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Bound state solutions of D-dimensional Klein−Gordon equation with hyperbolic potential

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Abstract

By using the basic supersymmetric quantum mechanics concepts and formalism, the energy eigenvalue equation and the corresponding wave function of the Klein−Gordon equation with vector and scalar potentials for an arbitrary dimensions are obtained together with hyperbolic potential using a suitable approximation scheme to the orbital centrifugal term. The non-relativistic limit is obtained and the numerical values for various values of D, n, α and ℓ are obtained.

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1. Introduction

The exact analytic solutions of the wave equations (relativistic and non-relativistic) are only possible for certain potentials of physical interest under consideration since they contain all the necessary information in the quantum system [1].

The analytical approximation methods to Klein−Gordon equation that describes relativistic spin 0 particles have attracted a great deal of interest in physics [2]. The solution of the Klein−Gordon equation plays an important role in the relativistic quantum mechanics. In the recent time, many authors have solved relativistic equations with physical potential models. These potentials include Rosen−Morse potential [3,4], Posch−Teller potential [5,6], five parameter exponential potential [7,8], Hulthen potential [9−13], Davidson potential [14], Wooden−Saxon potential [7,8]. Within the past three decades, however, the introduction of the concept of supersymmetric quantum mechanics (SUSY QM) has greatly simplified the problem in some cases [15] and others [16,17]. Apart from SUSY approach and its extension such as supersymmetric WKB and supersymmetric path integral formalism [18], many methods including Nikiforov − Uvarov method [1,14,19−21], asymptotic iteration method [22−27] are also used in solving the wave equation to obtain the energy equation. In this work, we study the.

Klein−Gordon equation in an arbitrary dimensional space with the hyperbolical potential. The hyperbolical

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potential is closely related to the Morse and Coulomb potential functions [28−30]. The hyperbolical potential has already been studied under Schrödinger equation and Dirac equation by various analytical tools. Here, bearing in mind the deeper physical insight that analytical methodologies provide into the physics of problem, we use the powerful SUSY QM in our calculations on the D-dimensions, for works in parallel on D-dimensional space [31−47] and references therein, one could see many papers.

2. The Klein − Gordon equation in D − dimensions

The time independent D-dimensional Klein−Gordon equation in the atomic units $\hbar = c = \mu = 1$, may be written as [48].

$$-\nabla_N^2\psi(r) + [M + S(r)]^2\psi(r) - [E - V(r)]^2\psi(r) = 0, \tag{1}$$

where M is the particle mass, E is the energy, V(r) and S(r) are vector and scalar potentials respectively. The D-dimensional Laplacian operator ∇_D^2 is given by Ref. [49].

$$\nabla_D^2 = r^{1-D}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) + \frac{L_D^2(\Omega_D)}{r^2}, \tag{2a}$$

where $L_D^2(\Omega_D)$ is the ground angular momentum [48]. In addition, we know that $L_D^2(\Omega_D)/r^2$ is a generalization of the centrifugal barrier for the D-dimensional space and involves angular coordinates Ω_D and the eigenvalues of the $L_D^2(\Omega_D)$ [49]. $L_D^2(\Omega_D)$ is a partial differential operator on the unit space S^{D-1} (Laplace Betrami operator or the ground orbital operator) define analogously to a three-dimensional angular momentum [50] as $L_D^2(\Omega_D) = -\sum_{i\geq j}^{D}(L_{ij}^2)$ where $L_{ij}^2 = x_i\partial/\partial x_j - x_j\partial/\partial x_i$ for all Cartesian component x_i of the D-dimensional vector $(x_1, x_2, \ldots\ldots, x_N)$. To eliminate the first order derivative, Hassanabadi et al. [49] defined the total wave function as

$$R_{n,\ell}(r) = r^{-\frac{(D+1)}{2}}U_{n,\ell}(r), \tag{2b}$$

then,

$$L_D^2 Y_l^m(\Omega_D) = l(l + D - 2)Y_l^m(\Omega_D). \tag{2c}$$

here, we studied the Klein−Gordon equation in the D − dimensions for vector and scalar potential given as [31,32,46,47].

