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Research Article

Dirac particles in the field of Frost-Musulin diatomic potential and the thermodynamic properties via parametric Nikiforov-Uvarov method.

C. A. Onate[1,2*] and M. C. Onyeaju[3]

[1]*Physics Department, University of Benin, PM B, 1154, Benin, Nigeria.*
[2]*PhysicsProgramme, Department of Physical Sciences, Landmark University,*
PM B, 1001, Omu-Aran, Nigeria.
[3]*Theoretical Physics Group, Department of Physics, University of Port Harcourt,*
PMB 5323, Choba Port Harcourt, Nigeria.

Abstract

The Frost-Musulin Diatomic molecular potential has been investigated under spin and pseudospin symmetries of the Dirac equation in a comparative study. By using a siutable approximation scheme, we have presented the energy eigenvalues for both the spin symmetry and pseudospin symmetry and the corresponding upper and lower spinor component of the wave functions in a close form. We have obtained the non-relativistic limit and compared our numerical eigenvalues with the one previousely obtained. Finally, we calculated thermodynamic properties of a particle under the Frost-Musulin diatomic potential.

Keywords: Frost-Musulin diatomic potential, Dirac equation, approximation scheme, Bound state solutions, Thermodynamic properties

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[*] Corresponding Author Email: oaclems14@physicist.net

1. INTRODUCTION

The exact analytical solutions of the relativistic/non-relativistic wave equations play an important role in quantum mechanics because they contain all the necessary information required to understand the quantum behaviour of the relativistic/ non-relativistic particles[1]. One of these wave equation is the Dirac equation which has received many theoretical and phenomenological attentions as it provides us with a relativistic background to study spin-1/2 particles. For instance, Ikhdair and Sever[2,3] solved approximately bound states of the Dirac equation with some physical quantum potential; approximate bound state solution of the Dirac equation with Hulthėn potential including Coulomb like tensor potential; Arda and Sever[4] obtained approximate solutions of Dirac equation with hyperbolic-type potential; Hassanabadi *et al.*[5] obtained Dirac equation under Manning Rosen potential and Hulthėn tensor interaction; Ikot *et al*.[6] obtained scattering states of Cusp potential in minimal length Dirac equation; Lütfüoğlu *et al.*[7] obtained scattering bound and quasi-bound states of the generalized symmetric woods-Saxon potential; Sari *et al.*[8] obtained solutions of Dirac equation for Eckart potential and trigonometric Manning-Rosen potential using asymptotic iteration method; Oyewumi and Akoshile[9] obtained bound states solutions of the Dirac-Rosen-Morse potential with spin and pseudospin symmetry; Hassanabadi *et al.*[10] investigated approximate solutions of the Dirac equation for hyperbolic scalar and vector potentials and a Coulomb tensor interaction by SUSY QM. Wei and Dong[11-15] studied Dirac equation extentensively in the frame-work of either spin symmetry or pseudospin symmetry or both with variour potential models such as Manning-Rosen potential, deformed generalized Pöschl-Teller potential, modified Pöschl-Teller potential, Second Pöschl-Teller potential and modified Rosen-Morse potential. The Dirac equation in this paper is studied under spin symmetry and pseudospin symmetry. The spin symmetry is relevaant to mesons[16] and it occurs when the difference of the scalar $S(r)$ and the vector $V(r)$ potentials are constant i.e. $\Delta(r)=C_s$ [17-22] while the pseudospin symmetry refers to quasi-degeneracy of single nucleon doublets with non-relativistic quantum number $\left(n,l,j=l+1/2\right)$ and $\left(n-1,l+2,j=l+3/2\right)$ where n,l and j denote the single nucleon radial orbital and total angular quantum numbers respectively[23, 24]. To the best of our knowledge, the symmetry limit of the Dirac equation and the thermodynamic properties with Frost-Musulin diatomic molecular potential has not been obtained yet. This is the most priority reason why the author attempts to study the behaviour of Frost-Musulin diatomic molecular potential with Dirac equation. The Frost-Musulin diatomic molecular potential function which has application in chemistry and molecular physics was first proposed by Frost and Musulin in 1954[25]. However, in 2012, Jia *et al.*[26] by using the equilibrium bond length and dissociation energy for a diatomic molecule as explicity parameter to generate improved expression for this poential.

