The focus of this book is on meeting the Mathematical needs of students in Senior High Schools who will be taking the West Africa Senior School Certificate Examination (WASSCE) and students preparing for the Private Candidates Examination. For the reason that the student-teacher ratio is uncomfortably high in our SHS, individual attention to students in the classroom is generally not practicable. Hence, the need for text books written for SHS to be necessarily detailed as this book to enable students follow it independently without any supervision. This book is also written to serve as an introductory text for undergraduates and other tertiary students.

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Part 1

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CONTENTS

CHAPTER 1

SETS AND OPERATIONS ON SETS

1.1 **Introduction**

A ***Set*** is any ***well-defined*** collection of objects possessing a common property that enables us to determine readily whether a given object is in the collection or not.
The ***objects*** of a set are referred to as its ***members*** or ***elements.*** Sets are denoted by ***upper case*** letters and the elements by ***lower case*** letters. The elements of a set are enclosed in ***curly brackets*** denoted by { }.

Example 1.1

1. $\{2,4,6,8\}$ is the set of even positive numbers less than ten
2. $\{2,3,5,7,\}$ is the set of prime numbers less than ten
3. $\{3,6,9...99\}$ is the set of multiples of three less than a hundred
4. $\{1,4,9,16,25,36...\}$ is the set of squares
5. {April, June, September, November} is the set of months with exactly 30 days

NB

1. $\in$ is used to mean "belong to" or "is an element of" or "is a member of".

2. $\notin$ is used to mean "does not belong to" or "is not a member of" or "is not an element of".
Thus, if ***S*** is the set of prime numbers, we write $37 \in S$ to mean 37 is an element of the set of prime numbers.
In contrast, $36 \notin S$ means 36 is not a member of the set of prime numbers.
3. When listing the elements of a set, we use a series of three dots to indicate a continuing pattern. For instance, the set of natural numbers is written as: N ={1, 2, 3, …}

1.2 **Specifying Sets**

There are essentially two ways to display the elements of a particular set. These include:

1.2.1 **The Listing Method**

In this case, some or all the elements of a set are listed out and enclosed in curly brackets. This method is illustrated in example 1.2 below.

Example 1.2

1. $\{2,4,6,8\}$ is the set of even positive numbers less than ten
2. $\{2,4,6,8,10,12,14,16,18\}$ is the set of even positive integers less than twenty
3. $\{1,4,9,16,25,36,...\}$ is the set of square numbers

1.2.2 **The Relation Method**

This method is also called the ***Set-builder notation*.** Thus, it is another method of describing sets by a certain rule or property satisfied by all the elements of the given set. In this notation, we use variables (a letter used to stand for some numbers) to represent the elements of the set. The set is then built from the variable and a description of the numbers that the variables represents. This method is illustrated in example 1.3 below.

Example 1.3

1. The set S = {1, 2, 3, …, 49} is written in set-builder notation as:
   S = {x:x is a natural number less than 50}
   This notation is read as "S is the set of numbers x such that x is a natural number less than 50.
2. We write T the set of integers between 1 and 90 in set-builder notation as: T = {x:x is an integer, and $1 < x < 90$}

NB

In the above examples, x is a symbol for a member of the sets S and T, and the colon **':'** means 'such that'.

1.3 **Types Of Sets**

1.3.1 **Finite Sets**

A Set is said to be finite if the process of counting the elements of the set ***terminates***. In short, a finite set is a set whose first and last elements ***are known***.

Example 1.4

1. $\{2,4,6,8\}$ 2. $\{2,3,5,7\}$ 3. $\{3,6,9...99\}$ are all finite sets.

1.3.2 **Infinite Sets**

A set is infinite if the process of counting the elements of the set ***does not*** terminate. In short, it is a set in which either the first or the last element or both are ***not known***.

Example 1.5

1. $\{1,4,9,16...\}$ is the set of square numbers
2. $\{...-3,-2,-1\}$ is the set of negative integers
3. $\{...-1,0,1,2,...\}$ is the set of integers

1.4 **Set Properties**

1.4.1 **The Null Set**

A set is said to be ***null or empty*** if it contains no element(s). It is denoted by ϕ or $\{\}$. Example 1.6 illustrates empty sets.

