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Research Article

Approximate Analytical Solutions of the Effective Mass Klein-Gordon Equation for Two Interacting Potentials.

Osarodion Ebomwonyi[1], Atachegbe Clement Onate[2*], Michael C. Onyeaju[3], Joshua Okoro[2], Matthew Oluwayemi[2]

[1]*Department of Physics, University of Benin, PMB 1154, Benin City, Nigeria.*

[2]*Department of Physical Sciences, Landmark University, PMB 1001, Omu-Aran, Nigeria.*

[3]*Theoretical Physics Group, Physics Department, PMB 5323 Choba, University of Port Harcourt, Nigeria*

Abstract

In this study, the approximate analytical solutions of the relativistic Klein-Gordon equation in the spatial dimensions with unequal Coulomb-inverse Trigonometry scarf scalar and vector potentials for an effective mass function is investigated in the framework of supersymmetric and shape invariance method by employing a suitable approximation scheme to the centrifugal term. The energy equation for some special cases such as the Coulomb potential and inverse Trigonometry scarf potential are obtained. Using a certain transformation, the non-relativistic energy equation is obtained which is identical to the energy equation of the Hellmann potential.

Keywords: Supersymmetric method; Klein-Gordon equation; Effective mass; wave function.

* Corresponding Author Email: oaclems14@physicist.net

https://orcid.org/0000-0002-9909-4718

1. INTRODUCTION

In the recent years, the analytical approximate methods to the relativistic wave equation such as Dirac equation and Klein-Gordon equation have attracted a great number of interest in Physics[1]. When a quantum system is in strong potential field, the relativistic effect is usually been taken into consideration which gives the correctness for the non-relativistic quantum mechanical system[2]. It is well known that the solutions of these equations play an essential role in the relativistic regime for some physical potential model such as Yukawa potential[3], Deng-Fan potential[4] which can easily be transformed into other useful potential like Tietz potential and Morse potential[5, 6]. The relativistic effects describe a potential field in the presence of either Dirac equation or Klein-Gordon equation. For example, Klein-Gordon equation describes relativistic spin-0 particles such as pion π^+, π^- and π^0. The Klein-Gordon equation is a relativistic version of the Schrödinger equation. It is the equation of motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. A great number of studies have been devoted to obtain the analytic solutions of the relativistic Klein equation with the well-known potentials such as Woods-Saxon potential[7, 8], Cusp potential[9], Hyperbolic Tangent potential[10]. It is understood that the exact solutions of the Klein-Gordon equation are only possible for some physical potential types. Thus, to obtain the solution of the Klein-Gordon equation with potentials such as Yukawa[3], Coulomb, Frost-Musulin[11], we imposed an approximation scheme as we are going to see later. Recently, different analytic techniques are employed to study the relativistic and nonrelativistic wave equations ranging from Nikiforov-Uvarov method, Factorization method, supersymmetric and shape invariance method to exact/proper quantization rule.

Motivated by the interest in higher dimensional spaces, we intend to investigate the N-dimensional space of the Klein-Gordon equation with a combination of Coulomb potential and inverse Trigonometry scarf potential. The purpose of this study is to investigate the spatial dimensions of the Klein-Gordon equation with a combination of Coulomb potential and inverse Trigonometric scarf potential with unequal vector and scalar potential for an effective mass function. The proposed Coulomb-inverse Trigonometry scarf potential is given as

$$V_{CITP}(r) = \frac{-C\sin^2(\alpha r) + r^2 B}{r\sin^2(\alpha r)} \tag{1}$$

where α and B are the potential parameters and r is the inter nuclear separation.

