{-}}(1 - e^{-\alpha r})^{\lambda}\, 2F_1(-n, n + 2(\lambda\!\!\!^{-} + \lambda); 2\lambda\!\!\!^{-} + 1, e^{-\alpha r}) \tag{50}$$

$$F_{p,nk}(r) = \frac{1}{M - E_{p,nk} + C_p}\left(\frac{d}{dr} - \frac{k}{r} + U(r)\right)G_{p.nk}(r) \tag{51}$$

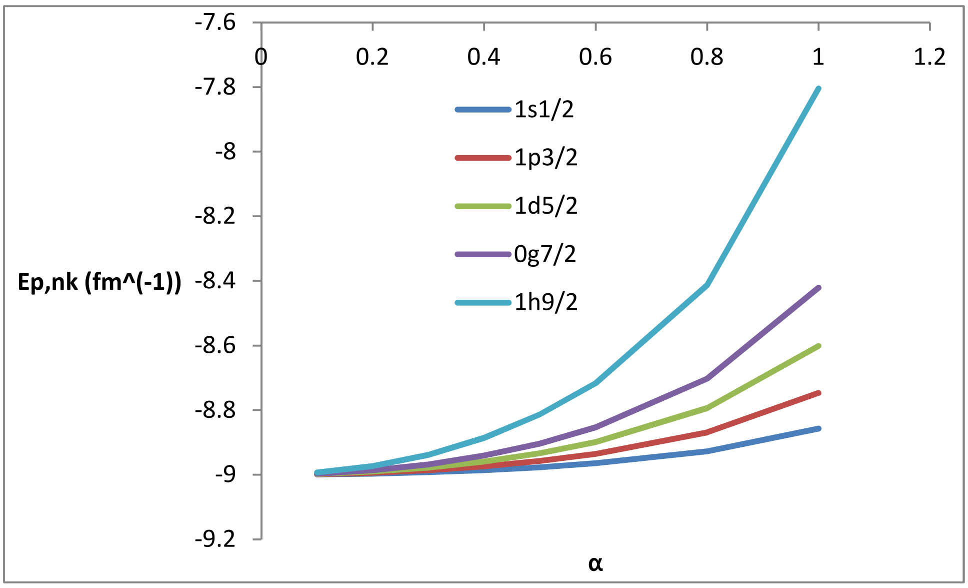

Fig.1: Energy in pseudo-spin symmetry $E_{p,nk}$ against α.

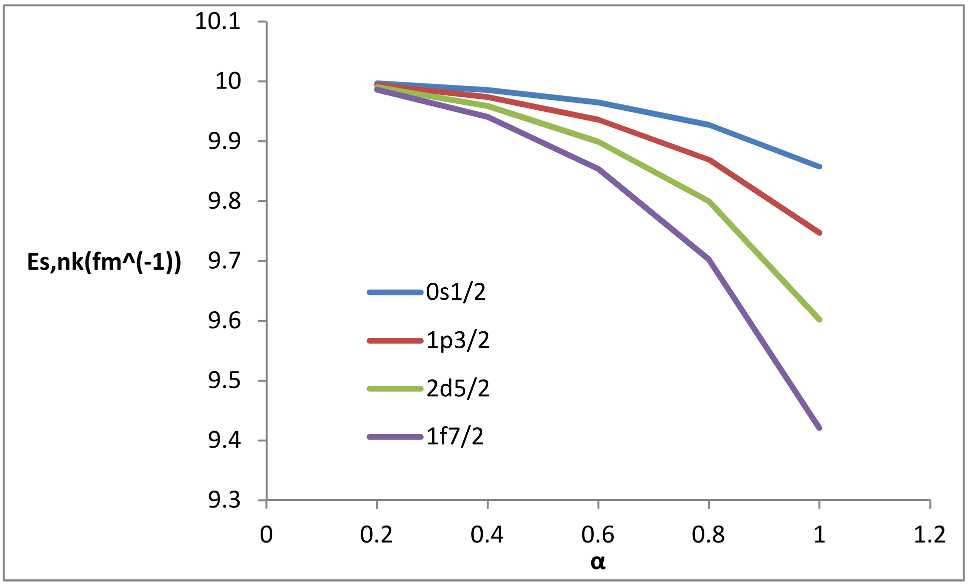

Fig.2: Energy in spin symmetry $E_{s,nk}$ against α.

