Exact Solutions of Feinberg-Horodecki Equation for Time-dependent Tietz-Wei Diatomic Molecular Potential

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In this study, we obtained exact solutions of the Feinberg-Horodecki equation under time-dependent Tietz-Wei di-atomic molecular potential. We have obtained the quantized momentum eigenvalues and the corresponding wave functions. It is shown that the solution can be obtained in the framework of super-symmetric quantum mechanics.

1. Introduction

The Tietz-Wei diatomic molecular potential was proposed as an inter-molecular potential and is considered as one of the best potential models that describes the vibrational energy of a diatomic molecules [1-3]. Various solutions of the wave equation with this potential have been obtained by many authors. For example, Falaye et al. [3] obtained Fisher's information entropy of the Tietz-Wei diatomic molecular model and Sun and Dong [4] presented bound state solutions of the relativistic Klein-Gordon equation with Tietz-Wei diatomic molecular potential. It is understood that there is no report of Teitz-Wei diatomic molecular potential with the Feinberg-Horodecki equation to the best of our knowledge. In this work, we examine the exact solution of the Feinberg-Horodecki equation for time-dependent Tietz-Wei diatomic molecular potential in the framework of super-symmetric quantum mechanics. The Feinberg-Horodecki time-dependent Schrödinger-like equation [5, 6] is given as

$$\left(-\frac{\hbar^2}{2mc^2}\frac{d^2}{dt^2}+V(t)\right)\psi_n(t)=cP_n\psi_n(t), \tag{1}$$

Where, c is the speed of light, t is the space-like parameter, $V(t)$ is the vector potential, m is the mass of the particle, and P_n is the quantized momentum. According to Hamzavi et al. [7], bound states of the above equation have not been considered yet. However, Molski [8] obtained Feinberg-Horodecki equation to demonstrate a possibility of describing the biological systems in terms of space-like quantum super-symmetry for non-harmonic oscillators, Hamzavi et al. obtained exact solutions of the Feinberg-Horodecki equation for time-dependent Deng-Fan molecular potential, Bera and Sil [9] found exact solutions of the Feinberg-Horodecki equation for time-dependent Wei-Hua oscillator and Manning-Rosen potentials by the Nikiforov-Uvarov method. The exact solution of the Feinberg-Horodecki equation for time-dependent Tietz-Wei diatomic molecular potential is given in the next section. In section three, we give the conclusion.

2. Exact solution of the Feinberg-Horodecki equation for time-dependent Tietz-Wei diatomic molecular potential

In this section, we obtain the exact solution of Eqn. (1) with the vector potential given as [9]

$$V(t)=D\left[\frac{1-e^{-b_h(t-t_e)}}{1-C_h e^{-b_h(t-t_e)}}\right]^2. \tag{2}$$

Substituting Eqn. (2) into Eqn. (1) results in the following second-order differential equation of the form

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$$\frac{d^2\psi_n(t)}{dt^2}=\left[\frac{V_{T_1}e^{-b_h t}}{1-C_h b e^{-b_h t}}+\frac{V_{T_2}e^{-b_h t}}{\left(1-C_h b e^{-b_h t}\right)^2}+\frac{V_{T_3}e^{-2b_h t}}{\left(1-C_h b e^{-b_h t}\right)^2}-\varepsilon_n\right]\psi_n(t), \tag{3}$$

Where

$$V_{T_1}=\frac{2mc^2bDC_h}{\hbar^2},\ V_{T_2}=\frac{2mc^2bD\left(C_h-2\right)}{\hbar^2},\ V_{T_3}=\frac{2mDc^2b^2}{\hbar^2},\ -\varepsilon_n=\frac{2mc^2\left(D-cP_n\right)}{\hbar^2},\ \ b=e^{b_h t_e}. \tag{4}$$