Example 1.6

1. {days of the week starting with Q}
2. {months of the year with less than 28 days}
3. {men with two left feet}

NB

1. A set that contains only an empty set ϕ is not empty. i.e. if $S = \{\phi\}$, then $S \neq \phi$
2. Again, the set $\{0\}$ is not an empty set since it has only one element, the zero (0).

1.4.2 **The Unit Set**

A set with exactly one member is called a ***unit set*** or a ***singleton***.

Example 1.7

1. The set of months of the year with less than 30 days ie. {February}
2. The set of prime even numbers ie. {2}

1.4.3 **The Universal Set**

This is the set of all objects under consideration. Usually, we can always find a bigger or larger set containing all the members of a particular set under consideration. This larger set is referred to as the universal set and denoted by U or ε.

Example 1.8

1. {Students of GHANASCO} is a universal set for {Science students of GHANASCO}
2. {Games} is the universal set for {soccer, tennis}

3. {European languages} is the universal set for {English, French}

1.4.4 **Equal Sets**

Two sets A and B are said to be ***equal*** denoted $A = B$ if and only if they contain exactly the same elements but possibly with different listings.

Example 1.9

1. If $A = \{1,2,3\}, B = \{3,2,1\}$, then $A = B$ since they have the same elements
2. Given that A = {factors of 10}, B = {1, 2, 5, 10} and C = {x:x divides 10}, then A = B = C since all have the same members
3. {x:3x=9} and {3} are said to be equal

NB

The ***order*** of a set ***A*** is the number of distinct elements in ***A*** denoted by $ord(A)$ or $n(A)$ or $|A|$

For instance, let $A = \{1,2,3,4,5\}, B = \{a,b,c,d\}$, then $|A| = 5$ $and |B| = 4$

1.4.5 **Equivalent Sets**

Two sets A and B are said to be ***equivalent*** denoted by $A \equiv B$ or $A \sim B$ if and only if they have the same ***order*** but possibly with different elements.

Example 1.10

1. If $A = \{1,2,3,4\}, B = \{a,b,c,d\}$ then $A \equiv B$ since $|A| = |B| = 4$
2. {days of the week} and {1, 2, 3,…} are equivalent sets
3. {factors of 14} and {factors of 8} are equivalent

1.4.6 **Subsets**

A set N is called a ***subset*** of set M or N is ***contained in*** M denoted by $N \subseteq M$ if every element of N is a member of M (with N possibly equal to M).

NB

N is said to be a ***Proper*** subset of M represented by $N \subset M$ if $N \subseteq M$ but $N \neq M$. ie. There is at least some element of M that is not in N.
If there is a possibility of $N = M$, then we say that $N \subseteq M$ is an improper subset.

Example 1.11

1. {odd integers} is a ***proper subset*** of {all integers}
2. {positive numbers less than ten} is a ***proper subset*** of {numbers less than or equal to ten}
3. If S = {Teachers in Navasco} and T = {Workers in Navasco}, then we can conclude that S is a ***proper subset*** of T
4. If $A = \{2,4,6,8\}, B = \{8,6,4,2\}$ then ***A*** is an ***improper subset*** of ***B*** and vise versa. i.e. $A \subset B$ *or* $B \supset A$ since $A = B$

NB

1. Every set is an improper subset of itself

2. The empty set ϕ is a subset of every set

1.4.7 **Power Sets**

Let A be a set. The family or collection of all subsets of A is called the ***power set*** of A, denoted by $P(A)$ or 2^A. If the order of A is n, i.e. $|A| = n,$ then the order of $P(A) = 2^n$
i.e. $|P(A)| = 2^{|A|}$

Example 1.12

1. If $A = \{1,2,3\}$, then $|A| = 3$
$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
which has $2^3 = 8$ elements

2. Let $B = \{a,c\}$ then $|B| = 2$
and $P(B) = \{\phi, \{a\}, \{c\}, \{a,c\}\}$
which has $2^2 = 4$ elements

1.5 **Operations On Sets**

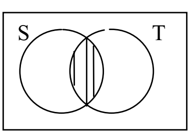
An operation on a set is called a ***unitary operation*** and that between two sets is called a ***binary operation.***
Here, we shall also represent elements of given sets on a diagram called the Venn diagram showing the relationships between the various sets under each operation. In each region of the diagram, we record the respective element with a dotted point (**.**) placed before the element but avoiding commas (,) inside the diagram.