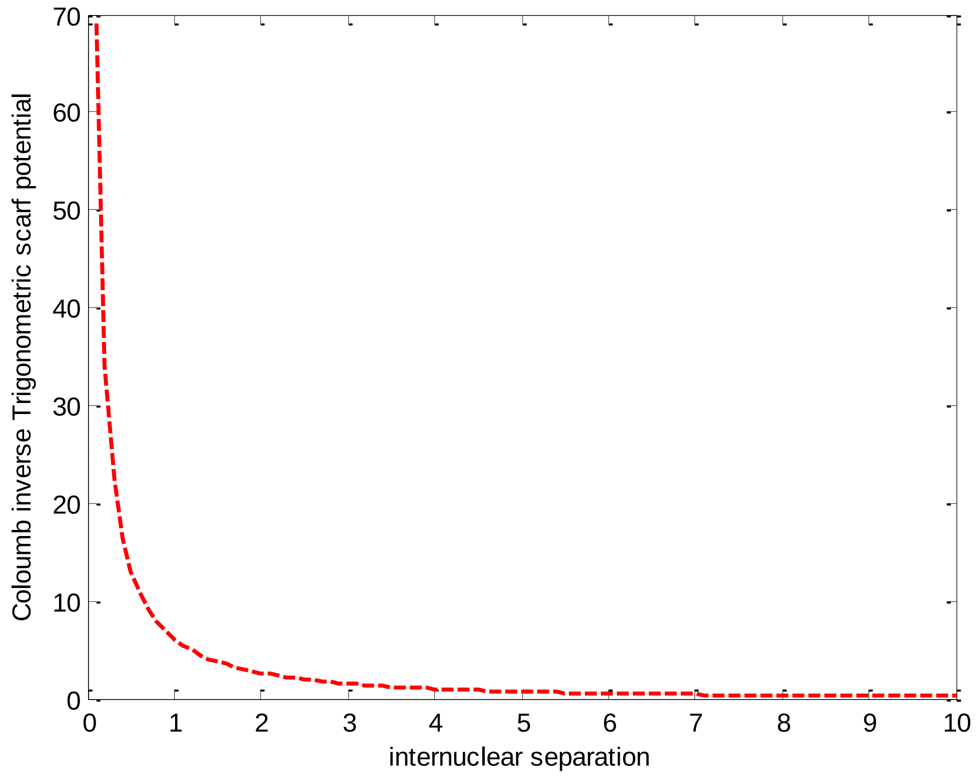

Figure 1: Coulomb inverse Trigonometric scarf potential $V_{CITP}(r)$.

2. BOUND STATE SOLUTION

The time independent N-Dimensional Klein-Gordon equation in the natural unit $c=\mu=\hbar=1$ with the scalar potential $S(r)$ and vector potential $V(r)$ is given as[12]

$$-\nabla_N^2\psi(r)+\left(M+S(r)\right)^2\psi(r)-\left(E_{n\ell}-V(r)\right)^2\psi(r)=0, \tag{2}$$

where M is the mass of the particle, $E_{n\ell}$ is the relativistic energy and N is the spatial dimensions. The N-Dimensional Laplacian operator ∇_N^2 is given as[13]

$$\nabla_N^2=r^{1-N}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial}{\partial r}\right)+\frac{L_N^2\left(\Omega_N\right)}{r^2}, \tag{3}$$

where $L_N^2\left(\Omega_N\right)$ is the ground angular momentum[12], and $\frac{L_N^2\left(\Omega_N\right)}{r^2}$ is a generalization of the centrifugal barrier for the N-Dimensional space that involves angular coordinates Ω_N and the eigenvalues of the $L_N^2\left(\Omega_N\right)^9$. $L_N^2\left(\Omega_N\right)$ is a partial differential operator on the unit space

S^{N-1}, define analogously to a three-dimensional angular momentum[14,15] as $L_N^2\left(\Omega_N\right) = -\sum_{i\geq j}^{N}\left(L_{ij}^2\right)$ where $L_{ij}^2 = x_i\dfrac{\partial}{\partial x_j} - x_j\dfrac{\partial}{\partial x_i}$ for all Cartesian component x_i of the N-dimensional vector $(x_1, x_2, ..., x_N)$. To eliminate the first order derivative, we set the total wave function as

$$R_{n\ell}(r) = r^{\frac{N-1}{2}}U_{n\ell}(r), \tag{4}$$

then,

$$L_N^2 Y_i^m\left(\Omega_N\right) = \ell(\ell+N-2)Y_\ell^m\left(\Omega_N\right). \tag{5}$$

Hence, the Klein-Gordon equation in N-dimensional space is written in the form

$$\left[-\frac{d^2}{dr^2} + E_{n\ell}^2 + V^2(r) - 2(V(r)E_{n\ell} + S(r)M) - M^2 - S^2(r) + \frac{(N+2\ell-1)(N+2\ell-3)}{4r^2}\right]U_{n\ell}(r) = 0, \tag{6}$$

where,

$$V(r) = -\frac{V_0}{r} + \frac{V_1\delta e^{-\delta r}}{1-e^{-\delta r}}, \tag{7}$$

$$S(r) = -\frac{S_0}{r} + \frac{S_1\delta e^{-\delta r}}{1-e^{-\delta r}}. \tag{8}$$

Here, V_0 and V_1 are the strength of the vector potential while S_0 and S_1 are the strength of the scalar potential and N is spatial. For proper elucidation, we have defined the following relations: $\delta = 2\alpha$, $V_0 = \dfrac{1}{\lambda}$ and $\delta^2 = \dfrac{4}{\lambda^2}$ in our proposed potential. Eq. (6) cannot be solved exactly $(\ell = 0)$ because of the presence of the centrifugal term $\dfrac{1}{r^2}$. Therefore, we have to adopt a suitable approximation type to the centrifugal term. It is noted that for the short-range potential, the following formula:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)}, \tag{9a}$$

is a good approximation to $\dfrac{1}{r^2}$. Such an approximation proposed by Greene and Aldrich[16] was to generate pseudo-Hulthén wave functions for arbitrary $\ell-$states. To show the validity of the adopted approximation (9a), we define the following function

$$U_{app}(r) = \frac{\alpha^2 \ell(\ell+1)}{\sinh^2(\alpha r)}, \tag{9b}$$

and then, show the plot of the approximation as a function of the variable r with three values of the potential parameter as shown in Figure 2.