The Frost-Musulin diatomic molecular potential function is written as[25, 26]

$$V_{FM}(r)=D_e\left[1-\frac{(r+\delta r_e(r-r_e))\,\mathrm{e}^{-\delta(r-r_e)}}{r}\right], \tag{1}$$

where δ is the range of the potential well, r_e is the equilibrium bound length and D_e is disociation energy. Jia *et al.*[26] found the minimum level of this potential at

$$r_e=\frac{1}{2}\frac{\delta p_1\sqrt{\delta p_1\left(\delta p_1+4p_2\right)}}{\delta p_2} \tag{2}$$

where

$$p_1=\frac{\delta D_e r_e^2}{e^{-\delta r_e}},\; p_2=\frac{D_e\left(1+\delta r_e\right)}{e^{-\delta r_e}}. \tag{3}$$

In the next step, we proceed to review Dirac equation and a brief of Nikiforov-Uvarov method. In section 3, we present bound state solutions while in section 4, we give the conclusion.

2. DIRAC EQUATION

In this section, we briefly review the Dirac equation. The Dirac equation with scalar $S(r)$ and $V(r)$ potentials in spherical coordinates is given as[27-34]

$$\left[\vec{\alpha}.\vec{p}+\beta\left(M+S(r)\right)-\left(E-V(r)\right)\right]\psi\left(\vec{r}\right)=0, \tag{4}$$

where $\vec{p}=-i\vec{\nabla}$ is the momentum operator, E denote the relativistic energy of the system, α and β are 4×4 usual Dirac matrice. For a particle in a spherical field, the total angular momentum operator j and the spin-orbit matrix operator $k=\left(\sigma.L+1\right)$, where σ and L

are the Pauli matrix and orbital angular momentum respectively, commute with the Dirac Hamiltonian. The eigenvalues of k are $k=-\left(j+1/2\right)$ for the aligned spin $\left(s_{1/2}, p_{3/2}, etc\right)$ and $k=\left(j+1/2\right)$ for the unaligned spin $\left(p_{1/2}, d_{3/2}, etc\right)$. The complete set of conservative quantities can be chosen as $\left(H, K, J^2, J_z\right)$. The Dirac spinor is

$$\psi_{nk}(r)=\begin{pmatrix} f_{nk}(r) \\ g_{nk}(r) \end{pmatrix}=\begin{pmatrix} \dfrac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\vartheta) \\ \dfrac{G_{nk}(r)}{r}Y_{jm}^{\tilde{l}}(\theta,\vartheta) \end{pmatrix}, \tag{5}$$

where $F_{nk}(r)$ and $G_{nk}(r)$ are the radial wave functions of the upper and lower components respectively with $Y_{jm}^{l}(\theta,\vartheta)$ and $Y_{jm}^{\tilde{l}}(\theta,\vartheta)$ for spin and pseudospin spherical harmonics coupled to the angular momentum on the $z-$ axis. Now substitute Eq. (5) into Eq. (4), we recast the following differential equations[1, 35, 36]

$$\left(\frac{d^2}{dr^2}+\frac{\kappa}{r^2}\right)F_{n\kappa}(r)=\left(M+E_{n\kappa}-V(r)+S(r)\right)G_{n\kappa}(r), \tag{6}$$

$$\left(\frac{d^2}{dr^2}-\frac{\kappa}{r^2}\right)G_{n\kappa}(r)=\left(M-E_{n\kappa}+V(r)+S(r)\right)F_{n\kappa}(r), \tag{7}$$

