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THE GAME THEORY AND SIMULATION WITH FINITE MIXED STRATEGY OF TWO-PLAYERS ZERO SUM GAME

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Method Article

ABSTRACT

We explain the concept and theory behind game theory and simulation. How game theory can be used to simulate competition existing in the market, business organisation, and even in the management setting. The game we consider here is the zero-sum game. We show types strategies game players in a real market situation should adopt. We show how we minimise our losses and maximise our gain by introducing the concept of maxmin and min-max. We analysed finite payoff matrix of a zero sum-game using some fundamental principles. We apply the theory of mixed strategies with a game without a saddle point. We show that the dual linear programming problem of one player is the linear programming problem of the other player. The four methods available in solving mixed strategy of two-players zero-sum game was analysed by showing the application of linear programming problem to game theory. We use these techniques to simulate possible mixed strategy of two-players zero-sum game using the Microsoft excel package.

Keywords: Games and decisions; payoff; strategies; saddle points; players.

Subject Classification: MSC2010 database 91A05, 91A06, 91A13

1 Literature Review

There are many research work on game theory and various method have been develop to cover game with and without saddle point [1,2,3]. In the highlight of the solution of $(n \times m)$ matrix by graphical method by [4], state that once the dominance property fails to get the solution, then the graphical method $(n \times m)$

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matrix also fails. Game theory is so important in the field of science and social science, it relevant in this field is more than fifty years [5]. In the game when one player gain is exactly equal to net losses of other player known as the competitor, we refer to such game as zero-sum game. [6], define game theory as a formal study of decision-making where several players must make choices that potentially affect the interests of other players. Turocy & von Stengel highlighted that the game theory was first presented on the study of duopoly by Antoine Cournt in 1838. However [7] discovered that the first known discussion of game theory occurred in a letter written by James Waldegrave in 1713. John von Nuemann in 1928 brought game theory to lamplight in the study of polar game. Game theory was further emphasis in 1949 when John Forbes Nash published his thesis titled Non Cooperative games, in this thesis he introduced the concept of Nash equilibrium point. [7] stated that game theory is generally used in economics, political science, and psychology, as well as logic and biology. Game theory constituent of n-player which was define [8] as an individual or group of individuals making a decision. [9] went on to outline the assumptions of the game theory as that, all players form beliefs based on analysis of what others might do, choose a best response given those beliefs, and adjust best responses and beliefs until they are equal. He also observed that these assumptions are sometimes violated, meaning that not every player behaves rationally in difficult situations. [10] also highlighted that the basic assumption that motivates the game theory is that decision-makers who are aware of their alternatives and result of their choices are rational and they reason strategically and intelligently. It was highlighted by [6] that game of one player is a decision problem. [11], presented an example for games as a scenario where firms compete with each other by setting prices, or a scenario where a group of consumers are bidding against each other at an auction. According to [10] game theory are divided into two branches namely; cooperative and non-cooperative, which will depend whether player can communicate [5]. In non-cooperative game player make choice out of their own interest [6]. In the model of non-cooperative game ordering and timing of players' choices are crucial in determining the outcome of a game. Dominance is explained by the use of Prisoner's Dilemma that was first introduced by Tucker in 1950. A dominant strategy has payoffs such that, regardless of the choices of other players, no other strategy would result in a higher payoff. [12] explained dominance using an example called the Prisoner's Dilemma. [13] also used the Prisoner's Dilemma to illustrate dominance of the game. [14] also pointed out that a game can have many Nash equilibriums, and some of these equilibriums may be unreliable compared to what should be the outcome of a game. Players have to know precisely what their opponents will choose [15]. [16] is of the view that mixed strategy equilibrium is then as common knowledge opportunities, this is because all the actions to which a strictly positive probability is assigned are ideal, given the beliefs. In a situation where there are more than one player, each player's payoff is generally affected by the actions of the other players [17]. [17] argued that such games, the decision of what to do at a point in time cannot generally be optimally made without considering decisions at all other points in time because other decisions affect the probabilities of being at different states at the current point in time. [18], says that zero-sum game is a mathematical representation of a situation in which a participant's gain or loss of utility is exactly balanced by the losses or gains of the utility of the other participant. The theory of von Neumann and Morgenstern is mostly applied in games such as two-person zero-sum games, that is games with only two players in which one player wins what the other player loses. In game theory, a game with only a few strategies can be easily represented by a matrix showing the payoff for each player along with the strategy they use [19]. Zero sum games can also be viewed as a closed system, meaning everything that someone wins must be lost by someone else [19]. Game theory has help in the analysis in auction according to [6]. Auction theory was established by the economist William Vickrey in 1961. It was practical applied in the early 90's, as result of billions of dollars auctions generated through radio frequency spectrum for mobile telecommunication. [20] views a game-theoretic auction model as a mathematical game represented by a set of players, a set of strategies available to each player, and a payoff vector corresponding to each combination of strategies. [5] pointed out that the first studies of game theory in the economics literature were the papers by Cournot in year 1838, Bertrand in year 1883 and Edgeworth in year 1897 on oligopoly pricing and production. [5] further outlined that game theory continued to grow in the 1970s in terms of both theoretical extensions as well as widened applications in areas of economics and that proved that game theory is here to stay. Game theory is a tool for making strategic decision; it is therefore used to make decisions when interdependence existsin Oligopoly. The application of game theory in economics is considerably wide, such as auctions, bargaining, oligopolies, externalities and public goods, market equilibrium, general equilibrium and other application [21]. It will not be of any surprise that optimal net investment wealth with discounted stochastic cash flows and efficient frontier and optimal variational portfolios with inflation protection strategy and efficient frontier of expected value of wealth for a defined contributory pension scheme, see [22] and [23].