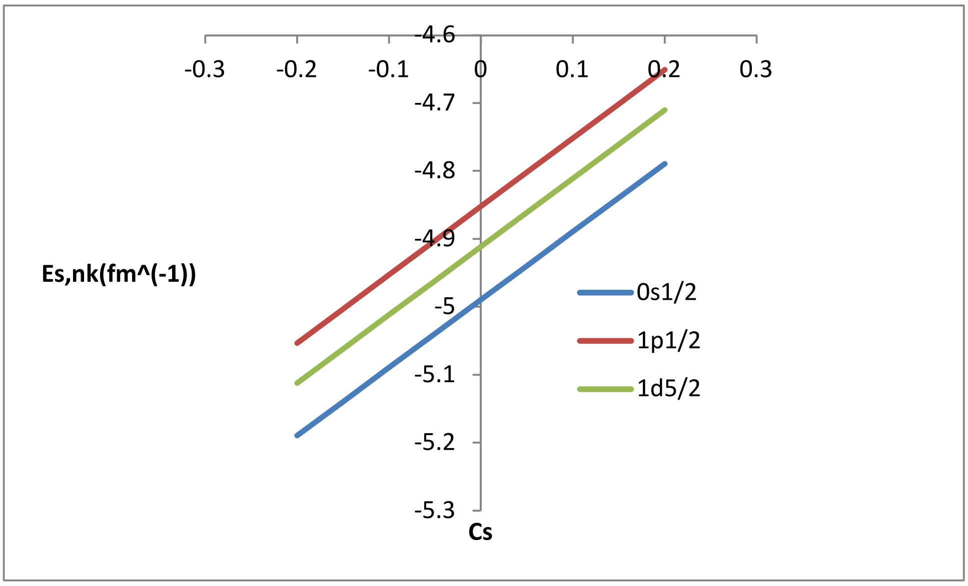

Fig.3: Energy $E_{s,nk}$ Vs. C_s for H = 1, M = 5, $D_1 = 10$ $D_2 = 1$ and α = 0.1.

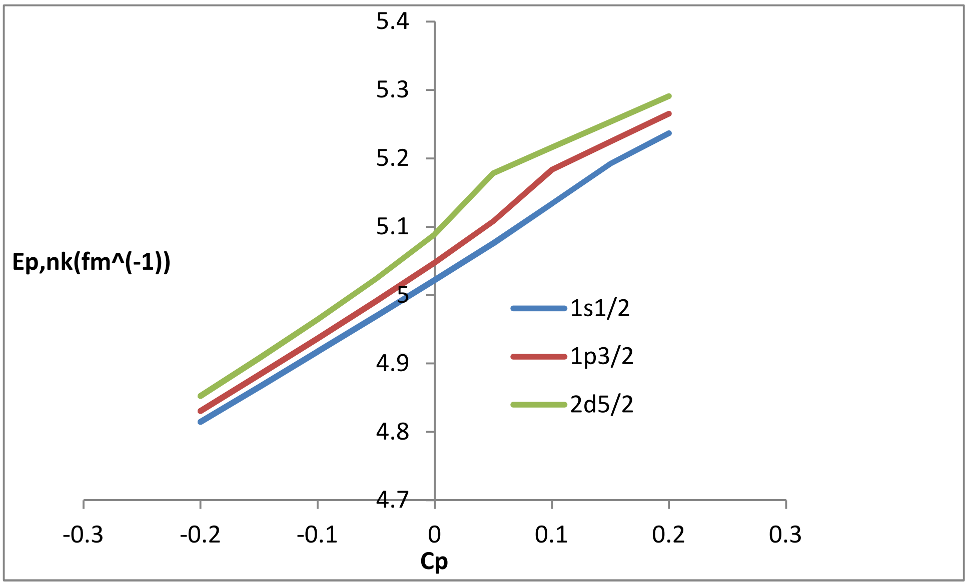

Fig.4: Energy $E_{p,nk}$ Vs. C_p for H = 1, M = 5, $D_1 = 10$ $D_2 = 1$ and α = 0.1.

In Tables 1, 2, 3 and 4, we have reported some degenerate states and the energy behavior for different values of *H*.