which later give

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-\left[\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\sum(r)\right)\right]+\frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}\right)\right\}F_{n\kappa}(r)=0, \tag{8}$$

for $\kappa(\kappa+1)=\ell(\ell+1),\ \ r\grave{o}(0,\infty)$,

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\left[\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\sum(r)\right)\right]-\frac{\frac{d\sum(r)}{dr}}{M-E_{n\kappa}+\sum(r)}\left(\frac{d}{dr}-\frac{\kappa}{r}\right)\right\}G_{n\kappa}(r)=0. \tag{9}$$

for $\kappa\left(\kappa-1\right)=\tilde{\ell}\left(\check{\ell}+1\right)$, $r\grave{o}(0,\infty)$. It is noted that $\Delta(r)=V(r)-S(r)$ and $\Sigma(r)=V(r)+S(r).$

2.1 Nikiforov-Uvarov method.

Given the following general form of the Schrödinger-like equation[37-41]

$$\psi''\left(r\right)+\left[\frac{\alpha_1-\alpha_2 s}{s\left(1-\alpha_3 s\right)}\right]\psi'\left(s\right)+\left[\frac{-\zeta_1 s^2+\zeta_2 s-\zeta}{\left(s\left(1-s\right)\right)^2}\right]\psi\left(s\right)=0, \tag{10}$$

where the condition for the energy equation is deduced as follows [37, 42-44]

$$n\alpha_2-\left(2n+1\right)\alpha_5+\left[n\left(n-1\right)+2\alpha_8\right]\alpha_3+\sqrt{4\alpha_8\alpha_9}+\left(2n+1\right)\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)=0. \tag{11}$$

and wave function is given as

$$\psi_{n,\ell}\left(s\right)=N_{n,\ell}s^{\alpha_{12}}\left(1-\alpha_3 s\right)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}\left(1-2\alpha_3 s\right). \tag{12}$$

$$\alpha_4=\frac{1-\alpha_1}{2},\quad \alpha_5\frac{\alpha_2-2\alpha_3}{2},\quad \alpha_6=\alpha_5^2+C_1,\quad \alpha_7=2\alpha_4\alpha_5-C_2,\quad \alpha_8=\alpha_4^2+C_3,$$

$$\alpha_9=\alpha_3\left(\alpha_7+\alpha_3\alpha_8\right)+\alpha_6,\ \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}-1>-1,$$

$$\alpha_{11}=1-\alpha_1-2\alpha_4+\frac{2}{\alpha_3}\sqrt{\alpha_9}>-1\ ,\alpha_3\neq 0,\qquad \alpha_{12}=\alpha_4+\sqrt{\alpha_8}>0,$$

$$\alpha_{13}=-\alpha_4+\frac{1}{\alpha_3}\left(\sqrt{\alpha_9}-\alpha_5\right)>0, \tag{13a}$$

In a special case when $\alpha_3=0,$ the wave function (12) becomes

$$\lim_{\alpha_3\to 0}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}\left(1-2\alpha_3 s\right)=L_n^{\alpha_{10}}\left(\alpha_{11}s\right),\quad \lim_{\alpha_3\to 0}\left(1-\alpha_3 s\right)^{\alpha_{13}}=e^{\alpha_{13}s},$$

$$\psi_{n,\ell}\left(s\right)=N_{n,\ell}s^{\alpha_{12}}e^{\alpha_{13}s}L_n^{\alpha_{10}}\left(\alpha_{11}s\right), \tag{13b}$$

where $L_n(x)$ is a Laguerre polynomial.