2 Introduction

Game theory is a game play with n-payer where $n \geq 2$. This n −player are competitor for the same goal. For example in a business organization we talk about union and management, in most case $n = 2$. Here union is taken care of the employee management is taken care of the policy and the have competing goal in front of them. The primary gaol of management is to increase profit and the union is to improve the welfare and savings structure of their employee. Now if the management give more percentage savings to employee, definitely profit will get reduce by approximately to that amount. So, it is clear both management and employer has competing goals in front of them.

Generally, game theory exist in the real world market because competitors exist in the market, organisation, company, producer, manufacturer, etc. are launching new and new product by competing with one another, they have a common goal to improve their market share and for that they have different strategies and the strategies could be one player may give discount of product, free product for more purchase. Again a player might give advertisement in the radio channel, will the other player may give advertisement in the TV channel, also one player may be using magazine of their product, the other is using some other means so different player has different strategies and these strategies are disclose before they start playing the game, these strategies are available in the research papers, management magazine and one can take. It possible to take decision on which strategies to use in other for one position in the game to be optimised, by being in the optimal position one profit is maximised and lost minimised. This is the particular aim in front of any player while playing a game. A player take a decision based on the strategies adapted by other players, he changes his strategies the moment the other player changes his strategies. In this presentation we will say how we minimised our lost and maximised our gain or profit. When playing a particular goal should be set or defined and all competitor with the same goal identify, before setting up your different strategies to be adopted. It is important to understand this mathematical took for us to be able to do proper optimisation in the market.

A game theory is concern with a type of decision for a problem characterised by conflict or competition among two or more competitors, for example workers versus management, supplier versus supplier (to one particular factory), political leader versus another political leader that to win a particular election, etc. Game theory provides systematic quantitative methods for analysing competitive (or players) make use of logical processes and techniques in order to determine an optimal strategy for winning. Any logical, rational, critical and intelligent thinker will not want to play a game and lose it. He or she will want to win at all time, rather he or she will really want to minimise lost at all times. Game theory can be classify as follows; first we have n −player, where n stand for the number of players and $n \geq 2$, secondly we have the sum of gain and losses are normally zero or non-zero sum game, that is the gain and losses may sum up to zero or non-zero, thirdly, what strategies employed. Strategies is defined as an alternative course of action available to the player in advance, by which the player decides his action plan. We have two type of strategies; the pure and the mixed strategy. The pure strategies is when or if the player selects the same strategies each time, which deterministic in nature. The mixed Strategies is when the players use a combination of strategies and each player is kept guessing as to which course of action is to be adopted by other player on a particular occasion, thus, mixed strategy is a probabilistic situation, and the objectives of the player is to maximised expected gain or minimised expected losses. As earlier mention rule are fixed before starting the game. The rule of the game is that the players act rationally and intelligently, meaning that they know the different strategies available to them and they know the different strategies available to their competitor, they know the result for playing different strategies by other players and by them, they are logical, they are playing to win the game they are not playing to lose the game. Each players has a finite set of possible courses of action, that is each player has finite number of strategies available to him, the effect of such strategies is also known to him, the player. Again the players attempt to maximize gains and minimize losses. All relevant information is known to each player. Players take individual decision without direct communication with the other competitors. The players simultaneously select their respective courses of action, that if one take a strategy the other take another based on e strategy taken by their competitor. The pay-off is fixed and determined in advance, based on that we take our action. Mathematically, a mixed strategy, for a player with two or more possible courses of action, is denoted by the set S, where S is equal to $(x_1, x_2, \dots, x_n)$ subject to constraints that $x_1 + x_2 + \cdots + x_n = 1$, where $x_1, x_2, \dots, x_n$ are probability of player different strategy, such that $x_i \geq 0; j = 1, 2, 3, .., n$. **Optimal strategies** is the course of action or a complete game plan that leaves a player in the most preferred position regardless of the actions of his competitors. The **preferred position** is a situation where by that any deviation from the optimal strategy or plan would result in decreased pay off. Payoff is a quantitative measure (e.g. money percent market share, etc.) of satisfaction, a player gets at the end of the game is called payoff or outcome. The value of the game refers to the expected outcome for the players when the players follow their optimum strategy.

3 Payoff Matrix

The payoff matrix is shown as Table 3.1 below

Table3.1 General Payoff Matrix

Player A@s strategy	Player B's Strategies			
	B_1	B_2	B_j	B_n
A_1	a_{11}	a_{12}	a_{1j}	a_{1n}
A_2	a_{21}	a_{22}	a_{2j}	a_{2n}
A_i	a_{i1}	a_{i2}	a_{ij}	a_{in}
A_m	a_{m1}	a_{m2}	a_{mj}	a_{mn}

Where A and B are the player and A_i *and* B_j are the strategies taken by each players and a_{ij} are their respective payoff for taking such strategies, where $i = 1, 2, \dots, m$ *and* $j = 1, 2, \dots, n$ and a_{ij} is the gain of the player A and losses of Player B. note that pure strategies deal with saddle points.