Table 1: Energies in the spin symmetry for $M = D_2 = 1$, $D_1 = 10$, $C_s = D_1$, $C_0 = 1/12.0015$ and $\alpha = 0.1$.

ℓ	n	k	(ℓ, j)	$E_{s,nk}\,(fm^{-1})(H=0)$	$E_{s,nk}\,(fm^{-1})(H=0.5)$	$E_{s,nk}\,(fm^{-1})(H=1)$
0	0	-1	$0s_{\frac{1}{2}}$	9.096544831 9.001818235	9.000668746 9.046586529	9.096544831 9.001818235
0	1	-1	$1s_{\frac{1}{2}}$	9.344251321 9.007346936	9.302083862 9.004726082	9.344251321 9.007346936
0	2	-1	$2s_{\frac{1}{2}}$	9.582870226 9.016746499	9.012920466 9.556922146	9.016746499 9.582870226
0	3	-1	$3s_{\frac{1}{2}}$	9.030252759 9.756080283	9.740357052 9.025417034	9.030252759 9.756080283
1	0	-2	$0p_{\frac{3}{2}}$	9.291804764 9.006044390	9.003626592 9.186751041	9.096544831 9.001818235
1	1	-2	$1p_{\frac{3}{2}}$	9.521119646 9.015278720	9.010990772 9.429110425	9.344251321 9.007346936
1	2	-2	$2p_{\frac{3}{2}}$	9.705416938 9.028473810	9.640504773 9.022246255	9.582870226 9.016746499
1	3	-2	$3p_{\frac{3}{2}}$	9.045904874 9.835532752	9.792797936 9.037636805	9.756080283 9.030252759
2	0	-3	$0d_{\frac{5}{2}}$	9.503554296 9.012734808	9.009077713 9.399918784	9.291804764 9.006044390
2	1	-3	$1d_{\frac{5}{2}}$	9.025809506 9.688489781	9.608916712 9.020214010	9.015278720 9.521119646
2	2	-3	$2d_{\frac{5}{2}}$	9.822802831 9.043060012	9.035406999 9.767335050	9.028473810 9.705416938
2	3	-3	$3d_{\frac{5}{2}}$	9.913125281 9.064856730	9.054974746 9.876603947	9.045904874 9.835532752
3	0	-4	$0f_{\frac{7}{2}}$	9.021963320 9.682984914	9.017025903 9.598521233	9.503554296 9.012734808
3	1	-4	$1f_{\frac{7}{2}}$	9.039050322 9.819059353	9.758551075 9.032081740	9.688489781 9.025809506
3	2	-4	$2f_{\frac{7}{2}}$	9.860624717 9.911723436	9.051456242 9.870905823	9.822802831 9.043060012
3	3	-4	$3f_{\frac{7}{2}}$	9.087193436 9.969904947	9.075582205 9.944240773	9.913125281 9.064856730
1	0	1	$0p_{\frac{1}{2}}$	9.291804764 9.006044390	9.009077713 9.399918784	9.012734808 9.503554296

1	1	1	$1p_{\frac{1}{2}}$	9.521119646 9.015278720	9.020214010 9.608916712	9.025809506 9.688489781
1	2	1	$2p_{\frac{1}{2}}$	9.028473810 9.705416938	9.035406999 9.767335050	9.822802831 9.043060012
1	3	1	$3p_{\frac{1}{2}}$	9.045904874 9.835532752	9.876603947 9.054974746	9.913125281 9.064856730
2	0	2	$0d_{\frac{3}{2}}$	9.503554296 9.012734808	9.017025903 9.598521233	9.021963320 9.682984914
2	1	2	$1d_{\frac{3}{2}}$	9.025809506 9.688489781	9.032081740 9.758551075	9.039050322 9.819059353
2	2	2	$2d_{\frac{3}{2}}$	9.043060012 9.822802831	9.870905823 9.051456242	9.060624717 9.911723436
2	3	2	$2d_{\frac{3}{2}}$	9.913125281 9.064856730	9.944240773 9.075582205	9.969904947 9.087193436
3	0	3	$0f_{\frac{5}{2}}$	9.682984914 9.021963320	9.756577369 9.027561638	9.033837923 9.819746673
3	1	3	$1d_{\frac{5}{2}}$	9.039050322 9.819059353	9.870558899 9.046738094	9.055171483 9.913837303
3	2	3	$2f_{\frac{5}{2}}$	9.060624717 9.911723436	9.945724449 9.070599738	9.081421383 9.973504793
3	3	3	$3f_{\frac{5}{2}}$	9.087193436 9.969904947	9.990383869 9.099742005	9.113289533 10.00604235

Table 2: Energies in the pseudo-spin symmetry for $M = D_2 = 1$, $D_1 = 10$, $C_p = -D_1$, $\alpha = 0.1$ and $C_0 = 1/12.005$.