$$V(r) = S(r) = \delta[1 - \sigma_0\coth(\alpha r)]^2. \tag{3}$$

In order to solve Eq. (1) explicitly with orbital angular momentum $\ell \neq 0$, we apply a suitable approximation-type. The approximation apply get rids of the orbital centrifugal barrier. The approximation is given by Refs. [16,17,51].

$$\frac{1}{r^2} \approx \left(2\alpha e^{-\alpha r}\right)^2\left(1 - e^{-2\alpha r}\right)^{-2}. \tag{4}$$

which is valid for $ar \ll 1$. In the arbitrary dimension, we set

$$U_{n,\ell}(r) = r^{\frac{D-1}{2}}R_{n,\ell}(r). \tag{5}$$

Thus, Eq. (1) is written in a new form as

$$-\frac{d^2U_{n,\ell}(r)}{dr^2} + \left[-\left(E_{n,\ell} - V(r)\right)^2 + \left(M + S(r)\right)^2 + \frac{(D + 2\ell - 1)(D + 2\ell - 3)}{4r^2}\right]U_{n,\ell}(r) = 0. \tag{6a}$$

For a non-relativistic limit of potential +V, Eq. (6a) is written in the form

$$-\frac{d^2U_{n,\ell}(r)}{dr^2} + \left[M^2 - E_{n,\ell}^2 + V(r)(M + E_{n,\ell}) + \frac{(D + 2\ell - 1)(D + 2\ell - 3)}{4r^2}\right]U_{n,\ell}(r) = 0. \tag{6b}$$

Substituting Eqs. (3) and (4) into Eq. (6b), we easily have

$$\frac{d^2U_{n,\ell}(r)}{dr^2} = \left(\xi\sigma_0 + \Lambda\alpha^2\right)csch^2(\alpha r) - 2\xi\coth(\alpha r) - \overline{E}, \tag{7}$$

where

$$-\overline{E} = E_{n,l}^2 - M^2 + \frac{1}{2}\varepsilon - \xi\sigma_0, \tag{8a}$$

$$\xi = \delta\sigma_0(E_{n,\ell} + M), \tag{8b}$$

$$\varepsilon = -2\delta(E_{n,\ell} + M), \tag{8c}$$

$$\Lambda = \frac{(D + 2\ell - 1)(D + 2\ell - 3)}{4}, \tag{8d}$$

Eq. (7) is a non-linear Riccati equation that can be transform as:

$$W^2(r) - \frac{dW(r)}{dr} = \left(\xi\sigma_0 + \Lambda\alpha^2\right)csch^2(\alpha r) - 2\xi\coth(\alpha r) - \overline{E}, \tag{9}$$

In other to obtain the solution of Eq. (9), we simply write the superpotential of the supersymmetric quantum mechanics. The superpotential gives a solution to the Riccati equation given in Eq. (9). This is to ensure that the left hand side of Eq. (9) is compatible to the right hand side. The propose superpotential is written in the form:

$$W(r) = A - \frac{Bcosh(\alpha r)}{\sinh(\alpha r)}. \tag{10}$$

From the superpotential, the left hand side of Eq. (9) can easily be obtain as well as the two constants A and B in Eq. (10) as follows:

$$W'(r) = \alpha B csch^2(\alpha r), \tag{11}$$

$$W^2(r) = A^2 + B^2 csch^2(\alpha r) - 2ABcoth(\alpha r), \tag{12}$$

$$B = \frac{\alpha \pm \sqrt{\alpha^2(1+(D+2\ell-1)(D+2\ell-3)+4\sigma_0\xi}}{2}, \tag{13}$$

$$A = \frac{\xi}{B}. \tag{14}$$

The ground state wave function $U_{o,\ell}(r)$ is simply calculated from

$$U_{o,\ell}(r) = N_{o,l} \exp\left(-\int W(r)dr\right), \tag{15}$$

where N is the normalization constant. Now, to proceed to the next step, we construct the supersymmetric partner potentials $V_{\pm}(r) = W^2(r) \pm dW(r)/dr$, of the supersymmetric quantum mechanics

$$V_+(r) = A^2 + B^2 - \frac{2ABcosh(\alpha r)}{\sinh(\alpha r)} + \frac{B(B+\alpha)}{sinh^2(\alpha r)}, \tag{16}$$

$$V_-(r) = A^2 + B^2 - \frac{2ABcosh(\alpha r)}{\sinh(\alpha r)} + \frac{B(B-\alpha)}{sinh^2(\alpha r)}. \tag{17}$$