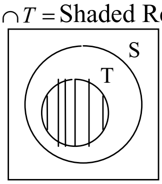
1.5.1 **Intersection Of Sets** $(\cap)$

Another way to form a new set from two known sets is by considering only those elements that the two sets have in common.
Thus, if ***S*** and ***T*** are two sets, then the intersection of ***S*** and ***T*** written as $S \cap T$ is the set of all elements ***common*** to both ***S*** and ***T***.
In notational form: $S \cap T = \{x : x \in S, x \in T\}$

Illustration

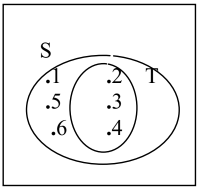
In the diagram below, $S \cap T =$ Shaded Regions





Example 1.13

Let $S = \{1,2,3,4,5,6\}, T = \{2,3,4\}$ then $S \cap T = \{2,3,4\} = T$
i.e. we pick the elements that are found in both S and T
Thus, we illustrate the result diagrammatically as below:



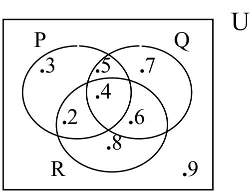
Example 1.14

$P, Q \, and \, R$ are subsets of $\{2,3,4,5,6,7,8,9\}$. If
$P = \{x : x < 6\}, Q = \{y : 3 < y < 8\}$ and
$R = \{Even \ numbers\}, \ find \ P \cap Q \cap R$

Solution

From question,
$U = \{2,3,4,5,6,7,8,9\}, P = \{x : x < 6\} = \{2,3,4,5\}, Q = \{y : 3 < y < 8\} = \{4,5,6,7\}$
$and \ R = \{Even \ numbers\} = \{2,4,6,8\}$
$\Rightarrow P \cap Q \cap R = \{4\}$

Illustration



Example 1.15

Given that A, B, C are subsets of the universal set μ of real numbers such that:

$A = \{1, 2, ..., 16\}$ $\quad B = \{x : 0 < x < 16\}$ *where x is an odd integer*,

$C = \{P : P < 16\}$, *where P is prime.*

(i) List all the elements of B

(ii) Find $B \cap C$

(iii) Find $(A \cap B)'$

Solution

$(i)\ B = \{x : 0 < x < 16\} = \{1,3,5,7,9,11,13,15\}$

$(ii)\ C = \{P : P < 16\} = \{2,3,5,7,11,13\}$

$\Rightarrow B \cap C = \{3,5,7,11,13\}$

$(iii)\ A \cap B = \{1,3,5,7,9,11,13,15\}$

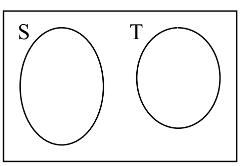
$\Rightarrow (A \cap B)' = \{2,4,6,8,10,12,14,16\}$

1.5.2 **Disjoint Sets**

If two sets ***S*** and ***T*** are such that $S \cap T = \phi$, we say that ***S*** and ***T*** are ***disjoint***.
i.e. there is no element common to ***S*** and ***T***.

Illustration

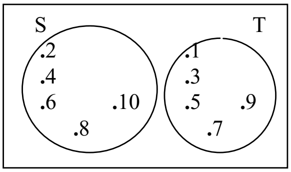
In the diagram below, the sets ***S*** and ***T*** are disjoint.



Example 1.16

Let $S = \{2,4,6,8,10\}, T = \{1,3,5,7,9\}$ then $S \cap T = \phi$, hence ***S*** and ***T*** are disjoint sets.

