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Research Article

Dirac Equation with Unequal Scalar and Vector Potentials under Spin and Pseudospin Symmetry

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Abstract

An approximate solution of the Dirac equation in the *D*-dimensional space is obtained under spin and pseudospin symmetry limits for the scalar and vector inversely quadratic Yukawa potential within the framework of parametric Nikiforov-Uvarov method using a suitable approximation scheme to the spin-orbit centrifugal term. The two components spinor of the wave function and their energy equations are fully obtained. Some numerical results are obtained for the energy level with various dimensions (*D*), quantum number (*n*), vector potential V_0 and scalar potential S_0. The results obtained under spin symmetry using either V_0 or S_0 are equal to the results obtained using $V_0 + S_0$. But under the pseudospin symmetry, the results obtained using V_0 or S_0 are not equal to the results obtained using $V_0 - S_0$.

Keywords: Dirac equation; Spin symmetry, Pseudospin symmetry, Parametric Nikiforov-Uvarov method.

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1. INTRODUCTION

Dirac equation is a relativistic wave equation derived in particle Physics by Paul Dirac in 1928 which described the spin -1/2 massive particles such as electrons and quarks[1,2]. This equation is consistent with the principles of quantum mechanics and the theory of special relativity. It was the first theory to completely describe the special relativity in quantum mechanics[3]. In the framework of the Dirac equation, the relativistic symmetries (spin and pseudospin) of the Dirac Hamiltonian has been discovered which have been recently recognized empirically in spherical atomic nuclei and hadronic spectroscopic[4-6]. The pseudospin symmetry was used to explain some certain features such as deformed nuclei[7], superdeformation and effective shell coupling scheme [8,9]. The spin symmetry is relevant to meson[10]. The pseudospin symmetry refers to a quasi-degeneracy of single nucleon doublets with non-relativistic quantum number $(n, \ell, j = \ell + 1/2)$ and $(n-1, \ell+2, j = \ell + 3/2)$, where n denote the single nucleon radial number, ℓ is the orbital angular quantum number and j is the total angular quantum number[4,5]. The total angular quantum number $j = \overline{\ell} + \overline{s}$, where $\overline{\ell} = \ell + 1$ is pseudo-angular momentum and $\overline{s}$ is pseudospin angular momentum[11]. The pseudospin symmetry occurs when the sum of the potential for the repulsive Lorentz vector potential $V(r)$ and the attractive Lorentz scalar potential $S(r)$ is a constant, that is $\sum(r) = V(r) + S(r) =$ constant while the spin symmetry occurs as the difference of the potential between the repulsive Lorentz vector potential $V(r)$ and the attractive Lorentz scalar potential $S(r)$ is a constant, that is $\Delta(r) = V(r) - S(r) =$ constant[12-14]. In line with the importance and usefulness of the Dirac equation, a great number of studies have been recently devoted to obtain the analytic solutions of the relativistic Dirac equation with the well-known potential models in the framework of the spin and pseudospin symmetry limits in the presence or absence of tensor potential. For instance, Maghsoodi *et al.*[15], studied Dirac particles in the presence of the Yukawa potential plus a tensor interaction in SUSY QM framework. Oyewumi *et al.*[16], investigated k states solutions for the fermionic massive spin-1/2 particles interacting with double ring-shaped Kratzer and oscillator potentials. Onate *et al.*[17], obtained approximate solutions of the Dirac equation for Second Pöschl-Teler like scalar and vector potentials with a Coulomb tensor interaction. Eshghi *et al.*[18-20], studied relativistic Killingbeck energy states under external magnetic fields, bound states of (1+1)-dimensional Dirac equation with kink-like vector potential and delta interaction and the relativistic bound states of a non-central potential.

Hosseinpour and Hassanabadi[21], studied scattering states of Dirac equation in the presence of cosmic string for Coulomb interaction. Ikot *et al.*[22], obtained bound state solutions of the Dirac equation for Eckart potential with Coulomb-like Yukawa-like tensor interactions. Ikot, Maghsoodi, Ibanga, Ituen and Hassanabadi[23], obtained bound states of the Dirac equation for Modified Mobius square potential within the Yukawa-like tensor interaction. Ikot, Zarrinkamar, Zare and Hassanabadi[24], investigated relativistic Dirac attractive radial problem with Yukawa-like tensor interaction via SUSY QM. Onate and Onyeaju[25], studied Dirac particles in the field of Frost-Musulin diatomic potential and the thermodynamic properties via parametric Nikiforov-Uvarov method. Onate and Ojonubah[26], investigated Dirac equation in the presence of the interaction of hyperbolic and generalized Pöschl-Teller like potential model in the framework of spin and pseudospin symmetries. Hassanabadi *et al.*[27], studied Actual and General Manning-Rosen potential under spin and pseudospin symmetries of the Dirac equation. Dong and Ma[28], also studied exact solutions of the Dirac equation with a Coulomb potential in 2+1 dimensions. Soylu *et al.*[29], in their own investigation, obtained k states solutions of the Dirac equation for the Eckart potential with spin and pseudospin symmetry. Ikhdair and Sever[30], deduced two approximation schemes to the bound states of the Dirac-Hulthén problem. Bayrak and Boztosun[31], studied the pseudospin symmetric solution of the Morse potential for any k states. Hamzavi *et al.*[32], investigated exactly complete solutions of the Dirac equation with pseudoharmonic potential including linear plus Coulomb-like tensor potential, Suparmi *et al.*[33,34], studied Dirac equation for scarf and hyperbolic tangent potentials respectivelly, Cari *et al.*[35], obtained the solutions of Dirac equation for Cotangent potential with a new tensor potential. In all these studies and investigations, the authors considered only the case where the vector potential and scalar potential are equal under spin and pseudospin symmetry respectively. In this study, we considered the sum potential under spin symmetry as $\sum(r) = V(r) + S(r)$ and the difference potential under pseudospin symmetry as $\Delta(r) = V(r) - S(r),$ where $V(r)$ and $S(r)$ are the interacting potentials. This study considered the inversely quadratic Yukawa potential as the interacting potential in the arbitrary dimensions. The inversely quadratic Yukawa potential has not received much attention so far. This potential takes the form[36-38]