From Eqs. (16) and (17), it is seen that $V_+(r)$ and $V_-(r)$ are shape invariant and the relationship between the two partner potentials is written as [52].

$$R(a_1) = V_+(r, a_0) - V_-(r, a_1), \tag{18}$$

where the shape invariance holds via mapping of the form $B \rightarrow B - \alpha$ and a_1 is a function of a_0 which can be written as $a_1 = f(a_0) = a_0 - \alpha$ and $a_0 = B$ with $R(a_1)$ as the residual term and is independent of r. With the shape invariance approach [51,53−55], we can determine the approximate energy eigenvalues of the shape invariant potential of Eq. (17) and obtain the following results:

Here we plotted the graph of energy against α (See Figs. 1 and 2).

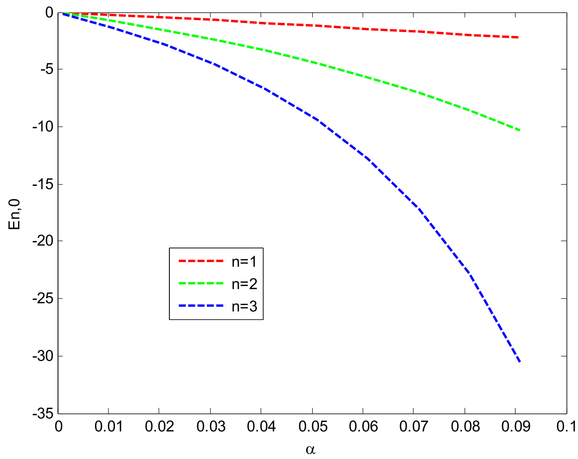

Fig. 1. $E_{n,0}$ against α with $\sigma_0 = 0.1$, $\delta = 10$ and $D = 3$.

$$R(a_2) = V_+(r, a_1) - V_-(r, a_2), \tag{19}$$

$$R(a_3) = V_+(r, a_2) - V_-(r, a_3), \tag{20}$$

$$R(a_n) = V_+(r, a_{n-1}) - V_-(r, a_n), \tag{21}$$

$$\overline{E}^-_{o,\ell} = 0, \tag{22}$$

$$\overline{E}^-_{n,\ell} = \sum_{k=1}^{n} R(a_k) = -\left(\left(\frac{\xi}{a_0}\right)^2 + a_0^2\right) - \left(\left(\frac{\xi}{a_n}\right)^2 + a_n^2\right), \tag{23}$$

where, $a_n = B - \alpha n$. Then,

$$\overline{E} = \overline{E}_{n,\ell} + \overline{E}^-_{0,l} = -\left(\left(\frac{\xi}{a_n}\right)^2 + a_n^2\right). \tag{24}$$

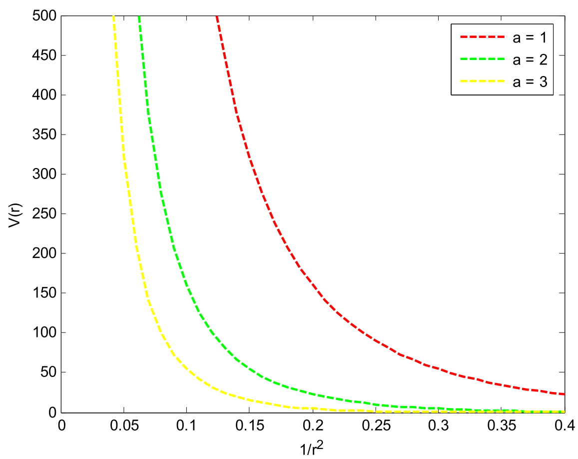

Fig. 2. $V(r)$ against $1/r^2$ with $\sigma_0 = 0.2$, $\alpha = 1$ and $\delta = 10$.