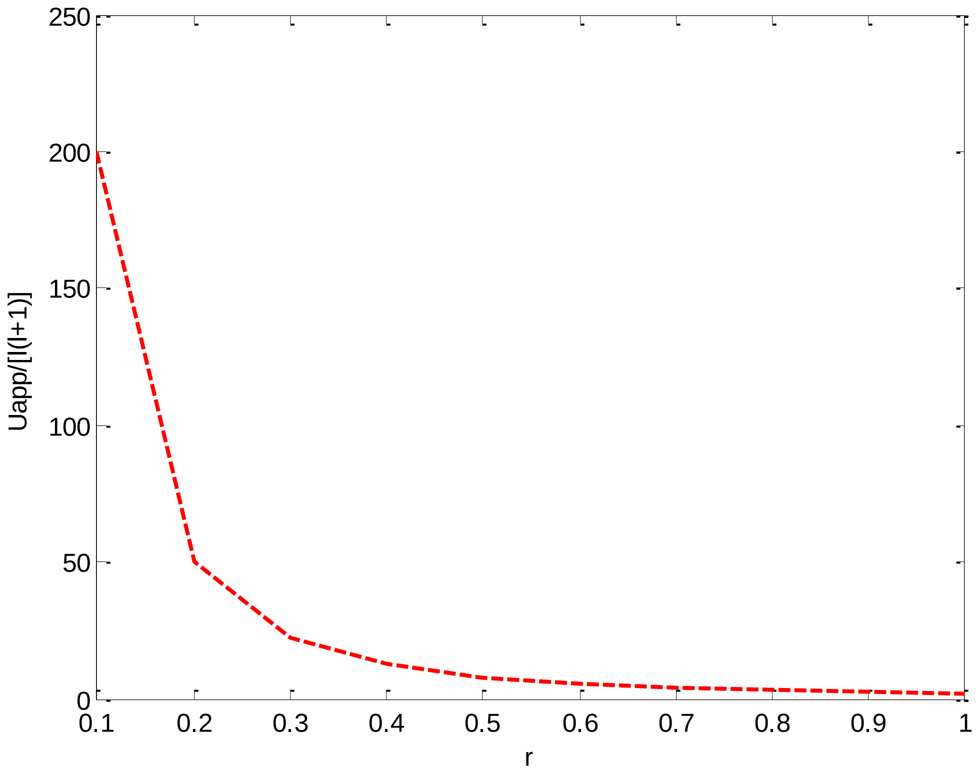

Figure 2: Centrifugal potential with $\alpha = 0.2$ and $l = 1$

Now, substituting Eqs. (1), (7), (8) and (9) into Eq. (6), we have a differential equation of the form

$$\frac{d^2 U_{n\ell}(r)}{dr^2} = \left[V_{n\ell} - \tilde{E}_{n\ell}\right] U_{n\ell}(r), \tag{10}$$

where

$$V_{eff} = \frac{V_{T_1} e^{-\delta r}}{1-e^{-\delta r}} + \frac{V_{T_2} + V_{T_3} e^{-\delta r} + V_{T_4} e^{-2\delta r}}{\left(1-e^{-\delta r}\right)^2}, \tag{11a}$$

$$V_{T_1} = \delta V_0 E_{n\ell} - m_1 + \delta S_0 m_0 - \delta S_1 m_0 + \delta S_0 m_1 - \delta V_1 E_{n\ell} + \alpha^2 \left(N + 2\ell - 1\right)\left(N + 2\ell - 3\right), \tag{11b}$$

$$V_{T_2} = \frac{\delta^2 \left(V_0^2 - S_0^2\right)}{4}, \tag{11c}$$

$$V_{T_3} = \frac{\delta^2\left(S_0S_1 - V_0V_1 + 2(N+2\ell-1)(N+2\ell-3)\right)}{2}, \tag{11d}$$

$$V_{T_3} = \frac{\delta^2 V_1^2 - \delta^2 S_1^2 + 4S_0m_1\delta - 4S_1m_1\delta}{4}, \tag{11e}$$

$$\tilde{E}_{n\ell} = E_{n\ell}^2 - m_0^2 + V_0E_{n\ell}\delta - S_1m_0\delta + (N+2\ell-1)(N+2\ell-3)\delta^2. \tag{12a}$$

In this study, we have taken the mass function as

$$M = m_0 + \frac{m_1e^{-\delta r}}{1-e^{-\delta r}}, \tag{12b}$$

where m_0 and m_1 in this distribution are two arbitrary positive parameters. The mass function of that form enable us to check out the results in the limit of the constant mass. Eq. (10) can be transform to a non-linear Riccati equation where a solution can be assumed based on the SUSY QM considerations. The non-linear Riccati equation is written as[17-22]

$$W^2(r) - \frac{dW(r)}{dr} = V_{eff} - \tilde{E}_{n\ell} \tag{13}$$

whose only solution is a propose superpotential function of the supersymmertic quantum mechanics which by a proper search, brings the compatibility of the property of the right hand side of Eq. (13). In order for the superpotential to satisfy Eq. (13), we proposed a superpotential function as follows:

$$W(r) = \rho_0 - \frac{\rho_1}{e^{\delta r}-1} \tag{14}$$

where ρ_0 and ρ_1 are two constants to be determine later. In this study, we are only concern with the bound state solutions that demand the wave function $U_{n\ell}(r)$ satisfying the boundary conditions: $U_{n\ell}(0) = U_{n\ell}(\infty) = 0.$ Thus, the regularity conditions imposed a restriction condition that $\rho_0 > 0$ and $\rho_1 < 0.$ Now, putting the restriction condition into consideration as we solve Eq. (13), we obtain the value of the two superpotential constants as

$$\rho_1 = \frac{\delta\left(1 \pm \sqrt{V_0^2 - S_0^2 + V_1^2 - S_1^2 + (N+2\ell-1)(N+2\ell-3) + 2S_0S_1 + 2V_0V_1 + \frac{4m_1\left(S_0+S_1\right)}{\delta}}\right)}{2}, \tag{15}$$

$$\rho_0 = \frac{V_1\left(E_{n\ell} + \frac{V_1}{4}\right)\delta + S_1\left(m_0 - m_1 - \frac{S_1}{4}\right)\delta - \delta\left(S_0m_0 + E_{n\ell}V_0\right) + m_1 - \rho_1^2 - \left(N+2\ell-1\right)\left(N+2\ell-3\right)\delta^2}{2\rho_1}. \tag{16}$$

$$\rho_0^2 = -\tilde{E}_{n\ell}\ . \tag{17}$$

Having proposed the supersymmetric superpotential function, it becomes very easy for us to construct a pair of supersymmetric partner potentials. First, we construct a supersymmetric quantum mechanical system by defining the Hamiltonians such that

$$H_{\pm} = -\frac{1}{2}\frac{d^2}{dr^2} + V_{\pm}(r), \tag{18a}$$

where $V_{\pm}(r)$ is the partner potentials related to the superpotential function of the supersymmetric quantum mechanics by

$$V_{\pm}(r) = \frac{1}{2}\left(W^2(r) \pm W^{'}(r)\right). \tag{18b}$$

In terms of the superpotential $W(r)$, we can now clearly express the partner potentials as

$$V_{+}(r) = W^2(r) + \frac{dW(r)}{dr} = \rho_0^2 + \frac{\rho_1(\rho_1 - 2\rho_0)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1+\delta)e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2}, \tag{19}$$

$$V_{-}(r) = W^2(r) - \frac{dW(r)}{dr} = \rho_0^2 + \frac{\rho_1(\rho_1 - 2\rho_0)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1-\delta)e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2}. \tag{20}$$

The Hamiltonian $H_{\pm}$ possess the same eigenvalues except for the zero energy ground state. For good SUSY, the ground state wave function $U_{0\ell}(r)$ is simply calculated from[23, 24]

$$U_{0\ell}(r) = N_{0\ell}\exp\left(-\int W(r)dr\right). \tag{21}$$

where $N_{0,\ell}$ is the normalization constant. The partner potentials are shape invariant[25, 26], that is $V_{+}(r)$ has the same functional form as $V_{-}(r)$ but different parameters except for the additive constant. Thus, the partner potentials of Eqs. (19) and (20) are shape invariants and hence, satisfied the shape invariance condition. It is then convenient to write the relationship between the partner potentials as all desired results are found[27, 28]

$$R(a_1) = V_{+}(a_0, r) - V_{-}(a_1, r) \tag{22}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via mapping of the form $F: a_0 \to a_1 = F(a_0)$ and the residual term $R(a_1)$ is an independent of the variable r. Now, considering the invariant potentials, as $\rho_0 \to \rho_0 - \delta$, the problem is simplified to a high degree of accuracy. Eq. (22) takes a new form

$$R(a_1) = \left(\frac{a_0^2 + (2\rho_0\rho_1 + \rho_0^2)}{2a_0}\right)^2 - \left(\frac{a_1^2 + (2\rho_0\rho_1 + \rho_0^2)}{2a_1}\right)^2, \tag{23}$$

$$R(a_2)=\left(\frac{a_1^2+(2\rho_0\rho_1+\rho_0^2)}{2a_1}\right)^2-\left(\frac{a_2^2+(2\rho_0\rho_1+\rho_0^2)}{2a_2}\right)^2, \tag{24}$$

$$R(a_3)=\left(\frac{a_2^2+(2\rho_0\rho_1+\rho_0^2)}{2a_2}\right)^2-\left(\frac{a_3^2+(2\rho_0\rho_1+\rho_0^2)}{2a_3}\right)^2, \tag{25}$$