4 Methodology

4.1 The dominance principle

The dominance principle says that if all elements in a column (showing the losses of player B) are greater than or equal to the corresponding elements in another column, then that column is dominated and can be deleted from the matrix, since player B do not want much losses. We say mathematically for any $B_{j_r} \geq B_j$ then B_{j_r} can be deleted. Similarly, if all the elements in a row(showing the gain of player A) are less than or equal to the corresponding elements in another row, then that row is dominated and can be deleted from the matrix since player A do not need less gain. We say mathematically for any $A_{i_r} \leq A_i$ then A_{i_r} can be deleted.The row are showing the gain of player A while the column are showing the losses of player B. Before finding solution of a gain problem we check for dominance.

Table 4.1. General payoff showing the Column Maxima, the Row minima, and the saddle point

Player A's strategy	Player B's Strategies				
	B_1	B_2	B_j	B_n	Row minima
A_1	$a_{1,1}$	$a_{1,2}$	$a_{1,j}$	$a_{1,n}$	a_{1,j_k}
A_2	$a_{2,1}$	$a_{2,2}$	$a_{2,j}$	$a_{2,n}$	a_{2,j_k}
A_i	$a_{i,1}$	$a_{i,2}$	$a^*_{i,j}$	$a_{i,n}$	a^*_{i,j_k}
A_m	$a_{m,1}$	$a_{m,2}$	$a_{m,j}$	$a_{m,n}$	a_{m,j_k}
Column Maxima	$a_{i_k,1}$	$a_{i_k,2}$	$a^*_{i_k,j}$	$a_{i_k,j}$	

Where $a_{i,j}$ are the expected payoff for each strategies A_i and B_j and $i = 1, 2, \ldots, m$ and $j = 1,2, \ldots .n$. We see that the maximin for the row minima is a^*_{i,j_k} and that the Minimax for the row maxima is $a^*_{i_k,j}$. When $a^*_{i,j_k} = a^*_{i_k,j}$, we say there exist a saddle point and the corresponding payoff is $a^*_{i,j}$ where i may not necessary be equal to j. When $a^*_{i,j_k} \neq a^*_{i_k,j}$, we say there exist no saddle point and we invoke mixed strategies, which is probabilistic model. Mixed strategies is a game without saddle point, while pure strategies are available as optimal strategies only for those game which have a saddle point is a game with saddle point. Game which do not have a saddle point can be solved by applying the concept of mixed strategies. The pure strategies really do not require any solution since there is only one strategies available to but player A and B.

4.2 Method of solving two-person-zero-sum game

Just like solving quadratic equation, game theory with mixed strategies can be solve by one of the following method; algebraically, graphically, using calculus or analytically, and linear programming method. It is important to note that, among all competitor, we have one closet competitor in a situation where there are one we arbitrarily chose one. Hence in most game theory we usually have $2-$players. In the game theory involving two players, algebraic method may be suitable when the number of strategies are two or can be reduce to two for both players, by dominance method or other suitable method. In using linear programming method you convert the game theory programming and solving by linear programming method.

4.3 The algebraic method

We have the two person zero sum game pay off matric for a mixed strategies game given in Table below;

Table 4.2. General Payoff Matrix with Strategies probabilities

Player X's strategy	**Probabilities**	**Player Y's Strategies**			
		Y_1	Y_2	Y_j	Y_n
		Probabilities			
		y_1	y_2	y_j	y_n
X_1	x_1	$a_{1,1}$	$a_{1,2}$	$a_{1,j}$	$a_{1,n}$
X_2	x_2	$a_{2,1}$	$a_{2,2}$	$a_{2,j}$	$a_{2,n}$
X_i	x_i	$a_{i,1}$	$a_{i,2}$	$a_{i,j}$	$a_{i,n}$
X_m	x_m	$a_{m,1}$	$a_{m,2}$	$a_{m,j}$	$a_{m,n}$

The player X uses strategy X_i, where $i = 1, 2, \ldots, m$ with the following respective probabilities x_i where $i = 1, 2, \ldots, m$ and player Y uses strategy X_j, where $j = 1, 2, \ldots, n$, with the following

respective probabilities y_i where $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{m} x_i = 1$ and $\sum_{j=1}^{n} y_i = 1$. We can solve for x_i, and y_j the probability of taking strategies X_i and Y_j respectively algebraically.