Illustration

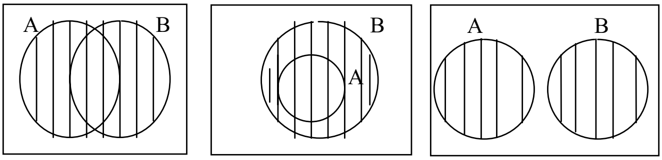


1.5.3 **Union Of Sets**

Any two sets can be combined to form a new set called their ***union*** that consists of all the elements of both sets avoiding repetition. Thus, the union of the two sets ***A*** and ***B*** denoted by $A \cup B$ is the set of elements which are either in ***A*** or ***B*** or both. i.e. $A \cup B = \{x | x \in A$ or $x \in B$ or $x \in A \cap B\}$

Illustration

The shaded regions in the Venn-diagrams below shows $A \cup B$



Example 1.17

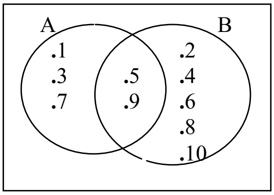
If $A = \{1,3,5,7,9\}$, $B = \{2,4,5,6,8,9,10\}$ *and* $C = \{0,6\}$
$A = \{1,3,5,7,9\}, B = \{2,4,6,8,10\}$, and C = $\{0, 6\}$ find (i) $A \cup B$ (ii) $A \cup C$

Solution

(i) Putting all the elements of A and B together but avoiding repetition gives;

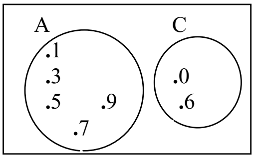
$A \cup B = \{1,2,3,4,5,6,7,8,9,10\}$

Illustration



(ii) *Similarly*, $A \cup C = \{0,1,3,5,6,7,9\}$

Illustration



Example 1.18

If $P = \{x : 0 \leq x < 9\}$, $Q = \{x : 2 < x < 5\}$ *and* $R = \{x : 4 \leq x \leq 6\}$,

find $P \cap (Q \cup R)$, *where x is an integer.*

Solution

First, $P = \{x : 0 \leq x < 9\} = \{0,1,2,3,4,5,6,7,8\}$, $Q = \{x : 2 < x < 5\} = \{3,4\}$

$R = \{x : 4 \leq x \leq 6\} = \{4,5,6\}$

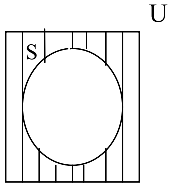
$\Rightarrow Q \cup R = \{3,4,5,6\} \therefore P \cap (Q \cup R) = \{3,4,5,6\}$

1.5.4 **Compliment Of A Set**

The compliment of a set ***S*** denoted by S^1 or S^c is the set of all elements not in ***S*** but which are in the universal set, ***U***.
i.e. $S^c = \{x | x \notin S\} = \{x | x \in U, x \notin S\}$

Illustration

The shaded region illustrates S^c



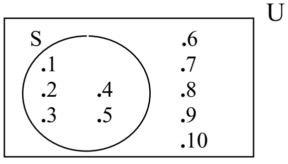
Example 1.19

If $U = \{1,2,3,...10\}$ *and* $S = \{1,2,3,4,5\}$ find S^c.

Solution

Picking the elements of U that are not in S we have: $S^c = \{6,7,8,9,10\}$
i.e. From the diagram below, we listed the elements in the rectangle that are not in the circle.

Illustration



Example 1.20

Given that $U = \{2,3,5,7\}$, $P = \{2,5\}$ *and* $Q = \{5,7\}$ find

$(i)\ (P \cap Q)^1$ $(ii)\ P^1 \cup Q^1$ and state the relationship between (i) and (ii)

Solution

$U = \{2,3,5,7\},\ P = \{2,5\}\ and\ Q = \{5,7\}$
$(i)\ P \cap Q = \{5\} \Rightarrow (P \cap Q)' = \{2,3,7\}$

$(ii)\ First,\ P^1 = \{3,7\}\ and\ Q^1 = \{2,3\}$
$\therefore P^1 \cup Q^1 = \{2,3,7\}$
The relationship between (i) and (ii) is that $(P \cap Q)^1 = P^1 \cup Q^1$

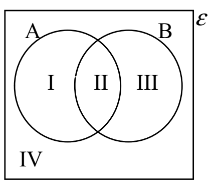
1.6 **Euler - Venn Diagrams**

An Euler - Venn diagram for sets consists of ***Circles*** (each representing a set) drawn inside a ***rectangle***, which represents the universal set relative to which the sets are being considered. We shall be looking at two and three set problems.