To obtain energy equation, we consider the invariant potential $V_-(r)$ and a relation

$$\tilde{E}_{n\ell}=\sum_{k=1}^{n}R(a_k)-\left(\frac{a_n^2+(2\rho_0\rho_1+\rho_0^2)}{2a_n}\right)^2, \tag{26}$$

where $a_n=\rho_1^2-\delta n.$ Substituting the values of $\tilde{E}_{n\ell}$ and a_n into Eq. (26), the energy equation is obtain as

$$E_{n\ell}^2-m_0^2+V_0E_{n\ell}\delta-S_1m_0\delta+(N+2\ell-1)(N+2\ell-3)\delta^2=$$
$$-\left[\frac{2V_1\left(E_{n\ell}+\frac{V_1}{4}\right)\delta+2S_1\left(m_0-m_1-\frac{S_1}{4}\right)\delta-2\delta\left(S_0m_0+E_{n\ell}V_0\right)+2m_1-2\left(N+2\ell-1\right)\left(N+2\ell-3\right)\delta^2-\left(\rho_1+2n\delta\right)^2}{2\left(\rho_1+2n\delta\right)}\right]^2. \tag{27}$$

Here, we compute the wave function via the standard function analysis method. Now, defining a variable of the form $s=\exp(-\delta r)$ and substitute it into Eq. (10), we have a second-order differential equation of the form

$$\frac{d^2U_{n\ell}(s)}{dr^2}+\frac{(1-s)}{s(1-s)}\frac{dU_{n\ell}(s)}{ds}+\frac{Qs^2+Ps+R}{s^2(1-s)^2}U_{n\ell}(s)=0, \tag{28}$$

where

$$Q=\frac{4E_{n\ell}^2+4m_0^2-4\delta\left(E_{n\ell}V_1+S_1m_0-S_1m_1\right)-4m_1+\delta^2\left(S_1^2-V_1^2\right)}{4\delta^2}, \tag{29}$$

$$P=\frac{4\delta\left(E_{n\ell}V_0+S_0m_0+V_1E_{n\ell}+S_1m_0-S_0m_1\right)+2V_0V_1-2S_0S_1+4m_1+8\left(E_{n\ell}^2-m_0^2\right)-\delta^2(N+2\ell-1)(N+2\ell-3)}{4\delta^2}, \tag{30}$$

$$C=\frac{E_{n\ell}^2-M^2-\delta V_0E_{n\ell}-\delta S_0m_0}{\delta^2}. \tag{31}$$

Now, let us analyze the asymptotic behavior of Eq. (28) at origin and at infinity. It can be tested when $r\to 0(s\to 1)$ and when $r\to\infty(s\to 0)$, Eq. (29) has a solution $U_{n\ell}(s)=s^{\lambda}(1-s)^{b}$ where

$$\lambda=\sqrt{\frac{4E_{n\ell}^2+4m_0^2-4\delta\left(E_{n\ell}V_1+S_1m_0-S_1m_1\right)-4m_1+\delta^2\left(S_1^2-V_1^2\right)}{4\delta^2}}, \tag{32}$$

and

$$b = \frac{1+\sqrt{V_0^2 - S_0^2 + V_1^2 - S_1^2 + (N+2\ell-1)(N+2\ell-3) + 2S_0S_1 + 2V_0V_1 + \dfrac{4m_1\left(S_0+S_1\right)}{\delta}}}{2}. \tag{33}$$

Taking the trial wave function $U_{n\ell}(s) = s^{\lambda}\left(1-s\right)^{b} f(s)$ and then substitute it into Eq. (28), we obtain

$$f''(s) + f'(s)\left[\frac{(2\lambda+1) - 2\left(\lambda + b + \dfrac{1}{2}\right)s}{s(1-s)}\right] - f(s)\left[\frac{\left(\lambda+b\right)^2 + Q}{s(1-s)}\right] = 0. \tag{34}$$

Eq. (34) is a well-known differential equation satisfied by the hypergeometric function whose solution is

$$f(s) = {}_2F_1\left(-n, n+2(\lambda+b); 2\lambda+1, s\right). \tag{35}$$

Using the hypergeometric function to replace the function $f(s)$, we obtain

$$U_{n\ell}(s) = N_{n\ell} s^{\lambda}\left(1-s\right)^{b} {}_2F_1\left(-n, n+2(\lambda+b); 2\lambda+1, s\right), \tag{36}$$

where $N_{n\ell}$ is a normalization factor.

2.1 Special cases:

In this section, we considered some special cases for our potential:

Now, let us consider the three-dimensional cases i.e. $N = 3,$ the energy equation turns to

$$E_{n\ell}^2 - m_0^2 + V_0E_{n\ell}\delta - S_1m_0\delta + \ell(\ell-1)\delta^2 =$$

$$-\left[\frac{2V_1\left(E_{n\ell} + \dfrac{V_1}{4}\right)\delta + 2S_1\left(m_0 - m_1 - \dfrac{S_1}{4}\right)\delta - 2\delta\left(S_0m_0 + E_{n\ell}V_0\right) + 2m_1 - 2\ell\left(\ell-1\right)\delta^2 - \left(\rho_1 + 2n\delta\right)^2}{2\left(\rho_1 + 2n\delta\right)}\right]^2. \tag{37}$$