Now, the payoff matrix before us is a mixed strategies two player game, us to have same expected strategy X_i and Y_j, what percentage do player X or Y have to use their strategies to maximised gain and minimised lost. For player X to solve for its strategy probability usage we define

$$E(X_1), E(X_2), \ldots, E(X_m)$$

that is $E(X_i)$ as the expected strategies of X and $i = 1, 2, \dots, m$

of which the following holds

$$E(X_1) = E(X_2) = \cdots = E(X_m)$$

is such that

$$\sum_{i=1}^{m} a_{i1}x_i = \sum_{i=1}^{m} a_{i2}x_i = \cdots = \sum_{i=1}^{m} a_{in}x_i$$

Using transitivity relation and closure properties, player X can easily build up a finite system of linear equation and the probabilities x_i can easily be solve under the condition that

$$\sum_{i=1}^{m} x_i = 1$$

Similarly

For player Y to solve for its strategy probability usage we define

$$E(Y_1), E(Y_2), \dots, E(Y_n)$$

that is $E(Y_j)$ as the expected strategies of Y and $j = 1, 2, \dots, n$
of which the following also holds

$$E(Y_1) = E(Y_2) = \cdots = E(Y_n)$$

is such that

$$\sum_{j=1}^{n} a_{1,j}y_j = \sum_{j=1}^{n} a_{2,j}y_j = \cdots = \sum_{j=1}^{n} a_{m,j}y_j$$

Using transitivity relation and closure properties, player Y can easily build up a finite system of linear equation and the probabilities y_i can easily be solve under the condition that

$$\sum_{j=1}^{n} y_j = 1$$

4.4 Analytical or Calculus method

From the general pay of matrix given in Table 3.1, we write the value of the game as V, as is define as given below

$$V = \begin{matrix} x_1y_1a_{1,1} + x_1y_2a_{1,2} + \cdots + x_1y_ja_{1,j} + \cdots + x_1y_na_{1,n} + \\ x_2y_1a_{2,1} + x_2y_2a_{2,2} + \cdots + x_2y_ja_{2,j} + \cdots + x_2y_na_{2,n} + \\ \cdot \\ \cdot \\ \cdot \\ x_iy_1a_{i,1} + x_iy_2a_{i,2} + \cdots + x_iy_ja_{i,j} + \cdots + x_iy_na_{i,n} + \\ \cdot \\ \cdot \\ \cdot \\ x_my_1a_{m,1} + x_my_2a_{m,2} + \cdots + x_my_ja_{m,j} + \cdots + x_my_na_{m,n} \end{matrix}$$

Now differentiating the value of the game V with respect to $x_1, x_2, \dots, x_m$ we have the following

$$\frac{\partial V}{\partial x_1} = y_1a_{1,1} + y_2a_{1,2} + \cdots + y_ja_{1,j} + \cdots + y_na_{1,n} = 0$$
$$\frac{\partial V}{\partial x_2} = y_1a_{2,1} + y_2a_{2,2} + \cdots + y_ja_{2,j} + \cdots + y_na_{2,n} = 0$$
$$\vdots$$
$$\frac{\partial V}{\partial x_i} = y_1a_{i,1} + y_2a_{i,2} + \cdots + y_ja_{i,j} + \cdots + y_na_{i,n} = 0$$
$$\vdots$$
$$\frac{\partial V}{\partial x_m} = y_1a_{m,1} + y_2a_{m,2} + \cdots + y_ja_{m,j} + \cdots + y_na_{m,n} = 0$$

We have system of linear equation and the value for y_j for $j = 1, 2, \dots, n$ can be find which is the probabilities of the strategies Y_j for player Y. This method can also be used to get the strategic probabilities for player X, but this time we have the following;

$$\frac{\partial V}{\partial y_1} = \begin{matrix} x_1a_{1,1} \\ x_2a_{2,1} \\ \cdot \\ \cdot \\ \cdot \\ x_ia_{i,1} \\ \cdot \\ \cdot \\ \cdot \\ x_ma_{m,1} \end{matrix} = 0 \;, \quad \frac{\partial V}{\partial y_2} = \begin{matrix} x_1a_{1,2} \\ x_2a_{2,2} \\ \cdot \\ \cdot \\ \cdot \\ x_ia_{i,2} \\ \cdot \\ \cdot \\ \cdot \\ x_ma_{m,2} \end{matrix} = 0, \cdots, \frac{\partial V}{\partial y_j} = \begin{matrix} x_1a_{1,j} \\ x_2a_{2,j} \\ \cdot \\ \cdot \\ \cdot \\ x_ia_{i,j} \\ \cdot \\ \cdot \\ \cdot \\ x_ma_{m,j} \end{matrix} = 0 \;, \dots, \frac{\partial V}{\partial y_n} = \begin{matrix} x_1a_{1,n} \\ x_2a_{2,n} \\ \cdot \\ \cdot \\ \cdot \\ x_ia_{i,n} \\ \cdot \\ \cdot \\ \cdot \\ x_ma_{m,n} \end{matrix} = 0$$

Which also is a system of linear equation. Here the probabilities of the strategies X_i can be find for player X. The probabilities strategies X_i or Y_j for player X and Y so represented as a system of linear can solve ordinarily or by the use of software like excel and matlab.

Note that in all cases, that is for the finding of the probabilities strategies X_i or Y_j for player X and Y, the following is sure, that $a_{i,j} \in \mathbb{R}$, such that $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, also the condition that $\sum_{i=1}^{m} x_i = 1$ and $\sum_{j=1}^{n} y_j = 1$ hold.