$\bar{\ell}$	n	k	(ℓ, j)	$E_{p,nk}\,(fm^{-1})(H=0)$	$E_{p,nk}\,(fm^{-1})(H=0.5)$	$E_{p,nk}\,(fm^{-1})(H=1)$
1	1	-1	$1s_{\frac{1}{2}}$	4.046704442 -8.998878226	3.841309570 -8.999192246	3.707068695 -8.999461324
2	1	-2	$1p_{\frac{3}{2}}$	4.588428600 -8.998115509	4.302469950 -8.998519311	4.046704442 -8.998878226
3	1	-3	$1d_{\frac{5}{2}}$	5.188500160 -8.997173226	4.887900160 -8.997666812	4.588428600 -8.998115509
4	1	-4	$1f_{\frac{7}{2}}$	5.762295750 -8.996051330	5.481755250 -8.996634718	5.188500160 -8.997173226
1	2	-1	$2s_{\frac{1}{2}}$	5.287843960 -8.997935528	5.159280070 -8.998389060	5.076873245 -8.998787394
2	2	-2	$2p_{\frac{3}{2}}$	5.640136863 -8.996903573	5.451889610 -8.997442014	5.287843960 -8.997935528
3	2	-3	$2d_{\frac{5}{2}}$	6.050155760 -8.995691923	5.842348650 -8.996320207	5.640136863 -8.996903573
4	2	-4	$2f_{\frac{7}{2}}$	6.459112740 -8.994300540	6.257233940 -8.995018689	6.050155760 -8.995691923
1	1	2	$0d_{\frac{3}{2}}$	4.046704442 -8.998878226	4.302469950 -8.998519311	4.588428600 -8.998115509
2	1	3	$0f_{\frac{5}{2}}$	4.588428600 -8.998115509	4.887900160 -8.997666812	5.188500160 -8.997173226
3	1	4	$0g_{\frac{7}{2}}$	5.188500160 -8.997173226	5.481755250 -8.996634718	5.762295750 -8.996051330
4	1	5	$0h_{\frac{9}{2}}$	5.762295750 -8.996051330	6.027031870 -8.995422990	6.274466560 -8.994749750
1	2	2	$1d_{\frac{3}{2}}$	5.287843960 -8.997935528	5.451889610 -8.997442014	5.640136863 -8.996903573
2	2	3	$1f_{\frac{5}{2}}$	5.640136863 -8.996903573	5.842348650 -8.996320207	6.050155760 -8.995691923
3	2	4	$1g_{\frac{7}{2}}$	6.050155760 -8.995691923	6.257233940 -8.995018689	6.459112740 -8.994300540
4	2	5	$1h_{\frac{9}{2}}$	6.459112740 -8.994300540	6.652838070 -8.993537420	6.836617950 -8.992729380

Table 3: Energies in the spin symmetry for $H=1$, $\alpha=0.1$, $D_2=1$, $D_1=C_s=10$, $r_e=0.4$ and $C_0=1/12.005$.

M	$0s_{\frac{1}{2}}$	$1s_{\frac{1}{2}}$	$2s_{\frac{1}{2}}$	$3s_{\frac{1}{2}}$
0	9.994984440 9.975386573	9.979514580 9.922079259	9.951331931 9.869341436	9.902112810
1	9.096544831 9.001818235	9.007346936 9.344251321	9.582870226 9.016746499	9.030252759 9.756080283
2	8.351857630 8.000736456	8.002954963 9.144583558	9.739061847 8.006672115	10.13537622 8.011908340
3	7.837013787 7.000462995	7.001855307 9.229737305	10.09284161 7.004182630	7.007451566 10.64422339
4	7.656319209 6.000337750	9.531638382 6.001352703	6.003047683 10.58058112	6.005425860 11.24244801
5	5.000265866 7.819881417	5.001064494 9.989222447	11.16231056 5.002397563	5.004266915 11.90426990

Table 4: Energies in the pseudo-spin symmetry for $H=1$, $\alpha=0.1$, $D_2=1$, $D_1=-C_s=10$, $r_e=0.4$ and $C_0=1/12.005$.