$$V_{IQY}(r) = -\frac{\lambda e^{-\delta r}}{r^2}, \tag{1}$$

where λ is the depth/strength of the potential, $\delta = 2\alpha$ and α is the screening parameter.

2. METHODOLOGY

2.1 Dirac Equation

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential and repulsive vector potential is given as

$$\left[\vec{\alpha}.\vec{p} + \beta(M + S(r)) - (E - V(\mathrm{r}))\right]\psi(\vec{r}) = 0, \tag{2}$$

where E is the relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three-dimensional momentum operator and M is the mass of the particle, $\vec{\alpha}$ and $\vec{\beta}$ are the 4×4 usual Dirac matrices. The spinor wave functions can be classified according to their angular momentum j, the spin-orbit quantum number k, and the radial quantum number n as follows

$$\psi_{nk}(\vec{r}) = \begin{pmatrix} f_{nk}(\vec{r}) \\ g_{nk}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{nk}(r)Y^{\ell}_{jm}(\theta,\varphi) \\ iG_{nk}(r)Y^{\bar{\ell}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{3}$$

where $f_{nk}(\vec{r})$ is the upper component and $g_{nk}(\vec{r})$ is the lower component of the Dirac spinors. $Y^{\ell}_{jm}(\theta,\varphi)$ and $Y^{\bar{\ell}}_{jm}(\theta,\varphi)$ are spin and pseudospin spherical harmonics respectively, and m is the projection of the angular momentum on the $z-$axis. Simplifying equation (3) further gives the two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{nk}(r)$ and $G_{nk}(r)$ as

$$\left(\frac{d}{dr} + \frac{k}{r}\right)F_{nk}(r) = (M + E_{nk} - V(r) + S(r))G_{nk}(r), \tag{4}$$

$$\left(\frac{d}{dr} - \frac{k}{r}\right)G_{nk}(r) = (M - E_{nk} + V(r) + S(r))F_{nk}(r), \tag{5}$$

Eliminating $G_{nk}(r)$ and $F_{nk}(r)$ from equations (4) and (5) respectively, we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components as

$$\left(\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}+\frac{\frac{d\Delta(r)}{dr}}{M+E_{nk}-\Delta(r)}\left(\frac{d}{dr}+\frac{k}{r}\right)-(M+E_{nk}-\Delta(r))(M-E_{nk}+\Sigma(r))\right)F_{nk}(r)=0, \quad (6)$$

$$\left(\frac{d^2}{dr^2}-\frac{k(k-1)}{r^2}\frac{\frac{d\Sigma(r)}{dr}}{M-E_{nk}+\Sigma(r)}\left(\frac{d}{dr}-\frac{k}{r}\right)-(M+E_{nk}-\Delta(r))(M-E_{nk}+\Sigma(r))\right)G_{nk}(r)=0. \quad (7)$$

2.2 Parametric Nikiforov-Uvarov Method.

To solve a second-order differential Schrödinger-like equation using parametric Nikiforov-Uvarov method, we first consider the differential equation of the form[39-43]

$$\left(\frac{d^2}{dr^2}+\frac{\alpha_1-\alpha_2}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\right)\psi(s)=0. \quad (8)$$

According to the parametric Nikiforv-Uvarov method, the condition for eigen energies and eigen functions respectively are[40, 44-47]

$$n\alpha_2-(2n+1)\alpha_5+\alpha_7+2\alpha_3\alpha_8+n(n-1)\alpha_3+(2n+1)\sqrt{\alpha_9}+\left(2\sqrt{\alpha_9}+\alpha_3(2n+1)\right)\sqrt{\alpha_8}=0, \quad (9)$$

$$\psi_{n,\ell}(s)=N_{n,\ell}s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \quad (10)$$

where the parametric constants in equations (9) and (10) above are defined in Appendix 1.