4.5 The graphical Method

The graphical method is applicable to two player and two strategies each, that is 2×2 or $2 \times n$ or $m \times 2$ mixed strategies of two player game, however is always combine with algebraic method and some form of linear programming idea can also be used. In giving the general analysis of the graphical method, we start we the $2 \times n$ mixed strategies of two player game. Given a 2×2 mixed strategies of two player game, that is a game played by two persons with two strategies each as shown below

Table 4.3. $2 \times n$ Mixed strategies of two player game

Player X's strategy	Probabilities	Player Y's Strategies				
		Y_1	Y_2	…	Y_j	…	Y_n
		Probabilities					
		y_1	y_2	…	y_j	…	y_n
X_1	x_1	$a_{1,1}$	$a_{1,2}$	…	$a_{1,2}$	…	$a_{1,n}$
X_2	x_2	$a_{2,1}$	$a_{2,2}$		$a_{2,j}$	…	$a_{2,n}$

Table 4.4. $m \times 2$ Mixed strategies of two player game

Player X's strategy	Probabilities	Player Y's Strategies	
		Y_1	Y_2
		Probabilities	
		y_1	y_2
X_1	x_1	$a_{1,1}$	$a_{1,2}$
X_2	x_2	$a_{2,1}$	$a_{2,2}$
⋮	⋮	⋮	⋮
X_i	x_i	$a_{i,1}$	$a_{i,2}$
⋮	⋮	⋮	⋮
X_m	x_m	$a_{m,1}$	$a_{m,2}$

The distance between the two adjacent line is one, since the strategic probability sum up to one for each players. The $a_{1,j}$ has there direct corresponding to the $a_{2,j}$. The dominance strategies are find above the highest point closet to the probability axis's which give the strategic probability x_i of player X at that point and a^*_{2,j_k} is the payoff, that is the maximin for player X. The rest probability can be find since $x_1 + x_2 = 1$. After finding x_1 and x_2 you can solve for $y_1, y_2, \dots, y_n$, by building a linear programming problem from the payoff matrix. We say u is unrestricted in sign because we do not know whether X at the end of the game will make be a profit or a lost u can be positive, zero or negative. We formulate the linear programming problem as follow;

For Player X linear Programming Problem formulation

Max $u = \min\,(u_1, u_2, \dots, u_n)$

Subject to

$$\begin{gathered} u_1 \le x_1 a_{1,1} + x_2 a_{2,1} \\ u_2 \le x_1 a_{1,2} + x_2 a_{2,2} \\ \vdots \\ u_j \le x_1 a_{1,j} + x_2 a_{2,j} \\ \vdots \\ u_n \le x_1 a_{1,n} + x_2 a_{2,n} \\ x_1 + x_2 = 1 \\ x_1, x_2 \ge 0 \\ u \text{ is unrestricted} \end{gathered}$$

For Player Y linear Programming Problem formulation of $2 \times n$ Mixed strategies of two player game

$$Min\ v = \max (v_1, v_2)$$

Subject to

$$v_1 \geq y_1 a_{1,1} + y_2 a_{1,2} \dots y_j a_{1,j} \dots y_n a_{1,n}$$
$$v_2 \geq y_1 a_{2,1} + y_2 a_{2,2} \dots y_j a_{2,j} \dots y_n a_{2,n}$$
$$y_1 + y_2 + \cdots + y_j + \cdots + y_n = 1$$
$$y_1, y_2, \dots, y_j, \dots, y_n \geq 0$$
$$v\ \ is\ unrestricted$$

First, we put linear programming X game player formulation in standard form and we refer to it as primal, we have;

Max $u = \min (u_1, u_2, \dots, u_n)$
Subject to

$$u_1 + x_1 a_{1,1} + x_2 a_{2,1} \leq 0$$
$$u_2 + x_1 a_{1,2} + x_2 a_{2,2} \leq 0$$
$$\vdots$$
$$u_j + x_1 a_{1,j} + x_2 a_{2,j} \leq 0$$
$$\vdots$$
$$u_n + x_1 a_{1,n} + x_2 a_{2,n} \leq 0$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$
$$u\ is\ unrestricted$$

Next we find the dual of the above primal

$$Min\ v = \max (v_1, v_2)$$

Subject to

$$y_1 + y_2 + \cdots + y_j + \cdots + y_n = 1$$
$$y_1 a_{1,1} + y_2 a_{1,2} \dots y_j a_{1,j} \dots y_n a_{1,m} + v_1 \geq 0$$
$$y_1 a_{2,1} + y_2 a_{2,2} \dots y_j a_{2,j} \dots y_n a_{2,m} + v_2 \geq 0$$
$$y_1, y_2, \dots, y_j, \dots, y_n \geq 0$$
$$v\ \ is\ unrestricted$$

The same is applicable to $m \times 2$ Mixed strategies of two player game.

Table, we see that the dual of X game player linear programming problem is actually the Y game player linear programming problem. So there is a primal-dual relationship between X problem and Y problem. So any $2 \times n$ or $m \times 2$ mixed strategies of two player game can be solved using the graphical method. The point intersect constitute two strategies and the closet point to the horizontal line gives the probability on same horizontal axis, since the sum of the probabilities is 1 the other probability can be easily be find. When the strategies are greater than 2 for the other player, the probability can be obtain by formulating the linear programming program of the player whose probability strategies is known and using the duality principle, the linear programming problem for the other player can be formulated, which can be solve using the simplex or the above mention mathematical software like excel, maple, mathematical or matlab.. Generally for $m \times n$ mixed strategies of two player game then we can formulate the linear programming problem for player X or Y, solve it and using principle of dual to get the other solution by using simplex method.