Now, substituting Eqs. (8a), (8b) and (8d) and the value of a_n into Eq. (24), we obtain the energy eigenvalue equation as

$$E_{n\ell}^2-\delta\left(1+\sigma_0^2\right)(E_{n\ell}+M)+\frac{(-2\delta\sigma_0(E_{n\ell}+M))^2}{\left(\alpha+2\alpha+\sqrt{\alpha^2[1+(D+2\ell-1)(D+2\ell-3)]+4\delta\sigma_0^2(E_{n\ell}+M)}\right)^2}$$
$$=M^2-\left[\frac{\alpha+2\alpha+\sqrt{\alpha^2[1+(D+2\ell-1)(D+2\ell-3)]+4\delta\sigma_0^2(E_{n\ell}+M)}}{2}\right]^2. \tag{25a}$$

3. Non-relativistic limit

The relativistic Klein–Gordon equation is spin-0 while the non-relativistic Schrődinger equation is bosonic in nature (spineless). It implicitly suggests that a relationship may exists between the solutions of these two important equations [56]. Alhaidari et al. [57] have shown the Klein–Gordon equation of potential V whose bound state in the non-relativistic limit can easily be obtained. The essence of the approach was that, in the non-relativistic limit, the Schrődinger equation may be derived from the relativistic one when the energies of the potentials S(r) and V(r) are small compared to the rest mass mc^2 [56], then, the non-relativistic energies can be determined by taking the non-relativistic limit values of the relativistic eigenenergies. By using the transformation $E_{n\ell}+M=2\mu/\hbar^2$ and $M-E_{n\ell}=-E_{n\ell}$, the relativistic energy Eq. (25a) reduces to

$$E_{n\ell}+\frac{\hbar^2}{2\mu}\frac{(-4\delta\sigma_0)^2}{\left(\alpha+2\alpha+\sqrt{\alpha^2[1+(D+2\ell-1)(D+2\ell-3)]+8\delta\sigma_0^2}\right)^2}$$
$$=\delta\left(1+\sigma_0^2\right)-\frac{\hbar^2}{2\mu}\left[\frac{\alpha+2\alpha+\sqrt{\alpha^2[1+(D+2\ell-1)(D+2\ell-3)]+8\delta\sigma_0^2}}{2}\right]^2 \tag{25b}$$

Eq. (25b) is identical to Eq. (27) of Ref. [55]. In other to obtain the wave functions, we define a variable of the form $y=e^{-2\alpha r}$ and substituting it into Eq. (7), we have

$$\frac{d^2U_{n\ell}(y)}{dy^2}+\frac{1}{y}\frac{dU_{n\ell}(y)}{dy}+\frac{Py^2+Ry+Q}{(y(1-y))^2}U_{n\ell}(y)=0, \tag{26}$$

where

$$P=\frac{E_{n\ell}^2-M^2-2\delta[(E_{n\ell}-M)(1+\sigma_0(\sigma_0+2)]}{4\alpha^2}$$

$$R=\frac{2\delta(E_{n\ell}-M)[1+\sigma_0(\sigma_0-2)]-\left(E_{n\ell}^2-M^2\right)-\alpha^2(D+2\ell-1)(D+2\ell-3)}{4\alpha^2}$$

$$Q=\frac{E_{n\ell}^2-M^2-\delta(E_{n\ell}-M)[1+\sigma_0(\sigma_0-2)]}{4\alpha^2}.$$

Analyzing the asymptotic behavior of Eq. (26) at origin and at infinity, it can be tested when $r\to 0(y\to 1)$, Eq. (7) thus has a solution $U_{n\ell}(y)=(1-y)^z$ with

$$z = \frac{1}{2} + \sqrt{\frac{1}{4} + (D + 2\ell - 1)(D + 2\ell - 3) + \frac{\delta(E_{n\ell} - M) - \left(E_{n\ell}^2 - M^2\right) + \delta\sigma_0(\sigma_0 - 2)}{\alpha^2}}. \tag{27}$$