3. SOLUTIONS OF DIRAC EQUATION

3.1 The Spin Symmetry Limit.

Under the spin symmetry limit, $\frac{d\Delta(r)}{dr}=0$ and $\Delta(r)=C_s$, [48, 49]. The sum potential in this case is the summation of the scalar potential and the vector potential. Thus, the interacting potential becomes

$$\Sigma(r)=V(r)+S(r)=-\frac{(V_0+S_0)e^{-\delta r}}{r^2}. \quad (11)$$

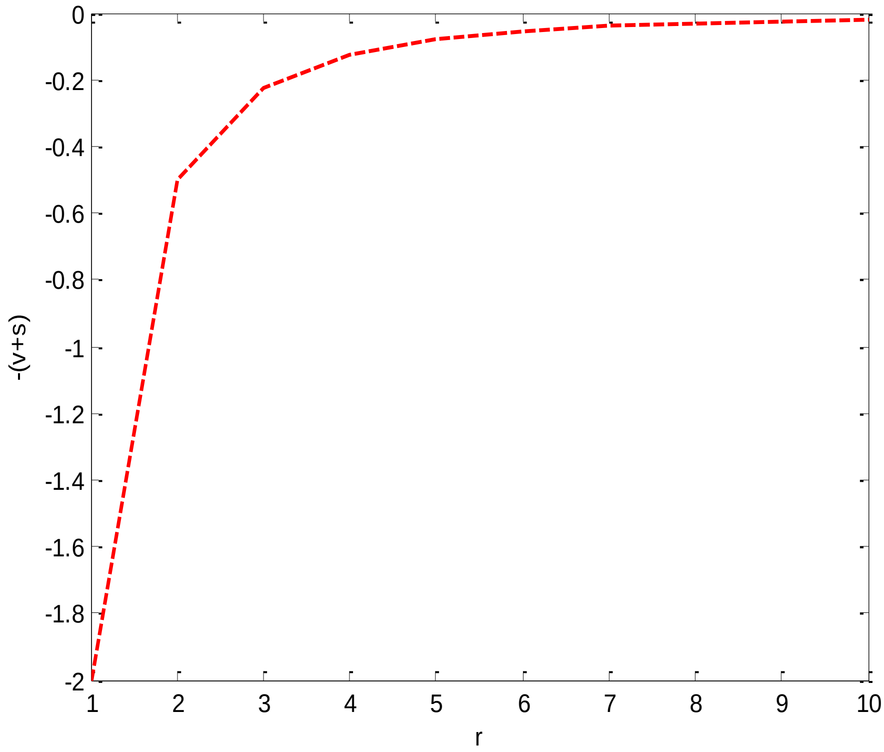

Figure 1: The sum potential $\left(-V_0 + -S_0 = -\left(V_0 + S_0\right)\right)$ with $V_0 = 2$ and $S_0 = 1$.

It is noted that equation (6) is a second-order differential equation containing a spin-orbit centrifugal term $k\left(k+1\right)/r^2$ which has strong singularity at $r = 0$. Thus, this needs a careful treatment while performing the approximation. Taking into account the spin-orbit centrifugal term, the radial Dirac equation can no longer be solved in a closed form for the exact solutions thus, it becomes obvious to resort to the approximate solutions. Different approximation schemes have been applied to the spin-orbit centrifugal term over the years. The choice of approximation applied depends on the interacting potential. Considering the potential given in equation (11) which also has a strong singularity at $r = 0$, we resort to use the following approximation scheme for a short potential range[50-53] that is a good approximation to the centrifugal term

$$\frac{1}{r^2} = \frac{\delta^2}{\left(1 - e^{-\delta r}\right)^2}. \tag{12}$$

which is valid for $\delta << 1$

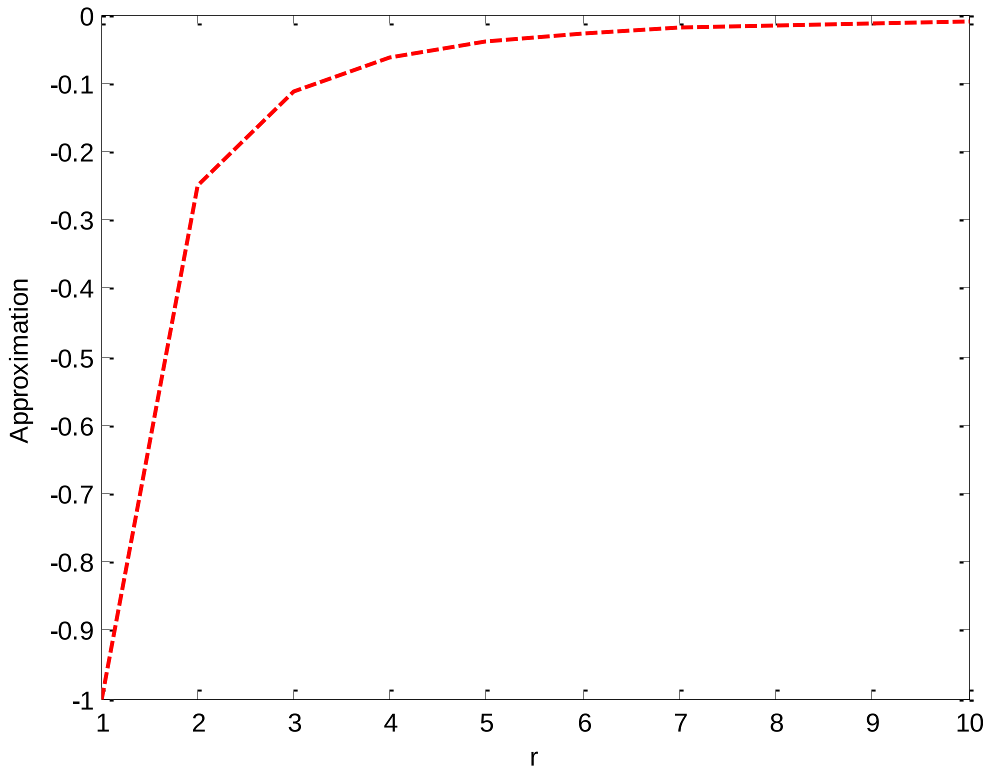

Figure 2: Potential (1) in the presence of approximation scheme given in equation (12)