3. BOUND STATE SOLUTION.

In order to obtain the solutions of the Dirac equation with potential (1), we employ a suitable approximation scheme. It is noted that such approximation given by Greene-Aldrich[45] and Zhang *et al.*[46]

$$\frac{1}{r^2} \approx \frac{\delta^2}{\left(1-e^{-\delta r}\right)^2}, \tag{14a}$$

is a suitable approximation to the centrifugal term. This approximation is valid for

$\delta << 1$[47, 48]. Thus, the Frost-Musulin potential (1) reduces to

$$V_{FM}(r) = D_e\left[1 - \mathrm{e}^{-\delta(r-r_e)} - \delta r_e\, \mathrm{e}^{-\delta(r-r_e)} + \frac{\delta^2 r_e^2\, \mathrm{e}^{-\delta(r-r_e)}}{1-\mathrm{e}^{-\delta r}}\right]. \tag{14b}$$

3.1 Spin symmetry limit for the Frost-Musulin Diatomic potential.

Under the spin symmetry, $\frac{d\Delta(r)}{dr} = 0$ and $\Delta(r) = C_s$ = constant[1, 49] for $\sum(r) = V_{FM}(r).$ Substituting potential (1) and approximation (14a) into Eq. (8) and by defined a variable of the form $z = e^{-\delta r}$, we have

$$\frac{d^2 F_{n,\mathrm{k}}(z)}{dz^2} + \frac{1-z}{z(1-z)}\frac{dF_{n,\mathrm{k}}(z)}{dz} + \left[\frac{Az^2 + Bz + C}{\left(z(1-z)\right)^2}\right] F_{n,\mathrm{k}}(r) = 0, \tag{15}$$

where

$$A = \frac{\left(M + E_{s,nk} - C_s\right)}{\delta^2}\left[D_e(b\delta r_e(1+r_e) + b - 1) + (M - E_{s,nk})\right], \tag{16a}$$

$$B = -\frac{\left(M + E_{s,nk} - C_s\right)}{\delta^2}\left[D_e(b\delta r_e(2+r_e) + 2b - 2) + 2(M - E_{s,nk})\right], \tag{16b}$$

$$C = \frac{\left(M + E_{s,nk} - C_s\right)}{\delta^2}\left[D_e(br_e + b - 1) + (M - E_{s,nk})\right] - \mathrm{k}(\mathrm{k}+1),,\ b = e^{\delta r_e}. \tag{16c}$$

Comparing Eq. (15) with Eq. (10), we obtain the following for α_i, $i=1,2,3,......,13.$ $\alpha_1=\alpha_2=\alpha_3=1$, $\alpha_4=0$, $\alpha_5=-0.5$, $\alpha_6=0.25-A$, $\alpha_7=-B$, $\alpha_8=-C$, $\alpha_9=\left(0.5-\mathrm{k}\right)^2$, $\alpha_{10}=1-\sqrt{-4C}$, $\alpha_{11}=3-2\left(k+\sqrt{-C}\right)$, $\alpha_{12}=-\sqrt{-C}$ and $\alpha_{13}=-1+k+\sqrt{-C}$. Substituting the values of α_i into Eqs. (11) and (12),we obtain the energy spectrum and the wave function for spin symmetry as

$$\left(M+E_{s,n\mathrm{k}}-C_s\right)\left(M-E_{s,nk}\right)-\left(M+E_{s,n\mathrm{k}}-C_s\right)\left[D_e(1-b\delta^2r_e^2)-bD_e(1+\delta r_e)\right]$$

$$=-\delta^2\left[\frac{\left(\ell+n+1\right)^2+\ell(\ell+1)+\left(M+E_{s,n\mathrm{k}}-C_s\right)bD_er_e^2}{2(\ell+n+1)}\right]^2. \tag{17}$$

$$F(\mathrm{z})=N_nz^{-\sqrt{-C}}\left(1-z\right)^{2-C}P_n^{2(1-k-\sqrt{-C})}(1-2\,\mathrm{z}). \tag{18}$$