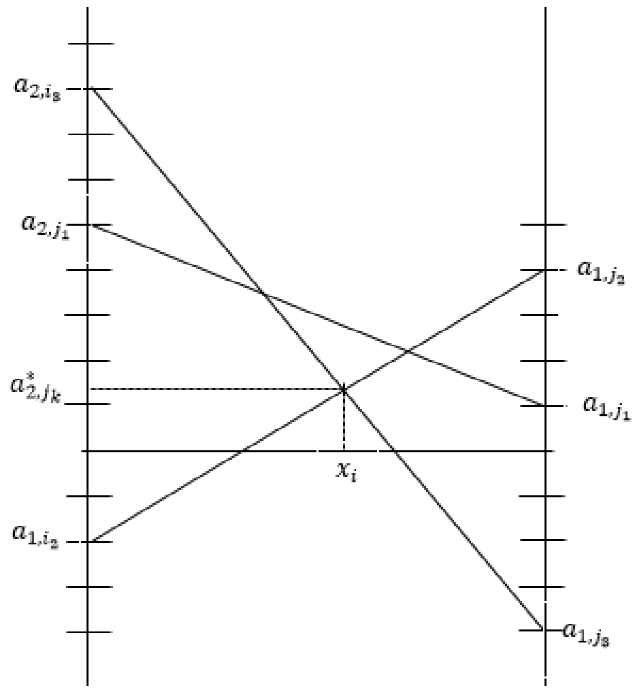

Fig. 4.5. Graph showing $2 \times n$ Mixed strategies of two player game

There are two firm having the payoff matrix below determine the probabilities for the mixed the strategies of the game and the expected payoff of player X and Y

4.6 Linear programming problem in Game Theory

Given $m \times n$ Mixed strategies of two player game as shown in Fig 4.5, we formulate the linear programming problem for player X and Y as follow

4.6.1 Linear programming problem for Player *X*

Max $u = \min\,(u_1, u_2, \dots, u_n)$

Subject to

$$\begin{gathered} u_1 \le x_1 a_{1,1} + x_2 a_{2,1} + \dots + x_i a_{i,1} + \dots + x_m a_{m,1} \\ u_2 \le x_1 a_{1,2} + x_2 a_{2,2} + \dots + x_i a_{i,2} + \dots + x_m a_{m,2} \\ \vdots \\ u_j \le x_1 a_{1,j} + x_2 a_{2,j} + \dots + x_i a_{i,j} + \dots + x_m a_{m,j} \\ \vdots \\ u_n \le x_1 a_{1,n} + x_2 a_{2,n} + \dots + x_i a_{i,n} + \dots + x_m a_{m,n} \\ x_1 + x_2 + \dots + x_j + \dots + x_m = 1 \\ x_1, x_2, \dots, x_m \ge 0 \\ u\ is\ unrestricted \end{gathered}$$

4.6.2 Linear programming problem for Player *Y*

$$Min\ v = \max\,(v_1, v_2, \dots, v_n)$$

Subject to

$$v_1 \geq y_1 a_{1,1} + y_2 a_{1,2} \dots y_j a_{1,j} \dots y_n a_{1,n}$$
$$v_2 \geq y_1 a_{2,1} + y_2 a_{2,2} \dots y_j a_{2,j} \dots y_n a_{2,n}$$
$$\vdots$$
$$v_i \geq y_1 a_{i,1} + y_2 a_{i,2} \dots y_j a_{i,j} \dots y_n a_{i,n}$$
$$\vdots$$
$$v_n \geq y_1 a_{n,1} + y_2 a_{n,2} \dots y_j a_{,j} \dots y_n a_{m,n}$$
$$y_1, y_2, \dots, y_j, \dots, y_n \geq 0$$
$$y_1 + y_2 + \cdots + y_j + \cdots + y_n = 1$$
$$v \ \ is\ unrestricted$$

So generally,

The value of the game for player X

Max Z =min $\left\{\sum_{i=1}^{m} a_{i,1}\, x_i, \sum_{i=1}^{m} a_{i,2}\, x_i, \dots, \sum_{i=1}^{m} a_{i,n}\, x_i\right\}$

Subject to

$$\sum_{i=1}^{m} x_i = 1$$
$$0 \leq x_i \leq 1;\ \ i = 1, 2, \dots, m$$

The value of the game for player Y

Min Z =max $\left\{\sum_{j=1}^{n} a_{1,j}\, y_j, \sum_{j=1}^{n} a_{2,j}\, y_j, \dots, \sum_{j=1}^{n} a_{m,j}\, y_j\right\}$

Subject to

$$\sum_{j=1}^{n} y_i = 1$$
$$0 \leq y_j \leq 1;\ \ j = 1, 2, \dots, n$$

5 Some Application

In this section we took the payoff matrix two hypothetical mixed strategy of two-players zero sum game, using micro excel we show graphically that the value of the maxmin and the minimax does have some relationship with the proportion of the payoff, see graph below

5.1 Example 1

The payoff matrix below consist of two companies with three (3) each. We need to find out what is the proportion of time that the company X will player its strategies in order to maximised its minimum profit and company Y should play its strategies in order to minimised its maximum lost.