Substituting equations (11) and (12) into equation (6) and by change of variable of the form $y = e^{-\delta r}$ for the spatial dimensional space, we decompose the spin symmetric Dirac equation (6) into the Schrödinger-type equation satisfying the upper-spinor component in the following form:

$$\left(\frac{d^2}{dy^2} + \frac{1-y}{y(1-y)}\frac{d}{dy} + \frac{\Re_{s1}y^2 + \Re_{s2}y + \Re_{s3}}{y^2(1-y)^2}\right)F_{nk}(y) = 0, \qquad (13)$$

where $F_{nk}(r) \equiv F_{nk}(y)$ and the following are used for mathematical simplicity

$$\Re_{s1} = (E-M)\beta_s, \qquad (14a)$$

$$\Re_{s2} = \left(V_0 + S_0 - 2(E-M)\right)\beta_s \quad , \qquad (14b)$$

$$\Re_{s3} = (E-M)\beta_s - \frac{\eta_s(\eta_s - 2)}{2},$$

$$\beta_s = M + E - C_s, \tag{14c}$$

$$\eta_s = D + 2k - 1. \tag{14d}$$

Comparing equation (13) with equation (8), we deduce the values of the parametric constants as shown in Appendix 2. Substituting the values of the parametric constants in Appendix 2 into equations (9) and (10), we have the energy equation for the spin symmetry as follows

$$M^2 - E_{nks}^2 + C_s(E_{nks} - M) + \eta_s(\eta_s - 2)\delta^2 = \left[\frac{\delta\left(n+\frac{1}{2}\right)\sqrt{1+2\eta_s(\eta_s-2)-4(V_0+S_0)\beta_s} + \aleph}{2n+1+\sqrt{1+2\eta_s(\eta_s-2)-4(V_0+S_0)\beta_s}}\right]^2, \tag{15}$$

and the upper component of the wave function for the spatial dimensions as

$$F_{n,\mathrm{k}}(y) = N_{n,\mathrm{k}} y^{a_s} (1-y)^{\frac{1}{2}(1+b_s)} P_n^{(2a_s,\mathrm{b}_s)}(1-2y), \tag{16}$$

where

$$a_s = \sqrt{(M-E)\beta_s + \frac{\eta_s(\eta_s-2)}{2}}, \tag{17}$$

$$b_s = \sqrt{1+2\eta_s(\eta_s-2)-4(V_0+S_0)\beta_s}, \tag{18}$$

$$\aleph = \frac{1}{2}\delta\left[2n(n+1) + 2\eta_s(\eta_s-2) - 2(V_0+S_0)\beta_s + 1\right], \tag{19}$$

and $P_n^{(2\mathrm{a},\mathrm{b})}$ are the orthogonal Jacobi polynomials. The lower component of the wave function is given as

$$G_{nk}(r) = \frac{\left(\frac{d}{dr} + \frac{k}{r} - U(r)\right)}{M + E_{nk} - C_s} F_{nk}(r). \tag{19a}$$

However, when $C_s = 0,$ the lower spinor exist if $M \neq -E$ which gives positive energy eigenvalues for the spin symmetry. Recall that $y = e^{-\delta r}$, as $r \to \infty, e^{-\delta r} \to 0$ and then $y \to 0.$ This reduces the wave function $F_{n,ks}(y) = 0.$ Similarly, when $r \to 0, e^{-\delta r} \to 1$ and then $y \to 1$ which makes $1 - y = 0,$ and thus, $F_{n,ks}(y) = 0.$

3.2 The Pseudospin Symmetry Limit.

Under the pseudospin symmetry limit, $\frac{d\Sigma(r)}{dr}=0$ and $\Sigma(r)=C_{ps}$, [48,49]. The sum potential for a case when the scalar potential is not equal to zero, the interacting potential becomes

$$\Delta(r)=V(r)-S(r)=-\frac{\left(-S_0+V_0\right)e^{-\delta r}}{r^2}. \tag{20}$$