1.6.1 **Two-Set Problems**

Two set problems involve two sets represented using the Euler-Venn diagram.
In general, if ***A*** and ***B*** are two sets, then
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Illustration of regions of two sets on the Venn diagram



In the region I of the above figure, the elements belong to A and not to B. This means the elements are in A **and** (**not** B). In set language, we write $A \cap B^1$.

Similarly, in region II, the elements belong to both A and B. This means the elements are in A **and** at the same time are also in B. In set language we write $A \cap B$.
Further, in the region III, the elements belong to B and not to A. This means the elements are in B **and** (**not** A). In set language, we write $A^1 \cap B$.
Lastly, the region IV represents all elements that belong to neither in A nor to B. This simply means the complement of A and that of B. In set language, we write $A^1 \cap B^1$ *or* $(A \cup B)^1$.

NB

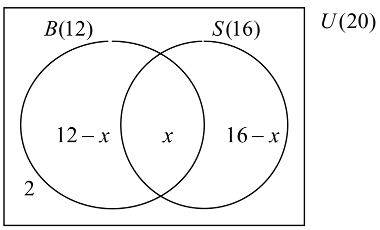
In drawing the Venn diagram for any two-set problem, we always have to first represent the region that belong to the two sets.

Example 1.21

In a class of 20 boys, 16 play Soccer, 12 play Basket ball and 2 are not allowed to play either games. How many play both Soccer and Basket ball?

Solution

Let ***S*** and ***B*** denote the set of boys playing Soccer and Basket ball respectively.
Then $n(U) = 20$
Let x denote the number playing both games



Since $(L \cup S)^c = 2$, we have;

$2+(16-x)+(12-x)+x=20$

$2+16-x+12-x+x=20$

$30-x=20$

$30-20=x$

$\therefore x=10$

Therefore ten boys play both games

Example 1.22

In a class of 40 students, 30 of them read Chemistry and 20 read Physics. What is the probability that a student chosen at random in the class reads both Chemistry and Physics?

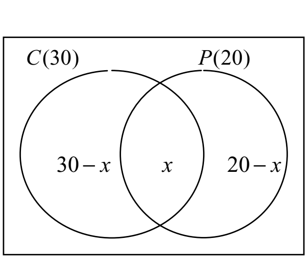
Solution

$U=\{No\ of\ students\ in\ class\} \Rightarrow n(U)=40$

$C=\{Chemistry\ students\} \Rightarrow n(C)=30$

$P=\{Physics\ students\} \Rightarrow n(P)=20$

$Let\ x = Students\ who\ read\ both\ Chemistry\ and\ Physics$



From Venn diagram,

$30-x+x+20-x=40$

$50-x=40$

$50-40=x$

$\Rightarrow x=10$

Therefore, the number of students who read both subjects were 10 and the total number of students in the class was 40.
Therefore, the probability that a student chosen at random

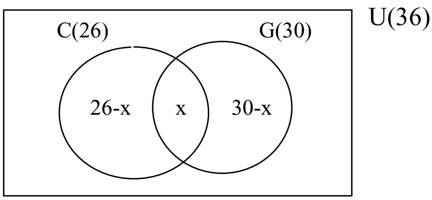
reads both subjects $= \frac{10}{40} = \frac{1}{4}$

Example 1.23

There are 36 students in a class. Each student offers at least one of Geography or Commerce. If 30 students offer Geography and 26 offer Commerce, how many students offer Commerce **only**?

Solution

$From\ question,$
$Let\ Geography\ students\ be\ n(G) \Rightarrow n(G) = 30$
$let\ Commerce\ students\ be\ n(C) \Rightarrow n(C) = 26$
$Let\ no\ of\ students\ who\ offer\ both\ G\ and\ C\ be\ x \Rightarrow n(C \cap G) = x$



From the Venn diagram,
$36 = 26 - x + x + 30 - x$
$36 = 56 - x$
$\Rightarrow x = 56 - 36 = 20$
$\Rightarrow n(Commerce\ only) = 26 - x = 26 - 20 = 6$
Therefore, the number of students who offer Commerce only is 6.