3.2 Pseudospin symmetry limit for the Frost-Musulin Diatomic potential.

Under the pseudospin symmetry, $\frac{d\sum(r)}{dr}=0$ or $\sum(r)=C_{ps}=$ constant [1] and $\Delta(r)=$ potential (1). Substitute potential (1) and approximation (14a) into Eq. (9) and using the transformation $z=e^{-\delta r}$, we recast a differential equation of the form

$$\frac{d^2G_{n,\mathrm{k}}(\mathrm{z})}{dz^2}+\frac{1-z}{z(1-z)}\frac{dG_{n,\mathrm{k}}(\mathrm{z})}{dz}+\left[\frac{A_1z^2+B_1z+C_1}{\left(z\left(1-z\right)\right)^2}\right]G_{n,\mathrm{k}}\left(r\right)=0, \tag{19}$$

where

$$A_1=-\frac{\left(M-E_{ps,nk}+C_{ps}\right)}{\delta^2}\left[D_e(b\delta r_e(1+r_e)+b-1)+(M+E_{ps,nk})\right], \tag{20a}$$

$$B_1=\frac{\left(M-E_{ps,nk}+C_{ps}\right)}{\delta^2}\left[D_e(b\delta r_e(2+r_e)+2b-2)+2(M+E_{ps,nk})\right], \tag{20b}$$

$$C_1=-\frac{\left(M-E_{ps,nk}+C_{ps}\right)}{\delta^2}\left[D_e(br_e+b-1)+(M+E_{ps,nk})\right]-k(k-1),,\ b=e^{\delta r_e}. \tag{20c}$$

Using the previous procedure, we obtain energy spectrum for the pseudospin symmetry as follows:

$$\left(M-E_{ps,n\mathrm{k}}+C_{ps}\right)\left(M+E_{ps,nk}\right)+\left(M-E_{ps,n\mathrm{k}}+C_{ps}\right)\left[D_e(1-b\delta^2r_e^2)-bD_e(1+\delta r_e)\right]$$

$$=-\delta^2\left[\frac{\left(\ell+n+1\right)^2+\ell(\ell+1)-\left(M-E_{ps,n\mathrm{k}}+C_{ps}\right)bD_er_e^2}{2(\ell+n+1)}\right]^2. \qquad (21)$$

and the wave function is given as

$$G(\mathrm{z})=N_nz^{-\sqrt{-C_1}}\left(1-z\right)^{2-C_1}P_n^{2(1-k-\sqrt{-C_1})}(1-2\mathrm{z}). \qquad (22)$$

3.3 Non-Relativistic Limit:

Here we obtain the non-relativistic limit of the energy equation of the spin symmetry. To obtain this, we make the following transformation $M-E_{s,nk}\to-E_{n\ell}$, $M+E_{s,n\mathrm{k}}\to=\frac{2\mu}{\hbar^2}$, $C_s=0$ and $k\to\ell$, with these, Eq. (17) turns to

$$E_{n,\ell}=D_e\left[1-b\left(1+\delta r_e(1+\delta r_e)\right)\right]-\frac{\delta^2\hbar^2}{2\mu}\left[\frac{\left(\ell+n+1\right)^2+\ell(\ell+1)+\frac{2\mu bD_er_e^2}{\hbar^2}}{2(\ell+n+1)}\right]^2. \qquad (23)$$

and the wave function is given as

$$R(\mathrm{z})=N_nz^{\eta}\left(1-z\right)^{\varepsilon}{}_2F_1\left(-n,n+2(\eta+\varepsilon);2\eta+1,\mathrm{z}\right). \qquad (24)$$

where, $\eta=\sqrt{\frac{-2\mu E_{n,\ell}}{\delta^2\hbar^2}-2\mu D_e\left(b+b\delta r_e-1\right)+\ell(\ell+)}$ and

$$\varepsilon=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{8\mu bD_er_e}{\delta^2\hbar^2}-8\mu bD_e\left(\delta r_e+1\right)+4\ell(\ell+)}.$$