When $V_1 = S_1 = 0,$ the potential reduced to Coulomb scalar and Coulomb vector potentials with energy equation

$$E_{n\ell}^2 - m_0^2 + V_0E_{n\ell}\delta + \ell(\ell-1)\delta^2 = -\left[\frac{-2\delta\left(S_0m_0 + E_{n\ell}V_0\right) + 2m_1 - 2\ell\left(\ell-1\right)\delta^2 - \left(\rho_1 + 2n\delta\right)^2}{2\left(\rho_1 + 2n\delta\right)}\right]^2. \tag{38}$$

with

$$\rho_1 = \frac{\delta\left(1 \pm \sqrt{V_0^2 - S_0^2 + (N+2\ell-1)(N+2\ell-3) + \dfrac{4m_1S_0}{\delta}}\right)}{2}. \tag{39}$$

Similarly, when $V_0 = S_0 = 0,$ the potential (7) and (8) turns to inverse Trigonometry scarf scalar and vector potentials with relativistic energy as

$$E_{n\ell}^2 - m_0^2 + -S_1 m_0 \delta + \ell(\ell-1)\delta^2 = -\left[\frac{2V_1\left(E_{n\ell} + \frac{V_1}{4}\right)\delta + 2S_1\left(m_0 - m_1 - \frac{S_1}{4}\right)\delta + 2m_1 - 2\ell(\ell-1)\delta^2 - (\rho_1 + 2n\delta)^2}{2(\rho_1 + 2n\delta)}\right]^2. \quad (40)$$

with

$$\rho_1 = \frac{\delta\left(1 \pm \sqrt{V_1^2 - S_1^2 + (N+2\ell-1)(N+2\ell-3) + \frac{4m_1 S_1}{\delta}}\right)}{2}. \quad (41)$$

For a case when $V(r) = S(r),\ m_0 = m_1 = M,\ M - E_{n\ell} = -E_{n\ell}$ and $M + E_{n\ell} = \frac{2\mu}{\hbar^2},$ equation (37) becomes

$$E_{n,\ell} = \delta\left(\frac{\delta\hbar^2\ell(\ell+1)}{2m} - C\right) - \frac{\hbar^2}{2m}\left[\frac{\frac{2m}{\hbar^2}(C-B) - \delta\left((\ell+n+1)\right)^2 - \ell(\ell+1)\delta}{2(\ell+n+1)}\right]^2. \quad (42)$$

Eq. (42) is identical to the energy equation of Hellmann potential obtained by Hamzavi et al.[29]

Table 1: Energy spectrum $(\pm E_{n\ell})$ for $S_0 = V_1 = 2$ and $S_1 = V_0 = 1$ with $\delta = 0.25,\ m_0 = 2$ and $m_1 = 1$.

δ	$E_{0,0}$	$E_{1,0}$	$E_{1,1}$	$E_{2,0}$	$E_{2,1}$	$E_{2,2}$
0.10	1.8051898 -2.1051540	1.8052651 -2.1051215	1.8052651 -2.1051215	1.8053600 -2.1051422	1.8053600 -2.1051422	1.7535522 -2.0533194
0.25	1.5392256 -2.2838626	1.5419778 -2.2858530	1.5419778 -2.2858530	1.5452356 -2.2886562	1.5452356 -2.2886562	1.4114093 -2.1545993
0.50	1.2058679 -2.6576673	1.2446611 -2.6886439	1.2446611 -2.6886439	1.2867318 -2.7286852	1.2867318 -2.7286852	1.0440534 -2.4839404
0.75	1.0887689 -3.1565361	1.2309305 -3.2871175	1.2309305 -3.2871175	1.3978205 -3.4482533	1.3978205 -3.4482533	1.1699347 -3.2127195
1.00	1.2294598 -3.7923208	1.5770673 -4.1131367	1.5770673 -4.1131367	1.9654386 -4.4888356	1.9654386 -4.4888356	1.8815857 -4.3856899
1.25	1.6614401 -4.5484771	2.3037160 -5.1402638	2.3037160 -5.1402638	2.9915931 -5.8052281	2.9915931 -5.8052281	3.1246190 -5.8990590
1.50	2.4117186 -5.3977414	3.4205924 -6.3266762	3.4205924 -6.3266762	4.4714416 -7.3426186	4.4714416 -7.3426186	4.8687649 -7.6706779

Table 2: Energy spectrum $(\pm E_{n\ell})$ for $S_0 = V_1 = 1$ and $S_1 = V_0 = 2$ with $\delta = 0.25$, $m_0 = 2$ and $m_1 = 1$.