Now, let us apply some basic concepts of the super-symmetric quantum mechanics formalism and shape invariance techniques to obtain the solution of Eqn. (4). For good SUSY (super-symmetry) [10, 11], the ground state function $\psi_0(t)$ can be written in the form:

$$\psi_0(t)=\exp\left(-\int W(t)dt\right), \tag{5}$$

Where, $W(t)$ is called the super-potential function in super-symmetric quantum mechanics [12-14]. Substituting Eqn. (5) into Eqn. (3), we obtain the following for the super-potential function

$$W^2(t)-\frac{dW(t)}{dt}=\frac{V_{T_1}e^{-b_h t}}{1-C_h b e^{-b_h t}}+\frac{V_{T_2}e^{-b_h t}}{\left(1-C_h b e^{-b_h t}\right)^2}+\frac{V_{T_3}e^{-2b_h t}}{\left(1-C_h b e^{-b_h t}\right)^2}-\varepsilon_n. \tag{6}$$

In order to make the super-potential function compatible with the right-hand side of the non-linear Riccati equation (Eqn. (6)), we propose a super-potential function of the form:

$$W(t)=\eta-\frac{\eta_0 e^{-b_h t}}{1-C_h b e^{-b_h t}}, \tag{7}$$

Where η and η_0 are two parametric constants that will be determined later. With the proposed super-potential function of Eqn. (7), we can now construct the two partner potentials $U_{\pm}(t)$ of the super-symmetric quantum mechanics as

$$U_+(t)=W^2(t)+\frac{dW(t)}{dt}=\eta^2-\frac{\eta_0\left(1+2\eta\right)e^{-b_h t}}{1-C_h b e^{-b_h t}}+\frac{\eta_0\left(\eta_0+b_h\right)e^{-b_h t}}{\left(1-C_h b e^{-b_h t}\right)^2}, \tag{8}$$

$$U_-(t)=W^2(t)-\frac{dW(t)}{dt}=\eta^2-\frac{\eta_0\left(1+2\eta\right)e^{-b_h t}}{1-C_h b e^{-b_h t}}+\frac{\eta_0\left(\eta_0-b_h\right)e^{-b_h t}}{\left(1-C_h b e^{-b_h t}\right)^2}. \tag{9}$$

To obtain the values of the two parametric constants in Eqn. (7), we compare Eqn. (9) with Eqn. (6) and then establish the following relationship between the two constants and other variables as

$$\eta^2=\frac{2mDc^2}{\hbar^2}+\frac{2mc^3P_n}{\hbar^2}. \tag{10}$$

Here, the author's interest is a solution, which demand the wave function to satisfy the boundary conditions

$$\frac{\psi_n(t)}{t}=\begin{cases}0, \mathrm{t}\to\infty\\ \infty, \mathrm{t}\to 0\end{cases}. \tag{11}$$

Now, taking the regularity conditions into consideration (or restriction conditions: $\eta>0$, $\eta_0>0$), we obtain the values of these constants as

$$\eta_0=\frac{b_h\pm\sqrt{b_h{}^2+\dfrac{8mbDc^2\left(C_h+b-2\right)}{\hbar^2}}}{2}, \tag{12}$$

$$\eta=\frac{\eta_0^2-\dfrac{2mbDc^2\left(C_h+b\right)}{\hbar^2}}{2\eta_0}. \tag{13}$$

From Eqns. (8) and (9), the two partner potentials are related by the equation

$$U_+(t,a_0)=U_+(t,a_1)+R(a_1), \tag{14}$$

Where, $a_0=\eta_0$ and a_1 is a function of a_0 i.e., $a_1=f(a_0)$ and the remainder $R(a_1)$ is independent of the space time t. This relationship can be expanded as

$$R(a_1)=\left(\frac{a_0^2-\rho}{2a_0}\right)^2-\left(\frac{a_1^2-\rho}{2a_1}\right)^2, \tag{15}$$

$$R(a_2)=\left(\frac{a_1^2-\rho}{2a_1}\right)^2-\left(\frac{a_2^2-\rho}{2a_2}\right)^2, \tag{16}$$

$$R(a_3)=\left(\frac{a_2^2-\rho}{2a_2}\right)^2-\left(\frac{a_3^2-\rho}{2a_3}\right)^2, \tag{17}$$

Consequently, we have

$$R(a_n)=\left(\frac{a_{n-1}^2-\rho}{2a_{n-1}}\right)^2-\left(\frac{a_n^2-\rho}{2a_n}\right)^2, \tag{18}$$