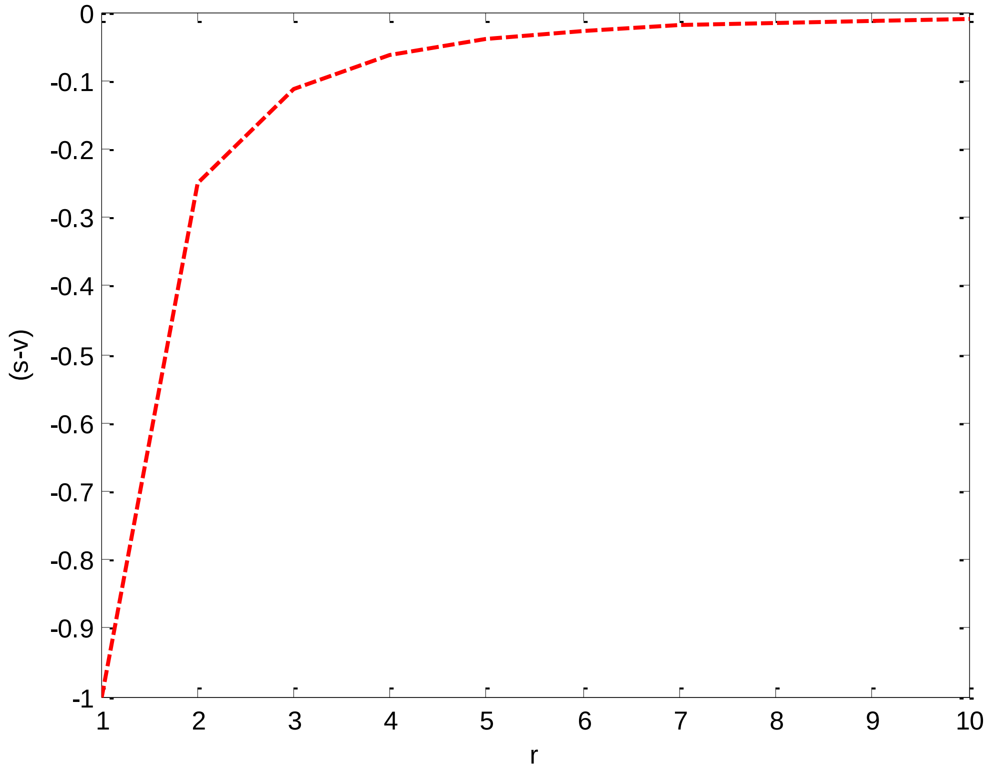

Figure 3: The difference potential $\left(-V_0--S_0=S_0-V_0\right)$ with $V_0=2$ and $S_0=1$.

Substituting equations (20) and (12) into equation (7) and using the change of variable as in the spin symmetry, we have

$$\left(\frac{d^2}{dy^2}+\frac{1-y}{y(1-y)}\frac{d}{dy}+\frac{-\Re_1 y^2+\Re_2 y-\Re_3}{y^2(1-y)^2}\right)G_{nk}(y)=0, \tag{21}$$

where we used the following for mathematical simplicity

$$\Re_1 = (E+M)\beta_{ps}, \tag{21a}$$

$$\Re_2 = \left(S_0 - V_0 + 2(E+M)\right)\beta_{ps}, \tag{21b}$$

$$\Re_3 = (E+M)\beta_{ps} - \frac{\eta_{ps}(\eta_{ps}-2)}{2}, \tag{21c}$$

$$\beta_{ps} = M - E + C_{ps}, \tag{22}$$

$$\eta_{ps} = D - 2k - 1. \tag{23}$$

Using the same procedures as in the spin symmetry, we easily obtain the negative component energy of the Dirac equation and its wave function in the form

$$M^2 - E_{nkps}^2 + C_s(E_{nkps}+M) + \eta_{ps}(\eta_{ps}-2)\delta^2 = \left[\frac{\delta\left(n+\frac{1}{2}\right)\sqrt{1+2\eta_{ps}(\eta_{ps}-2)+4(V_0-S_0)\beta_{ps}} + \aleph_0}{2n+1+\sqrt{1+2\eta_{ps}(\eta_{ps}-2)+4(V_0-S_0)\beta_{ps}}}\right]^2, \tag{24}$$

$$G_{n,\mathrm{k}}(y) = N_{n,\mathrm{k}} y^{a_{ps}} (1-y)^{\frac{1}{2}(1+b_{ps})} P_n^{(2a_{ps},\mathrm{b}_{ps})}(1-2y), \tag{25}$$

where

$$a_{ps} = \sqrt{(M+E)\beta_{ps} + \frac{\eta_{ps}(\eta_{ps}-2)}{2}}, \tag{26}$$

$$b_{ps} = \sqrt{1 + 2\eta_{ps}(\eta_{ps}-2) + 4(V_0 - S_0)\beta_{ps}}, \tag{27}$$

$$\aleph_0 = \frac{1}{2}\delta\left[2n(n+1) + 2\eta_{ps}(\eta_{ps}-2) + 2(V_0 - S_0)\beta_s + 1\right]. \tag{28}$$

The upper component of the wave function is given as

$$F_{nk}(r) = \frac{\left(\frac{d}{dr} - \frac{k}{r} + U(r)\right)}{M - E_{nk} + C_{ps}} G_{nk}(r). \tag{28a}$$

However, when $C_{ps} = 0,$ the lower spinor exist if $M \neq E$ which gives positive energy eigenvalues. When $r \to \infty, e^{-\delta r} \to 0$ and then $y \to 0.$ This reduces the wave function

$G_{n,kps}(y)=0.$ Similarly, when $r \to 0,\ e^{-\delta r} \to 1$ and then $y \to 1$ which makes $1-y=0,$ and thus, $G_{n,kps}(y)=0.$

Table 1: Energy for spin (E_{nks}) with $\delta=0.1,\ M=1\,fm^{-1}$ and $C_s=5\,fm^{-1}$.