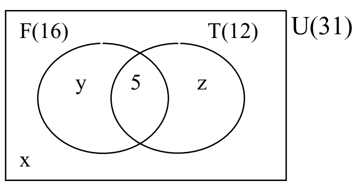
Example 1.24

In a class of 31 students, 16 play football, 12 play table-tennis and 5 play both games. Find the number of students who play:

(i) **at least** one of the games
(ii) **none** of the games

Solution

Let
U = {Students in the class} $\Rightarrow n(U) = 31$
F = {Students who play football} $\Rightarrow n(F) = 16$
T = {Students who play table-tennis} $\Rightarrow n(T) = 12$
But Students who play both games = 5
x = {Students who play none of the games}



From Venn diagram,
Considering region F,
$16 = y + 5 \Rightarrow 16 - 5 = y \Rightarrow y = 11$
Considering region T,
$12 = z + 5 \Rightarrow 12 - 5 = z \Rightarrow z = 7$
Therefore , students who play only football are 11 and those who play only table-tennis are 7

(i) Number of students who play at least one of the games
= those who play exactly one game + those who play both games
= 11 + 7 + 5 = 23

(ii) Number who play none of the games
= Total number of students in class – number that play at least one game
= 31 – 23 = 8

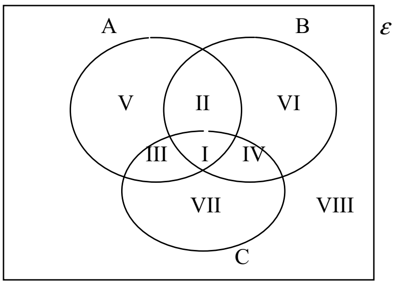
1.6.2 **Three-Set Problems**

These problems involve three sets represented using Euler - Venn diagrams.
In general,

$(A \cup B \cup C) =$

$n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Illustration of regions of three sets on the Venn diagram



In the region I of the diagram above, the elements belong to all A, B and C. Thus, the intersection of all the three sets. This is also referred to as the ***three-set region***. In the set language, we write $A \cap B \cap C$. In region II, the elements belong to A **and** B **only**. This means the elements are in A **and** B and (**not** C).

Thus, we write $A \cap B \cap C^1$.

Similarly, in region III, the elements belong to A **and** C **only**. This means the elements are in A **and** C and (**not** B).

Thus, we write $A \cap B^1 \cap C$.

Again, in region IV, the elements belong to B **and** C **only**. This means the elements are in B **and** C and (**not** A).

Thus, we write $A^1 \cap B \cap C$.

NB

The regions II, III and IV are all together referred to as the ***two-set region***.

In region V, the elements belong to A, not to B and not to C. This means the region can be described as A **and not** B and **not** C. In set language we write: $A \cap B^1 \cap C^1$.

We could also describe region V as A but neither B nor C meaning A **and not** (either B **or** C). This translate into set language as $A \cap (B \cup C)^1$. *Thus*, $A \cap (B \cup C)^1 = A \cap B^1 \cap C^1$.

In region VI, the elements belong to B, not to A and not to C. This means the region can be described as B **and not** A and **not** C. In set language we write: $A^1 \cap B \cap C^1$. *Or* $B \cap (A \cup C)^1$.

In region VII, the elements belong to C, not to A and not to B. This means the region can be described as C **and not** A and **not** B. In set language we write: $A^1 \cap B^1 \cap C$.

NB

The regions V, VI and VII are all together referred to as the ***one-set region***.

Region VIII describes all elements **not** in A, **not** in B **and not** in C but are in the universal set. This region is also referred to as the ***complementary region***.

Thus, we write $A^1 \cap B^1 \cap C^1$ *or* $(A \cup B \cup C)^1$.