M	$1s_{\frac{1}{2}}$	$1p_{\frac{3}{2}}$	$1d_{\frac{5}{2}}$	$1f_{\frac{7}{2}}$
0	-9.999513726 4.511522200	-9.998987300 4.831913074	-9.998298745 5.354214830	-9.997448091 5.946395640
1	-8.999461324 3.707068695	-8.998878226 4.046704442	-8.998115509 4.588428600	-8.997173226 5.188500116
2	-7.999396270 2.932821575	-7.998742813 3.293253874	-7.997888036 3.853301880	-7.996831982 4.457421200
3	-6.999313341 2.197127744	-6.998570210 2.579528144	-6.997598076 3.155183610	-6.996396998 3.757622440
4	-5.999203992 1.511696434	-5.998342649 1.916219458	-5.997215788 2.502074654	-5.995823473 3.094450940
5	-4.999053199 0.893323964	-4.998028895 1.317799893	-4.996688682 1.904041564	-4.995032624 2.474283250

The degenerate states in the spin symmetry limit for various H are as follows

For $H=0$

$0p_{\frac{3}{2}}=0p_{\frac{1}{2}}$, $1p_{\frac{3}{2}}=1p_{\frac{1}{2}}$, $2p_{\frac{3}{2}}=2p_{\frac{1}{2}}$,

$3p_{\frac{3}{2}}=3p_{\frac{1}{2}}$, $0d_{\frac{5}{2}}=0d_{\frac{3}{2}}$, $1d_{\frac{5}{2}}=1d_{\frac{3}{2}}$,

$2d_{\frac{5}{2}}=2d_{\frac{3}{2}}$, $3d_{\frac{5}{2}}=3d_{\frac{3}{2}}$, $0f_{\frac{7}{2}}=0f_{\frac{5}{2}}$,

$1f_{\frac{7}{2}}=1f_{\frac{5}{2}}$, $2f_{\frac{7}{2}}=2f_{\frac{5}{2}}$, $3f_{\frac{7}{2}}=3f_{\frac{5}{2}}$

For $H=0.5$

$0d_{\frac{5}{2}}=0p_{\frac{1}{2}}$, $1d_{\frac{5}{2}}=1p_{\frac{1}{2}}$, $2d_{\frac{5}{2}}=2p_{\frac{1}{2}}$,

$3d_{\frac{5}{2}}=3p_{\frac{1}{2}}$, $0f_{\frac{7}{2}}=0d_{\frac{3}{2}}$, $1f_{\frac{7}{2}}=1d_{\frac{3}{2}}$,

$2f_{\frac{7}{2}}=2d_{\frac{3}{2}}$, $3f_{\frac{7}{2}}=3d_{\frac{3}{2}}$

For $H=1$

$0s_{\frac{1}{2}}=0p_{\frac{3}{2}}$, $1s_{\frac{1}{2}}=1p_{\frac{3}{2}}$, $2s_{\frac{1}{2}}=2p_{\frac{3}{2}}$,

$3s_{\frac{1}{2}}=3p_{\frac{3}{2}}$, $0f_{\frac{7}{2}}=0p_{\frac{1}{2}}$, $1f_{\frac{7}{2}}=1p_{\frac{1}{2}}$,

$2f_{\frac{7}{2}}=2p_{\frac{1}{2}}$, $3f_{\frac{7}{2}}=3p_{\frac{1}{2}}$

The degenerate states in the pseudospin symmetry limit are as follows

For $H=0$

$1s_{\frac{1}{2}}=0d_{\frac{3}{2}}$, $1p_{\frac{3}{2}}=0f_{\frac{5}{2}}$, $1d_{\frac{5}{2}}=0g_{\frac{7}{2}}$,