Where, $\rho=\frac{2mbDc^2(C_h+b)}{\hbar^2}$

It is now clear that the two partner potentials satisfy the shape invariance conditions via mapping of the form: $\eta_0\to\eta_0-b_hn.$ Considering the shape invariant potential Eqn. (9) together with Eqns. (10), (12) and (13), the quantized momentum is obtained as

$$P_n=\frac{D}{c}-\frac{{b_h}^2\hbar^2}{2mc^3}\left[\frac{\left[\frac{1}{2}\left(1-2n-\sqrt{1+\frac{8mbDc^2(C_h+b-2)}{{b_h}^2\hbar^2}}\right)\right]^2-\frac{2mbDc^2(C_h+b)}{{b_h}^2\hbar^2}}{1-2n-\sqrt{1+\frac{8mbDc^2(C_h+b-2)}{{b_h}^2\hbar^2}}}\right]^2. \tag{19}$$

To obtain the corresponding wave function, we define a variable of the form $s=e^{-b_ht}$ and substitute it into Eqn. (3) to have

$$\frac{d^2\psi_n(s)}{ds^2}+\frac{1-s}{s(1-s)}\frac{d\psi_n(s)}{ds}+\frac{As^2+Bs+C}{\left(s(1-s)\right)^2}, \tag{20}$$

Where, $A=\frac{2mc^3P_n}{{b_h}^2\hbar^2}$, $B=\frac{2mc^2\left(\frac{2D-Db^2}{bC_h}-2cP_n\right)}{{b_h}^2\hbar^2}$,

$C=-\frac{2mc^2(D-cP_n)}{{b_h}^2\hbar^2}$.

The wave function is given by

$$U(s)=N_ns^{k_4}(1-\alpha_3s)^{k_5}\,{}_2F_1\left(-n,n+2(k_4+k_5)+\frac{\alpha_2}{\alpha_3}-1;2k_4+\alpha_1,\alpha_3s\right), \tag{21}$$

where $k_4=\frac{(1-\alpha_1)+\sqrt{(1-\alpha_1)^2-4C}}{2}$, $k_5=\frac{1}{2}+\frac{\alpha_1}{2}-\frac{\alpha_2}{2\alpha_3}+\sqrt{\left(\frac{1}{2}+\frac{\alpha_1}{2}-\frac{\alpha_2}{2\alpha_3}\right)^2-\left(\frac{A}{\alpha_3\alpha_3}+\frac{B}{\alpha_3}+C\right)}$,

The complete wave function is given as

$$\psi(s)=N_ns^{k_4}(1-s)^{k_5}\,{}_2F_1\left(-n,n+2(k_4+k_5);2k_4+1,s\right). \tag{22}$$

The quantized momentum (Eqn. (19)), which is a solution of the Feinber-Horodecki time-dependent Schrödinger-like equation (Eqn. (1)) through mapping gives the energy eigenvalues for time-independent Schrödinger equation. As $P_n\to E_n$ and $t\to r$, Eqn. (19) turns to

$$E_n = \frac{D}{c} - \frac{{b_h}^2\hbar^2}{2mc^3}\left[\frac{\left[\frac{1}{2}\left(1-2n-\sqrt{1+\frac{8mbDc^2\left(C_h+b-2\right)}{{b_h}^2\hbar^2}}\right)\right]^2 - \frac{2mbDc^2\left(C_h+b\right)}{{b_h}^2\hbar^2}}{1-2n-\sqrt{1+\frac{8mbDc^2\left(C_h+b-2\right)}{{b_h}^2\hbar^2}}}\right]^2. \quad (23)$$

which is the solution for s-wave for nonrelativistic Schrödinger equation.

3. Conclusion

In this work, we have obtained the exact solutions of the Feinberg-Horodecki equation for time-dependent Tietz-Wei diatomic molecular potential via super-symmetric approach. Through simple mapping, the exact solution of the nonrelativistic Schrödinger equation is obtained. It is therefore, worth mentioning that our method is elegant and powerful. Our results can be applied in biophysics and other branches of physics.

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