D	n	$V_0=5,\ S_0=1.$	$V_0=1,\ S_0=5.$	$V_0=S_0=1.$	$V_0=S_0=0.$	$V_0=1,\ S_0=0.$	$V_0=0,\ S_0=1.$
1	1	3.99918259	3.99918259	3.99916365	3.99666296	3.99812031	3.99812031
	2	3.99915520	3.99915520	3.99542468	3.99248116	3.99404179	3.99404179
	3	3.99463590	3.99463590	3.98965271	3.98660688	3.98817539	3.98817539
	4	3.98745301	3.98745301	3.98202543	3.97901995	3.98054584	3.98054584
	5	3.97803099	3.97803099	3.97255368	3.96969385	3.97113124	3.97113124
2	1	4.00874204	4.00874204	4.00416264	4.00062405	4.00249792	4.00249792
	2	4.00418674	4.00418674	3.99863764	3.99518533	3.99696864	3.99696864
	3	3.99736383	3.99736383	3.99139212	3.98798785	3.98972540	3.98972540
	4	3.98855742	3.98855742	3.98239819	3.97904341	3.98074411	3.98074411
	5	3.97782258	3.97782258	3.97161850	3.96833242	3.96999100	3.96999100
3	1	4.01904575	4.01904575	4.01243534	4.00835504	4.01046103	4.01046103
	2	4.01247232	4.01247232	4.00582203	4.00201805	4.00396080	4.00396080
	3	4.00412771	4.00412771	3.99744530	3.99378031	3.99564040	3.99564040
	4	3.99399426	3.99399426	3.98730097	3.98372369	3.98553200	3.98553200
	5	3.98203217	3.98203217	3.97535992	3.97185520	3.97362194	3.97362194
4	1	4.03166837	4.03166837	4.02392782	4.01952561	4.02177231	4.02177231
	2	4.02394949	4.02394949	4.01655713	4.01250278	4.01456061	4.01456061
	3	4.01448952	4.01448952	4.00727055	4.00340307	4.00538880	4.00538880
	4	4.00325777	4.00325777	3.99613833	3.99238470	3.99427810	3.99427810
	5	3.99020973	3.99020973	3.98316361	3.97949112	3.98134015	3.98134015
5	1	4.04696105	4.04696105	4.03856755	4.03398175	4.03630818	4.03630818
	2	4.03854161	4.03854161	4.03065538	4.02643375	4.02856849	4.02856849
	3	4.02824931	4.02824931	4.02064038	4.01662555	4.01865101	4.01865101
	4	4.01611764	4.01611764	4.00867166	4.00478404	4.00674196	4.00674196
	5	4.00213248	4.00213248	3.99479302	3.99099147	3.99290353	3.99290353
6	1	4.06501395	4.06501395	4.05626575	4.05159426	4.05395572	4.05395572
	2	4.05615866	4.05615866	4.04796639	4.04364742	4.04582609	4.04582609
	3	4.04524637	4.04524637	4.03737524	4.03326505	4.03533514	4.03533514
	4	4.03238796	4.03238796	4.02471031	4.02073038	4.02273245	4.02273245
	5	4.01761103	4.01761103	4.01005623	4.00616240	4.00811929	4.00811929

Table 2: Energy for pseudospin $(-E_{nkps})$ with $\delta = 0.1,\ M = 1 fm^{-1}$ and $C_s = -5 fm^{-1}$.

D	n	$V_0 = 5, S_0 = 1.$	$V_0 = 1, S_0 = 5.$	$V_0 = S_0 = 1.$	$V_0 = S_0 = 0.$	$V_0 = 1, S_0 = 0.$	$V_0 = 0, S_0 = 1.$
1	1	4.01599618	3.99854676	4.00835504	4.00835504	4.00611469	4.01046103
	2	4.00930463	3.99339634	4.00261805	4.00261805	3.99999230	4.00396079
	3	4.00089275	3.98576580	3.99378031	3.99378031	3.99186406	3.99564039
	4	3.99072321	3.97608199	3.98372369	3.98372369	3.98187536	3.98553200
	5	3.97875129	3.96449043	3.97185520	3.97185520	3.97005923	3.97362194
2	1	4.00686952	3.99093280	4.00062405	4.00062405	3.99853764	4.00249792
	2	4.00163701	3.98686016	3.99518533	3.99518533	3.99328587	3.99696864
	3	3.99451618	3.98030152	3.98798785	3.98798785	3.98617834	3.98972540
	4	3.98556854	3.97175799	3.97904341	3.97904341	3.97729537	3.98074411
	5	3.97478134	3.96137947	3.96833242	3.96833242	3.96664241	3.96999100
3	1	4.00000000	3.98640627	3.99666296	3.99666296	3.99478808	3.99812031
	2	3.99765466	3.98438026	3.99248116	3.99248116	3.99074108	3.99404179
	3	3.99233185	3.97940965	3.98660688	3.98660688	3.98494702	3.98817539
	4	3.98483989	3.97250737	3.97901995	3.97901995	3.97744965	3.98054584
	5	3.97533955	3.96395964	3.96969385	3.96969385	3.96824682	3.97113124
4	1	1.00541973	3.98592734	3.99576116	3.99576116	1.00801152	1.00584687
	2	1.01160620	1.02077901	3.99245287	3.99245287	1.01203405	1.01125303
	3	1.02066597	1.02934904	3.98743424	3.98743424	1.01710960	1.01873676
	4	1.03208880	1.03766194	3.98070411	3.98070411	1.02377879	1.02810091
	5	1.04565748	1.04653980	3.97224374	3.97224374	1.03217532	1.03929772
5	1	4.00000000	3.98640627	3.99666296	3.99666296	3.99478808	3.99812031
	2	3.99765466	3.98438026	3.99248116	3.99248116	3.99074108	3.99404179
	3	3.99233185	3.97940965	3.98660688	3.98660688	3.98494702	3.98817539
	4	3.98483989	3.97250737	3.97901995	3.97901995	3.97744965	3.98054584
	5	3.97533955	3.96395964	3.96969385	3.96969385	3.96824682	3.97113124
6	1	4.00686952	3.99093280	4.00062405	3.99666296	3.99853764	4.00249792
	2	4.00163701	3.98686016	3.99518533	3.99248116	3.99328587	3.99696864
	3	3.99451618	3.98030152	3.98798785	3.98660688	3.98617834	3.98972540
	4	3.98556854	3.97175799	3.97904341	3.97901995	3.97729537	3.98074411
	5	3.97478134	3.96137947	3.96833242	3.96969385	3.96664241	3.96999100