δ	$E_{0,0}$	$E_{1,0}$	$E_{1,1}$	$E_{2,0}$	$E_{2,1}$	$E_{2,2}$
0.10	1.7945878 -1.8947434	1.7946023 -1.8948379	1.7946023 -1.8948379	1.7946580 -1.8949486	1.7946580 -1.8949486	1.7396464 -1.8399496
0.25	1.4643803 -1.7217071	1.4674350 -1.7249128	1.4674350 -1.7249128	1.4711925 -1.7287605	1.4711925 -1.7287605	1.3080274 -1.5657650
0.50	0.8776490 -1.4467512	0.9348973 -1.5015529	0.9348973 -1.5015529	1.0018274 -1.5671822	1.0018274 -1.5671822	0.5737176 -1.1404728
0.75	0.4355009 -1.4282310	0.7273743 -1.7072785	0.7273743 -1.7072785	1.0158141 -1.9893449	1.0158141 -1.9893449	0.6090622 -1.5876110
1.00	0.5213244 -2.1093296	1.1093297 -2.6598297	1.1093297 -2.6598297	1.6598268 -3.1925824	1.6598268 -3.1925824	1.5393131 -3.0844589
1.25	0.9970548 -3.4190196	1.8510869 -4.1862053	1.8510869 -4.1862053	2.6753534 -4.9710594	2.6753534 -4.9710594	2.8085057 -5.1288729
1.50	1.6321110 -5.2060274	2.7852470 -6.1811478	2.7852470 -6.1811478	3.9226759 -7.2449262	3.9226759 -7.2449262	4.3078226 -7.6685262

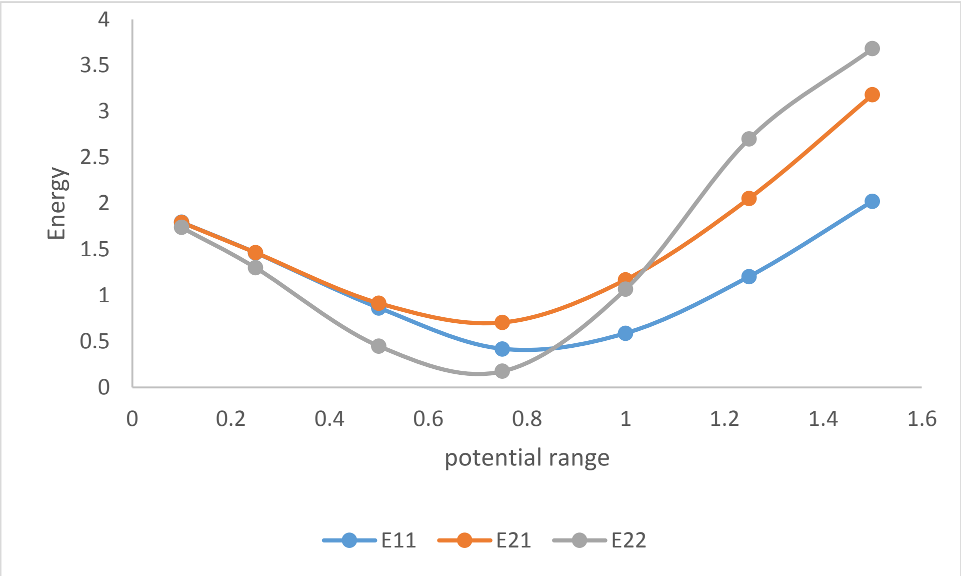

Figure 3: Energy eigenvalue E_{11}, E_{21} and E_{22} against the potential range with contant mass.

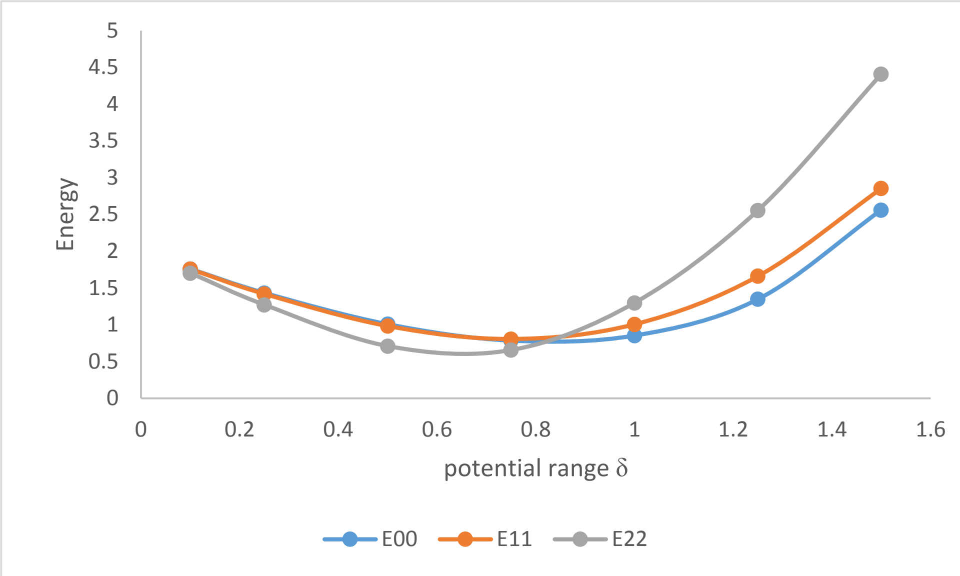

Figure 4: Energy eigenvalue E_{00}, E_{11} and E_{22} against the potential range with contant mass for coulomb potential