$1f_{\frac{7}{2}}=0h_{9}$, $2s_{\frac{1}{2}}=1d_{\frac{3}{2}}$, $2p_{\frac{3}{2}}=1f_{\frac{5}{2}}$,

$2d_{\frac{5}{2}}=1g_{\frac{7}{2}}$, $2f_{\frac{7}{2}}=1h_{\frac{9}{2}}$

For $H=0.5$

$1p_{\frac{3}{2}}=0d_{\frac{3}{2}}$, $1d_{\frac{5}{2}}=0f_{\frac{5}{2}}$, $1f_{\frac{7}{2}}=0g_{\frac{7}{2}}$,

$2p_{\frac{3}{2}}=1d_{\frac{3}{2}}$, $2d_{\frac{5}{2}}=1f_{\frac{5}{2}}$, $2f_{\frac{7}{2}}=2g_{\frac{7}{2}}$

For $H=1$

$1d_{\frac{5}{2}}=0d_{\frac{3}{2}}$, $1f_{\frac{7}{2}}=0f_{\frac{5}{2}}$, $2d_{\frac{5}{2}}=1d_{\frac{3}{2}}$,

$2f_{\frac{7}{2}}=1f_{\frac{5}{2}}$

4. Conclusion

Using the shape invariance concept and Pekeris approximation type, we have reported the approximate analytical solutions of the Dirac equation with the Shifted Deng-Fan potential under a tensor coupling term. We observed that the inclusion of the tensor interaction removes the energy degeneracy in both the spin and pseudo-spin symmetries.

Acknowledgments

The author thanks Drs. K. J. Oyewumi and B. J. Falaye, for their support during this research study.

References

[1] S. Zarrinkamar, A. A. Rajabi and H. Hassanabadi, Phys. Scr. **83**, 015009 (2011).

[2] W. Greiner, *Relativistic Quantum Mechanics* (Springer-Verlag, 1981).

[3] K. J. Oyewumi, B. J. Falaye, C. A. Onate, O. J. Oluwadare and W. A. Yahya, dx.doi.org/10.1080/00268976.2013.804960.

[4] C. A. Onate, K. J. Oyewumi and B. J. Falaye, Afr. Rev. Phys. **8**, 0020 (2013).

[5] J. N. Ginochio, Phys. Rep. **414**, 165 (2005).

[6] J. N. Ginochio and A. Leviatan, Phys. Lett. B. **425**, 1 (1998).

[7] H. Hassanabadi, E. Maghsoodi and S. Zarrinkamar, Eur. Phys. J. Plus **127**, 31 (2012).

[8] K. J. Oyewumi and C. O. Akoshile, Eur. Phys. J. A**45**, 311 (2010).

[9] C. S. Jia, P. Guo, Y. F. Diao, L. Z. Yi and X. J. Xie, Eur. Phys. J. A**34**, 41 (2007).

[10] H. Hassanabadi, E. Maghsoodi, S. Zarrinkamar and H. Rahimov, Mod. Phys. Lett. A **36**, 2703 (2011).

[11] S. M. Ikhdair and B. J. Falaye, doi.org/10.1016/j.chemphys.2013.05.021.

[12] C. S. Jia, T. Chen and S. He, Phys. Lett. A dx-doi-org/10.10161/j.physleta.2013.01.06.

[13] L. H. Zhang, X. P. Li and C. S. Jia, Int. J. Quant. Chem. **111**, 1870 (2011).
[14] S. Zarrinkamar, H. Hassanabadi and A. A. Rajabi, Int. J. Mod. Phys. A **26**, 1011 (2011).
[15] E. Maghsoodi, H. Hassanabadi and S. Zarrinkamar, Few Body Syst. DOI.10.1007/s00601-012-0314-5.
[16] H. Hassanabai, S. Zarrinkamar and H. Rahimov, Comm. Theor. Phys. **56**, 423 (2011).
[17] H. Feizi, A. A. Rajabi and M. R. Shojaei, ACTA Phys. Pol. B. **42**, 2143 (2011).
[18] H. Sun, Phys. Lett. A **374**, 116 (2009).
[19] S. M. Ikhdair, B. J. Falaye and A. G. Adepoju, 1308.0155v1 [quant-ph] Aug. 2013.

Received: 19 September, 2013
Accepted: 13 December, 2013