Table 1: Bound State energy eigenvalues in (eV **) for various** n **and** ℓ **quantum numbers in Frost-Musulin potential with** $\hbar = \mu = 1,\ \delta = 0.01$ **and** $D_e = 15$**.**

		$r_e = 0.4$		$r_e = 0.8$	
n	ℓ	Present	Ref. [50]	Present	Ref. [50]
0	0	-0.1210248969	-0.1210248969	-0.2475915200	-0.2475915200
1	0	-0.1208446620	-0.1208446620	-0.2441172793	-0.2441172793
	1	-0.1209491698	-0.1209491698	-0.2436925359	-0.2436925359
2	0	-0.1208668407	-0.1208668407	-0.2435294569	-0.2435294569
	1	-0.1210084134	-0.1210084134	-0.2435029510	-0.2435029510
	2	-0.1212431116	-0.1212431116	-0.2436806172	-0.2436806172
3	0	-0.1209402283	-0.1209402283	-0.2433893441	-0.2433893441
	1	-0.1211078347	-0.1211078347	-0.2434872003	-0.2434872003
	2	-0.1213627281	-0.1213627281	-0.2437199061	-0.2437199061
	3	-0.1217067487	-0.1217067487	-0.2440599865	-0.2440599865
4	0	-0.1210461962	-0.1210461962	-0.2433964919	-0.2433964919
	1	-0.1212382302	-0.1212382302	-0.2435550332	-0.2435550332
	2	-0.1215144449	-0.1215144449	-0.2438231879	-0.2438231879
	3	-0.1218773350	-0.1218773350	-0.2441887082	-0.2441887082
	4	-0.1223296446	-0.1223296446	-0.2446482043	-0.2446482043
5	0	-0.1211801479	-0.1211801479	-0.2434767635	-0.2434767635
	1	-0.1213964464	-0.1213964464	-0.2436755262	-0.2436755262
	2	-0.1216949462	-0.1216949462	-0.2439722530	-0.2439722530
	3	-0.1220782391	-0.1220782391	-0.2443609100	-0.2443609100
	4	-0.1225490830	-0.1225490830	-0.2448402933	-0.2448402933
	5	-0.1231098295	-0.1231098295	-0.2454108353	-0.2454108353

4. THERMODYNAMIC PROPERTIES AND THE FROST-MUSULIN POTENTIAL.

To study the thermodynamic properties of the Frost-Musulin diatomic molecular potential in which the pure vibrational states are considered, we write the energy equation E_n as

$$E_n = D_e\left[1-b\left(1+\delta r_e(1+\delta r_e)\right)\right]-\frac{\delta^2\hbar^2}{2\mu}\left[\frac{\left(n+1\right)^2+\frac{2\mu b D_e r_e^2}{\hbar^2}}{2(n+1)}\right]^2.$$

$$n=0,1,2,3,....<\left[\sqrt{-\varepsilon}+\sqrt{\xi}\pm\sqrt{\xi}\right], \tag{25}$$

and define the vibrational partition function $Z_{vib}(\beta)$ as [51-55]

$$Z_{vib}(\beta)=\sum_{n=0}^{|\tau|}e^{-\beta E_n}, \tag{26}$$

where $\beta=\left(kT\right)^{-1}$, k is a Boltzman constant and $|\tau|$ is the largest integer inferior to τ the potential depth and it is numerically given as

$$\tau=-\sqrt{\varepsilon}+\sqrt{\xi,} \tag{27}$$

$$\varepsilon=\frac{\delta^2\hbar^2}{2\mu}\left[\frac{\left(n+1\right)^2+\frac{2\mu b D_e r_e^2}{\hbar^2}}{2(n+1)}\right]^2. \text{ and } \xi=D_e\left[1-b\left(1+\delta r_e(1+\delta r_e)\right)\right].$$