Table 5.1 Payoff Matrix for Example 1

Company X's strategies	Probabilities	Company Y's Strategies		
		Y_1	Y_2	Y_3
		Probabilities		
		y_1	y_2	y_3
X_1	x_1	3	-1	-3
X_2	x_2	-3	3	-1
X_3	x_3	-4	-3	3

We formulate a linear programming problem for Company X from the payoff matrix

$$Max\ v = \min(v_1, v_2, v_3)$$

Subject to

$$v_1 = 3x_1 - 3x_2 - 4x_3$$
$$v_2 = -1x_1 + 3x_2 - 3x_3$$
$$v_3 = -3x_1 - 1x_2 + 3x_3$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Using Microsoft Excel we have the following results. The tables below, consist of probabilities for the strategies, the probability proportion of the payoff, and the value of the game for each player.

Table 5.2 Table showing solution of company X. This solution consist of probabilities for the strategies, the probability proportion of the payoff, and the value of the game of player X.

x_1	0.444444	1.333333	-0.44444	-1.33333	
x_2	0.244445	-0.73333	0.733334	-0.24444	
x_3	0.311111	-1.24444	-0.93333	0.933333	
sum	1	v_1	v_2	v_3	
		-0.64444	-0.64444	-0.64445	-0.64445

We formulate a linear programming problem for Company Y from the payoff matrix

$$Min\ w = \max(w_1, w_2, w_3)$$

Subject to

$$w_1 = 3y_1 - 1y_2 - 3y_3$$
$$w_2 = -3y_1 + 3y_2 - 1y_3$$
$$w_3 = -4y_1 - 3y_2 + 3y_3$$
$$y_1 + y_2 + y_3 = 1$$
$$y_1, y_2, y_3 \geq 0$$

Again using Microsoft Excel we have the following results

Table 5.3. Table showing solution of company Y. This solution consist of probabilities for the strategies, the probability proportion of the payoff, and the value of the game of player Y.

y_1	0.311112	0.933336118	-0.24445	-1.333319992	
y_2	0.244448	-0.933336118	0.733344	-0.444439997	
y_3	0.44444	-1.244448157	-0.73334	1.333319992	
sum	1	w_1	w_2	w_3	MINIMAX
		-0.644431838	-0.64443	-0.644472054	-0.64443

5.2 Example 2

The payoff matrix below consist of two companies with three (3) each. We need to find out what is the proportion of time that the company X will player its strategies in order to maximised its minimum profit and company Y should play its strategies in order to minimised its maximum lost.

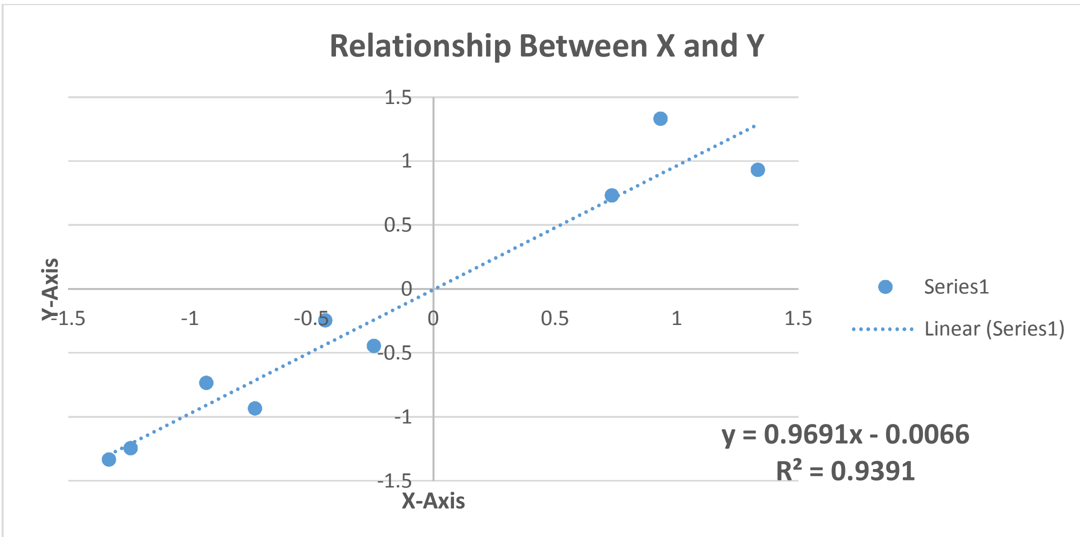

Fig 5.4. Graph showing the relationship between the probability proportions of payoff.

Table 5.5. Payoff Matrix for Example 2

Company X's strategies	**Probabilities**	**Company Y's Strategies**		
		Y_1	Y_2	Y_3
		Probabilities		
		y_1	y_2	y_3
X_1	x_1	3	3	0
X_2	x_2	3	0	3
X_3	x_3	0	1	3

We formulate a linear programming problem for Company X from the payoff matrix

$$Max\ v = \min(v_1, v_2, v_3)$$

Subject to

$$v_1 = 3x_1 + 3x_2 + 0x_3$$
$$v_2 = 3x_1 + 0x_2 + 1x_3$$
$$v_3 = 0x_1 + 3x_2 + 3x_3$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Using Microsoft Excel we have the following results. The tables below, consist of probabilities for the strategies, the probability proportion of the payoff, and the value of the game for each player.