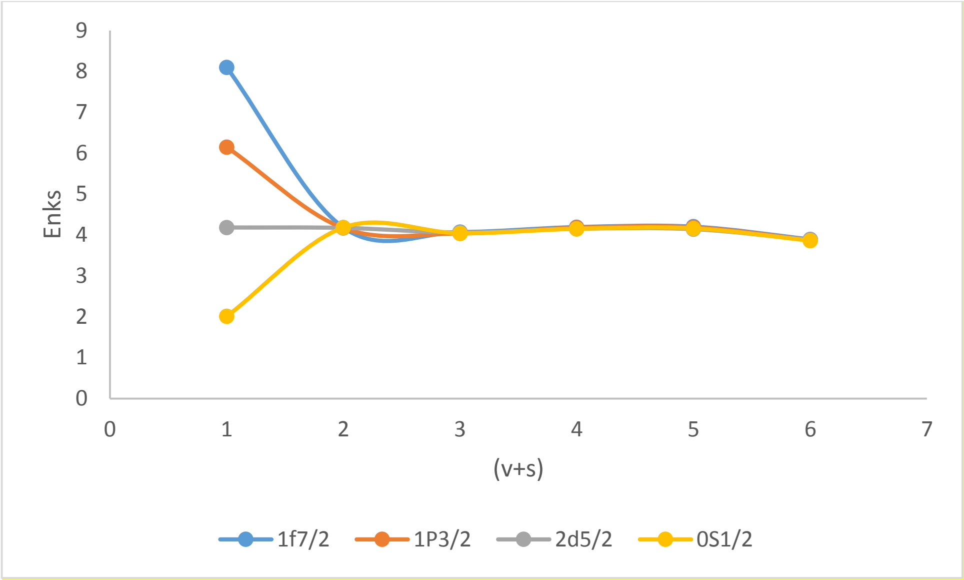

Figure 4: Energy of the spin symmetry against the sum of the potential depth for $0s_{1/2}$, $2d_{5/2}$, $1p_{3/2}$ and $1f_{7/2}$.

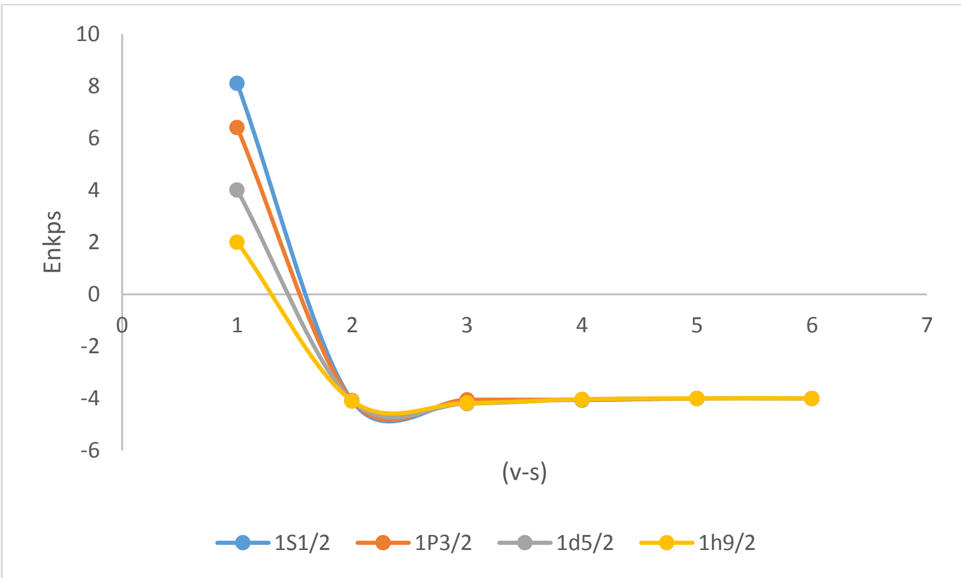

Figure 5: Energy of the pseudospin symmetry against the sum of the potential depth for $1h_{9/2}$, $1d_{5/2}$, $1p_{3/2}$ and $1s_{1/2}$.

4. RESULTS AND DISCUSSION

The energy eigenvalues equations obtained for spin symmetry and pseudospin symmetry in equations (16) and (24) respectively are quadratic algebraic equations according to E_{nk}. It therefore becomes necessary to obtain the solutions of these algebraic equations with respect to energy by choosing some numerical values for n and k.