Substituting Eq. (25) into Eq. (26), we have

$$Z_{vib}(\beta)=\sum_{n=0}^{\tau}e^{-\frac{\beta\xi\delta^2\hbar^2}{2\mu}+\frac{\beta\delta^2\hbar^2}{8\mu\left(n+\varepsilon\right)^2}+\frac{\beta\delta^2\hbar^2}{8\mu}\left(n+\varepsilon\right)^2}. \tag{28}$$

In the classical limit at high temperature T for large $|\tau|$, the sum can be replaced by an integral and $|\tau|$ can be replaced by τ. Thus, Eq. (28) turns to

$$Z_{vib}(\beta)=\int_0^{\tau}e^{-\frac{\beta\xi\eta}{2\mu}+\frac{\beta\eta}{8\mu\left(n+\varepsilon\right)^2}+\frac{\beta\eta}{8\mu}\left(n+\varepsilon\right)^2}dn, \quad \eta=\delta^2\hbar^2 \tag{29}$$

$$Z_{vib}(\beta)=\int_0^{\tau}e^{-\frac{\beta\xi\eta}{2\mu}+\frac{\beta\eta}{8\mu\lambda^2}+\frac{\beta\eta}{8\mu}\lambda^2}d\lambda, \quad \lambda=n+\varepsilon,$$

$$Z_{vib}(\beta)=e^{-\beta\mathbb{N}_3}\sqrt{\pi}\left[\frac{\left[1-Erf(\mathbb{N}_3\sqrt{\beta})\right]-e^{\mathbb{N}_3\beta}\left[1-Erf(\mathbb{N}_2\sqrt{\beta})\right]}{\mathbb{N}_5\sqrt{\beta}}\right], \tag{30}$$

where,

$$\mathbb{N}_1=\frac{1}{\lambda}\sqrt{-\frac{\eta}{8\mu}}-\lambda\sqrt{-\frac{\eta}{8\mu}},\quad \mathbb{N}_2=\frac{1}{\lambda}\sqrt{-\frac{\eta}{8\mu}}+\lambda\sqrt{-\frac{\eta}{8\mu}},\quad \mathbb{N}_3=-\frac{\eta\xi}{2\mu},\quad \mathbb{N}_4=\frac{\eta}{2\mu},$$

$\mathbb{N}_5=\sqrt{-2\eta},\quad \varepsilon\le\lambda\le(\varepsilon+\tau)$. Eq. (30) gives the vibrational partition function. Having obtained the partition function, it is now straight forward to determine the following:

4.1 The vibrational mean energy U :

$$U(\beta)=-\frac{\partial}{\partial\beta}InZ(\beta)=e^{-\beta\mathbb{N}_1^2}\left\{-\frac{2\sqrt{\beta}\left[\mathbb{N}_1-\mathbb{N}_2e^{\beta\left(\mathbb{N}_1^2-\mathbb{N}_2^2+\mathbb{N}_4\right)}\right]}{2\beta\sqrt{\pi}\left[Erfc\left(\mathbb{N}_1\sqrt{\beta}\right)-e^{\beta\mathbb{N}_4}Erfc\left(\mathbb{N}_2\sqrt{\beta}\right)\right]}\right.$$

$$\left.-\frac{Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)+e^{\beta\mathbb{N}_4}\left[-1+2\beta(\mathbb{N}_3+\mathbb{N}_4)\right]Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)}{2\beta\sqrt{\pi}\left[Erfc\left(\mathbb{N}_1\sqrt{\beta}\right)-e^{\beta\mathbb{N}_4}Erfc\left(\mathbb{N}_2\sqrt{\beta}\right)\right]}\right\}. \tag{31}$$