We formulate a linear programming problem for Company Y from the payoff matrix

$$Min\ w = \max(w_1, w_2, w_3)$$

Subject to

$$w_1 = 3y_1 + 1y_2 + 0y_3$$
$$w_2 = 3y_1 + 0y_2 + 3y_3$$
$$w_3 = 0y_1 + 1y_2 + 3y_3$$
$$y_1 + y_2 + y_3 = 1$$
$$y_1, y_2, y_3 \geq 0$$

Again using Microsoft Excel we have the following results.

Table 5.6. Table showing solution of company *X*. This solution consist of probabilities for the strategies, the probability proportion of the payoff, and the value of the game of player *X*

$\boldsymbol{x_1}$	**0.42874**	1.286221	1.286221	0	
$\boldsymbol{x_2}$	**0.142529**	0.427586	0	0.427586	
$\boldsymbol{x_3}$	**0.428731**	0	0.428731	1.286192	
sum	1	v_1 1.713808	v_2 1.714952	V_3 1.713779	MAXMIN 1.713779

Table 5.7. Table showing solution of company *Y*. This solution consist of probabilities for the strategies, the probability proportion of the payoff, and the value of the game of player *Y*

y_1	0.14287	0.428608819	1.285697	0	1.714306
y_2	0.428566	0.428608819	0	1.285693756	1.714303
y_3	0.428565	0	0.428566	1.285693756	1.71426
sum	1	w_1 1.714306244	w_2 1.714303	w_3 1.714259564	MINIMAX 1.714306

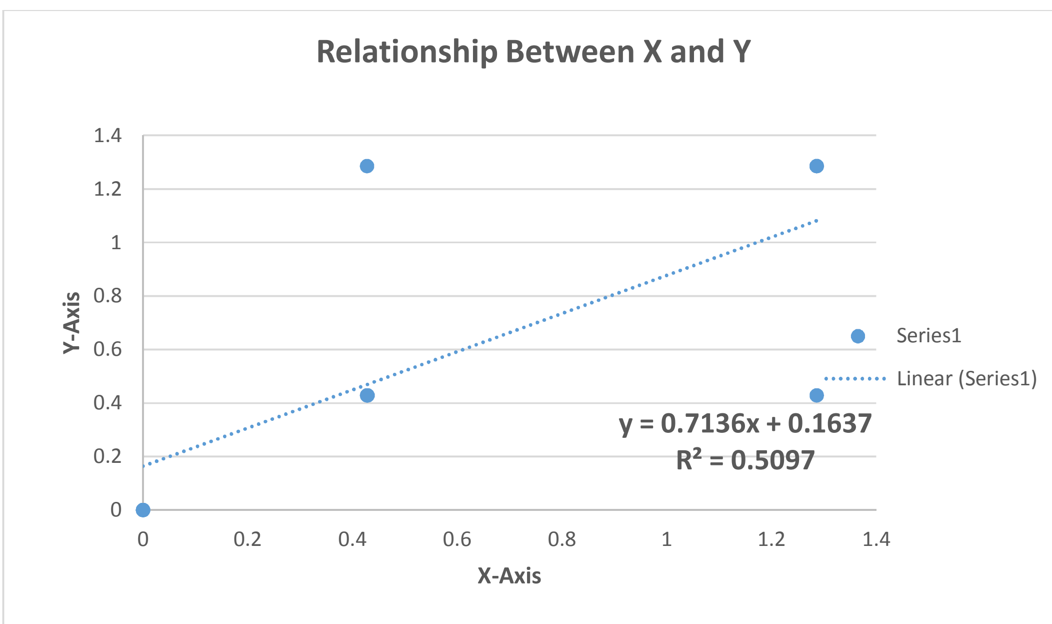

Fig. 5.8. Graph showing the relationship between the probability proportions of payoff

6 Conclusion

In this work, we saw that there are strategies that can be applied to the game theory the most popular of them all are the pure, mixed and optimal strategies. A two-person game player using the mixed strategies, there must be the need for the players to know the percentages of each strategy to be used. In determining these probabilities, the following tools are employed in solving for these possibilities, these include solving algebraically, graphically, using differential calculus and the linear programming method. We saw that the system of linear equation may be a bit complex to build when the number of strategies involve is much and it is even impossible to determine the strategies probabilities graphically, when we have a payoff matrix of 3×3 mixed strategies of two player game, that player 1 and player 2 have three(3) strategies each. For the case when it is 2×3 or 3×2 mixed strategies of two player game, graphical method one of the sets of strategies and later converted to dual of the linear programming of the other player, which can be solved algebraically. The best method to solve for the probability of $m \times n$ mixed strategies of two player game is by setting up a linear programming problem from it and using a mathematical software to determine the objective function of each player, putting the condition that these probabilities must sum up to one(1). Using Microsoft Excel, we saw that the higher of the difference between the value of the game of the two players, the lesser the linear correlation of the probability\proportion of the payoff. This is shown graphically. Other areas of green are the area of research to give generalise procedure that determined the equilibrium point of cooperative and non-cooperative game and how their strategic probabilities can be determined.

Competing Interests

Authors have declared that no competing interests exist.

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