In Table 1, we obtained the energy eigenvalues for spin symmetry limit with $C_s = 5\,fm^{-1}$, $M = 1\,fm^{-1}$, $\delta = 0.1$ for $D = 1$, $V_0 = 5$, $S_0 = 1$ and $n = 1, 2, 3, 4, 5$. We repeated this for $D = 2, 3, 4, 5$ and 6 with $V_0 = 1$, $S_0 = 5$; $V_0 = S_0 = 1$; $V_0 = S_0 = 0$; $V_0 = 1$, $S_0 = 0$ and $V_0 = 0$, $S_0 = 1$. It is readily observed from the table that the energy obtained with $V_0 = 5$ and $S_0 = 1$ are equal to the energy obtained with $V_0 = 1$ and $S_0 = 5$. This is due to the fact that $V_0 + S_0 = S_0 + V_0$ since in the energy equation for the spin symmetry we have $-(V_0 + S_0)$. This trend is also observed when $V_0 = 1$, $S_0 = 0$ and when $V_0 = 0$, $S_0 = 1$. It is also observed that the energy obtained decreases as n increases. Similarly, as D increases, the energy eigenvalue increases. This feature is also seen as the value of $V_0 + S_0$ increases. In Table 2, we obtained the energy eigenvalues for the pseudospin symmetry limit with $C_s = -5\,fm^{-1}$, $M = 1\,fm^{-1}$, $\delta = 0.1$ for $D = 1$, $V_0 = 5$, $S_0 = 1$ and $n = 1, 2, 3, 4, 5$. We repeated this for $D = 2, 3, 4, 5$ and 6 with $V_0 = 1$, $S_0 = 5$; $V_0 = S_0 = 1$; $V_0 = S_0 = 0$; $V_0 = 1$, $S_0 = 0$ and $V_0 = 0$, $S_0 = 1$. The same process in the spin symmetry was repeated. In this symmetry, as n increases, the energy obtained equally increases. This same feature is also observed as D increases from 1 to 3. It is readily observed that the energy obtained with $V_0 = S_0 = 1$ and that obtained with $V_0 = S_0 = 0$ are equal. This is due to the fact that $V_0 - S_0 = 0$ since in the energy equation for the pseudospin symmetry we have $(V_0 - S_0)$. However, the energies obtained with $D = 2$ are equal to the energies obtained with $D = 6$. Similarly, the energies obtained with $D = 3$ are equal to the energies obtained with $D = 5$. In Figures 1-3, we graphically described the sum potential, approximation scheme and difference potential respectively. It can be seen that the three figures are similar. As it can be seen from the

Figures, no matter the value of r, the value of $-(V+S)$ or $S-V$ cannot exceed 0. This is also justified by equations (11), (12) and (20). In Figures 4 and 5, we plotted energy in spin symmetry and pseudospin symmetry respectively against the potential depth for some states. Figure 4 shows that the energy for the spin symmetry are purely positive while the revise is seen in Figure 5 for the pseudospin symmetry.

5. CONCLUSION

In this study, we obtained the bound state solutions of the *D*-dimensional Dirac equation with spin and pseudospin symmetry in the presence of both the scalar and vector inversely quadratic Yukawa potential. The two spinor components of Dirac equation and their corresponding energy equation are obtained in a closed and compact form in view of the parametric Nikiforov-Uvarov method. Some numerical results are obtained for each of the symmetry. The energies increase as both *D* and n increases. The results obtained using either vector potential or scalar potential are equivalent to the results obtained by using the sum of the vector and scalar potentials under the spin symmetry. But under the pseudospin symmetry, the results obtained using either vector potential or scalar potential are not equivalent to the results obtained using the difference between the vector and scalar potentials.

APPENDIX 1. Parametric constants.

$$\alpha_4=\frac{1-\alpha_1}{2},\ \alpha_5=\frac{\alpha_2-2\alpha_3}{2},\ \alpha_6=\alpha_5^2+\xi_1,\ \alpha_7=2\alpha_4\alpha_5-\xi_2,\ \alpha_8=\alpha_4^2+\xi_3,$$

$$\alpha_9=\alpha_3\left(\alpha_7+\alpha_3\alpha_8\right)+\alpha_6,\ \ \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\ \ \alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right),$$

$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\ \ \alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right).$$

APPENDIX 2. **Values of the parametric constants**.

$$\alpha_1=\alpha_2=\alpha_3=1,\ \ \alpha_4=0,\ \ \alpha_5=-\frac{1}{2},\ \ \alpha_6=\frac{1}{4}+(M-E)\beta_s,\ \ \alpha_7=\left[2(E-M)-(V_0+S_0)\right]\beta_s,$$

$$\alpha_8 = (M-E)\beta_s + \frac{\eta_s(\eta_s-2)}{2},\ \alpha_9 = \frac{1}{4} + \frac{\eta_s(\eta_s-2)}{2} - (V_0+S_0)\beta_s,$$

$$\alpha_{10} = 1 + 2\sqrt{(M-E)\beta_s + \frac{\eta_s(\eta_s-2)}{2}},$$

$$\alpha_{11} = 2\left(1 + \frac{1}{2}\sqrt{1+2\eta_s(\eta_s-2)-4(V_0+S_0)\beta_s} + \sqrt{(M-E)\beta_s + \frac{\eta_s(\eta_s-2)}{2}}\right),$$

$$\alpha_{12} = \sqrt{(M-E)\beta_s + \frac{\eta_s(\eta_s-2)}{2}},$$

$$\alpha_{13} = -\frac{1}{2}\left(1 + \sqrt{1+2\eta_s(\eta_s-2)-4(V_0+S_0)\beta_s} + 2\sqrt{(M-E)\beta_s + \frac{\eta_s(\eta_s-2)}{2}}\right).$$

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