Similarly, when $r \to \infty (y \to 0)$, Eq. (7) has a solution $U_{n\ell}(y) = y^{\mathbb{C}}$ with

$$\mathbb{C} = \frac{1}{2\alpha}\sqrt{E_{n\ell}(E_{n\ell} - 2\delta) + M(\delta - M) - \delta\sigma_0(\sigma_0 - 2)(E_{n\ell} - M)}. \tag{28}$$

Taking a trial wave function of the form $U_{n\ell}(y) = y^{\mathbb{C}}(1-y)^z f(y)$ and substituting it into Eq. (7), we obtain

$$f''(y) + f'(y)\left[\frac{(2\mathbb{C}+1) - y(2\mathbb{C}+2z+1)}{y(1-y)}\right] - f(y)\left[\frac{(\mathbb{C}+z)^2 + Q}{y(1-y)}\right] = 0. \tag{29}$$

Eq. (29) is satisfied by the hypergeometric function whose solution is found as

$$f(y) = {}_2F_1(-n,\, n + 2(\mathbb{C}+z); 2\mathbb{C}+1,\, y). \tag{30}$$

Now, replacing the function $f(y)$ with the hypergeometric function, the complete radial wave function is given as

$$U_{n\ell}(y) = N_{n\ell}y^{\mathbb{C}}(1-y)^z{}_2F_1(-n, n+2(\mathbb{C}+z); 2\mathbb{C}+1, y), \tag{31}$$

where $N_{n\ell}$ is the normalization factor.

Table 1
Energy eigenvalues $(-E_{n,\ell})$ with $M = 1$, $\delta = 10$, $\sigma_0 = 0.2$ and $\alpha = 0.25$.

D	$E_{0,0}$	$E_{1,0}$	$E_{1,1}$	$E_{2,0}$	$E_{2,1}$	$E_{2,2}$
1	0.997887	0.984754	0.984754	0.957910	0.957910	0.925690
2	0.999879	0.990193	0.974822	0.964918	0.943621	0.904706
3	0.997887	0.984754	0.962012	0.957910	0.925690	0.880800
4	0.993188	0.974822	0.946425	0.943621	0.904706	0.853970
5	0.985794	0.962012	0.928060	0.925690	0.880800	0.824220
6	0.975706	0.946425	0.906910	0.904706	0.853970	0.791500
7	0.962900	0.928060	0.882950	0.880800	0.824220	0.755760
8	0.947370	0.906910	0.856110	0.853970	0.791500	0.716960
9	0.929080	0.882950	0.826400	0.824220	0.755760	0.675030
10	0.908020	0.856110	0.793720	0.791500	0.716960	0.629910

Table 2
Energy eigenvalues $[\![(+E]\!]_{n,\ell})$ with $M = 1$, $\delta = 10$, $\sigma_0 = 0.2$ and $\alpha = 0.25$.

D	$E_{0,0}$	$E_{1,0}$	$E_{1,1}$	$E_{2,0}$	$E_{2,1}$	$E_{2,2}$
1	1.67852	3.92528	3.92528	5.13021	5.13021	5.27399
2	1.56459	3.89098	4.02361	5.11106	5.18600	5.38800
3	1.67852	3.92528	4.17433	5.13021	5.27399	5.52132
4	1.96540	4.02361	4.36289	5.18600	5.38800	5.66731
5	2.33627	4.17433	4.57553	5.27399	5.52132	5.82042
6	2.73549	4.36289	4.80094	5.30800	5.66731	5.97594
7	3.13610	4.57553	5.03071	5.52132	5.82042	6.13014
8	3.52537	4.80094	5.25869	5.66731	5.97594	6.28035
9	3.89722	5.03071	5.48053	5.82042	6.13014	6.42424
10	4.24881	5.25869	5.69355	5.97594	6.28035	6.56050

Table 3
Energy eigenvalues with $M = 1$, $\delta = 10$, $\sigma_0 = 0.1$ and $D = 3$.