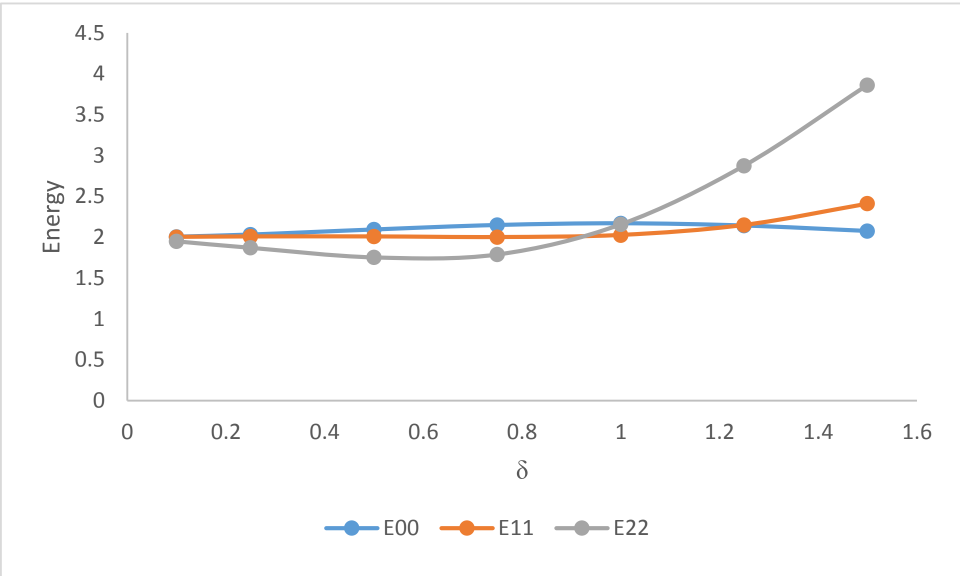

Figure 5: Energy eigenvalue E_{00}, E_{11} and E_{22} against the potential range with contant mass for inverse Trigonometry scarf potential

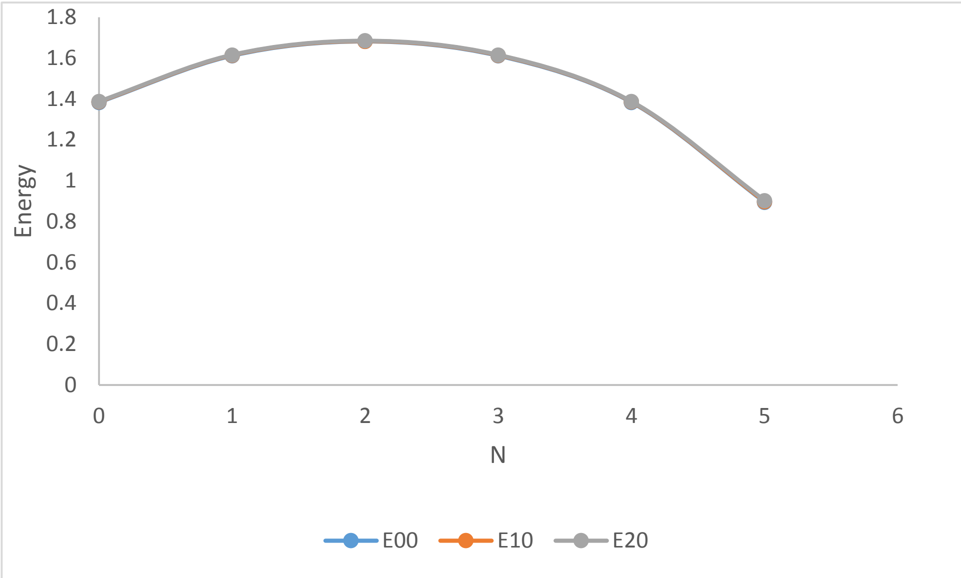

Figure 6: Energy eigenvalue E_{00}, E_{10} and E_{20} against N with contant mass

3. CONCLUSION

In this work, we critically examined the Klein-Gordon equation with unequal scalar and vector potentials using supersymmetric approach. The energy of relativistic Klein-Gordon equation for other useful potentials are obtained by changing the numerical values of the potential parameters. Some numerical results are obtained as presented in Tables 1 and 2. In Table 1, the numerical values obtained with $S_0 = V_1 = 2$ and $S_1 = V_0 = 1$ are greater than the energy eigenvalues obtained in Table 2 with $S_0 = V_1 = 1$ and $S_1 = V_0 = 2$. In Figires 1 and 2, we presented the Coulomb inverse Trigonometric scarf potential and the approximation potential respectively. In Figures 3-6, we plotted energy of some states against the potential range for Coulomb-inverse Trigonometry scarf potential, Coulomb potential and inverse Trigonometry scarf potential respectively. The non-relativistic energy obtained is identical to the results of Hellmann potential. In Figure 4, we showed energy of some states against the spatial dimensions. One of the interesting applications of this new potential is in many-body Physics such as Haldane-Shastry model. We hope to get other applications of this study in different fields of sciences as our results are not only interesting to pure theoretical physicists but also to the experimental physicists.

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