4,2 The vibrational specific heat C :

$$C(\beta)=-\frac{\partial}{\partial T}U=-k\beta^2\frac{\partial}{\partial T}U\;=ke^{-\beta\left(\mathbb{N}_1^2+\mathbb{N}_2^2\right)}\left\{\frac{e^{-\beta\left(2\mathbb{N}_1^2+\mathbb{N}_2^2\right)}}{2\pi\left[Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]^2}\right.$$

$$+\frac{\sqrt{\pi}Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)\left[\sqrt{\beta}e^{\beta\mathbb{N}_1^2}\left(\mathbb{N}_1+2\beta\mathbb{N}_1^3\right)-e^{\beta\left(\mathbb{N}_1^2+\mathbb{N}_4\right)}\mathbb{N}_2\left(1+2\beta\mathbb{N}_2^2-4\beta\mathbb{N}_4\right)\right]}{2\pi\left[Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]^2}$$

$$+\frac{\left[\sqrt{\pi}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\sqrt{\beta}e^{\beta\left(\mathbb{N}_1^2+\mathbb{N}_4\right)}\left(e^{\beta\left(\mathbb{N}_1^2+\mathbb{N}_4\right)}\left(\mathbb{N}_2+2\beta\mathbb{N}_1^3\right)-e^{\beta\mathbb{N}_2^2}\mathbb{N}_1\left(1+2\beta\mathbb{N}_1^2+4\beta\mathbb{N}_4\right)\right)\right]}{2\pi\left[Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]^2}$$

$$\left.-\frac{\left[\sqrt{\pi}Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)2e^{\beta\left(2\mathbb{N}_1^2+\mathbb{N}_2^2+\mathbb{N}_4\right)}\left(1+\beta^2\mathbb{N}_4^2\right)\sqrt{\pi}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]}{\pi\left[Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]^2}\right\}. \qquad (32)$$

4.3 The vibrational mean free energy F :

$$F(\beta)=-kTInZ_{vib}(\beta)=-\frac{1}{\beta}In\left\{e^{\beta\mathbb{N}_3}\sqrt{\pi}\left[\frac{\left[1-Erfc\left(\mathbb{N}_1\sqrt{\beta}\right)\right]-e^{\beta\mathbb{N}_4}\left[1-Erfc\left(\mathbb{N}_4\sqrt{\beta}\right)\right]}{\mathbb{N}_5\sqrt{\beta}}\right]\right\}. \qquad (33)$$

4.4 The vibrational entropy S :

$$S(\beta)=-kInZ_{vib}(\beta)+kT\frac{\partial}{\partial T}InZ_{vib}(\beta)$$

$$=k\left\{\log\left[\frac{e^{\beta\mathbb{N}_3}\sqrt{\pi}\left(Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right)}{\mathbb{N}_5\sqrt{\beta}}\right]\right.$$

$$+\frac{\sqrt{\pi}\left(\left(1-2\mathbb{N}_3\right)Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)+e^{\beta\mathbb{N}_4}\left(-1+2\beta\left(\mathbb{N}_3+\mathbb{N}_4\right)\right)Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right)}{2\sqrt{\pi}\left[Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]}$$

$$\left.-\frac{e^{-\beta\left(\mathbb{N}_1^2+\mathbb{N}_2^2\right)}\sqrt{\beta}\left(-e^{\beta\mathbb{N}_2^2}\mathbb{N}_1+e^{\beta\left(\mathbb{N}_1^2+\mathbb{N}_4\right)}\mathbb{N}_2\right)}{\sqrt{\pi}\left[Erfc\left(\sqrt{\beta}\mathbb{N}_1\right)-e^{\beta\mathbb{N}_4}Erfc\left(\sqrt{\beta}\mathbb{N}_2\right)\right]}\right\}. \qquad (34)$$

5. CONCLUSION.

In this study, we obtained approximate solutions of the Dirac equation with the Frost-Musulin potential via parametric Nikiforov-Uvarov method by employing a suitable approximation type. In other to test the accuracy of our results, we obtained the non-relativistic limit of the spin symmetry and obtained the numerical result. It is seen from the table that upto ten significant figures, our numerical results are equal in magnitude with the numerical results obtained using Function analysis method by Adepoju and Eweh. We have equally calculated the thermodynamic properties such as mean energy, heat capacity, entropy and free mean energy.

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