n	ℓ	α	$-E_{n,\ell}$	$+E_{n,\ell}$
0	1	0.05	0.999512	1.28726
		0.10	0.997982	1.87928
		0.15	0.995312	2.53566
1	1	0.05	0.998655	2.61063
		0.10	0.994508	3.73250
		0.15	0.987392	4.63130
0	2	0.05	0.998662	1.60426
		0.10	0.994540	2.59240
		0.15	0.987470	3.54579
2	1	0.05	0.997363	3.49838
		0.10	0.989293	4.88860
		0.15	0.975538	5.88063
1	2	0.05	0.997384	2.77253
		0.10	0.989378	4.08832
		0.15	0.975730	5.13925
0	3	0.05	0.997392	1.96491
		0.10	0.989420	3.29332
		0.15	0.975830	4.46169

Table 4
Bound-state energy spectrum for the non-relativistic limit as a function of α for 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states with $\mu = \hbar = 1$, $\sigma_0 = 0.1$ and $\delta = 10$.

n	ℓ	State	α	$E_{n\ell}$, $D = 2$	$E_{n\ell}$, $D = 3$	$E_{n\ell}$, $D = 4$
0	1	2p	0.10	2.22867	2.61556	3.09295
			0.15	3.25891	3.89830	4.59834
			0.20	4.17855	4.99062	5.77337
			0.25	4.97336	5.86611	6.62542
1	1	3p	0.10	4.50927	4.73223	5.01090
			0.15	5.73690	6.03829	6.37740
			0.20	6.58259	6.90394	7.22586
			0.25	7.16626	7.46417	7.72454
0	2	3d	0.10	3.09295	3.61747	4.15235
			0.15	4.59834	5.27263	5.87450
			0.20	5.77337	6.43684	6.96110
			0.25	6.62542	7.19574	7.59462
2	1	4p	0.10	5.86217	5.99969	6.17297
			0.15	6.95176	7.10812	7.28613
			0.20	7.56902	1.70634	7.84211
1	2	4d	0.10	5.01090	5.32177	5.64405
			0.15	6.37774	6.71441	7.02293
			0.20	7.22586	7.50672	7.73066
0	3	4f	0.10	4.15235	4.67061	5.15502
			0.15	5.87450	6.38708	6.81066
			0.20	6.96110	7.35782	7.64781
3	1	5p	0.10	6.71189	6.80027	6.91213
2	2	5d	0.10	6.17297	6.36810	6.57244
1	3	5f	0.10	5.64405	5.96159	6.26319
0	4	5g	0.10	5.15502	5.59631	5.99096
4	1	6p	0.10	7.26309	7.32099	7.39434
3	2	6d	0.10	6.91213	7.03872	7.17190
2	3	6f	0.10	6.57244	6.77575	6.97058
1	4	6g	0.10	6.26319	6.54204	6.79465

4. Discussion

From the numerical results obtained, it can be seen from Tables 1 and 2 that energy degeneracy occurred for some values of n and ℓ. For example, for$E_{0,0}$, $E_{1,0}$ and $E_{2,0}$, the energy obtained with $D = 1$ are equal to the energy obtained with $D = 3$. i.e. $E_{0,0}(D = 1) = E_{0,0}(D = 3)$, $E_{1,0}(D = 1) = E_{1,0}(D = 3)$ and $E_{2,0}(D = 1) = E_{2,0}(D = 3)$. These degeneracies occurred only when $\ell = 0$.

In Table 3, it can be seen that as α increases for all n, ℓ and D, the energy obtained increases. This trend observed in Table 3, are also observed in Table 4.

5. Conclusion

We obtained the solutions of the D − dimensional Klein − Gordon equation with hyperbolic potential using supersymmetric quantum mechanics (SUSY QM) after applying a proper approximation to the centrifugal term. The eigenfunction was equally obtained. The numerical results for both negative and positive energy were also obtained for different states. It is seen from Table 3 that energy increases with increasing α for both $-E_{n,\ell}$ and $+E_{n,\ell}$.

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