Example 1.25

100 vehicles took the ministry of transport test and 60 passed. Among the remainder, faults in brakes, lights and steering occurred as follows:

Brakes only **12**

Brakes, steering and light **3**

Steering and light only **2**

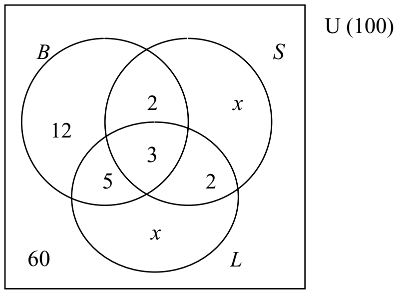
Brakes and steering **5**

Brakes and light **8**

Equal numbers of cars having one fault because of steering or lighting deficiency. How many cars had faulty lights? How many cars had only one fault?

Solution

Let ***B***, ***S*** and ***L*** represent the set of cars having brakes, steering and light faults respectively. Let x denote the number of cars having only one fault due to steering or lighting deficiency.



$n(B \cup S \cup L) = 100 - 60 = 40$ Since 60 passed.

$x + x + 12 + 2 + 3 + 5 + 2 = 40$

$2x + 24 = 40$

$2x = 40 - 24$

$\frac{2x}{2} = \frac{16}{2} = 8$

$\therefore x = 8$

Therefore, 8 had faulty lighting and $(8+8+12) = 28$ cars had only one fault.

Example 1.26

100 members of a community were asked to state the activities they undertake during the day. 38 go to school, 54 go for fishing, 50 engage in trading, 10 go to school and also fish, 18 go to school and also trade, 22 go for fishing and also trade. Each of these members undertakes at least one of the activities. The number of people who go to school only is the same as the number who engage in trading only. Use the information to find the number of people who (a) undertake all the three activities

(b) go to school only

Solution

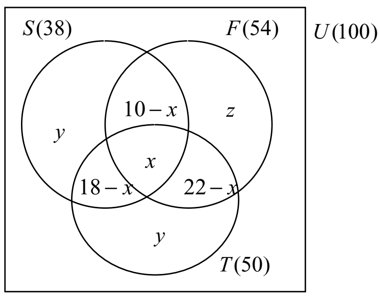
From question,

$n(U) = 100,\ n(S) = 38,\ n(F) = 54\ \text{and}\ n(T) = 50$

$n(S\cap F)=10,\ n(S\cap T)=18\ and\ n(F\cap T)=22$

$Let\ x=n(S\cap F\cap T),\ y=n(S\cup F'\cup T')=n(T\cup S'\cup F')\ and$

$z=n(F\cup S'\cup T')$



Considering region S,

$$y=38-(10-x+x+18-x)$$
$$=38-(28-x)$$
$$=38-28+x$$
$$=10+x$$
$$\Rightarrow y=10+x$$

Considering region F,

$$z=54-(10-x+x+22-x)$$
$$\Rightarrow z=22+x$$

Replacing $y\ and\ z$ and finding the union we have;

$$50+10+x+10-x+22+x=100$$
$$92+x=100$$
$$x=100-92=8$$

Therefore, the number of students who

(a) undertake all the three activities is 8

(b) go to school only is $y=10+x=10+8=18$

Example 1.27

A group of 34 women sells at least one of the following foodstuffs: yam, maize and plantain. Of these, 22 sell yam, 14 sell maize, 18 sell plantain, 7 sell both yam and maize, 9 sell yam and plantain and no one sells all the three items.

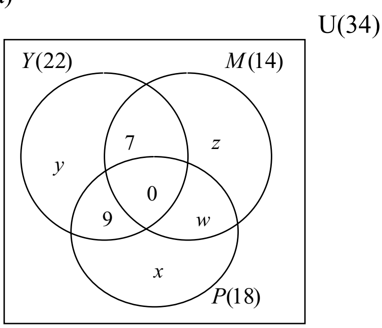
(a) Draw a Venn diagram to illustrate this information

(b) Find the number who sell maize and plantain

(c) What is the probability that a woman selected at random from the group sells plantain ***only***?

Solution

(a)



From set Y we get;

$y = 22 - 16 = 6$

Again, considering the whole universal set;

$14 + 6 + 9 + x = 34$

$x = 34 - 29 = 5$

From P;

$w = 18 - 14 = 4$

(b) Therefore, the number who sell maize and plantain are 4.

(c) P(Plantain only) = $\frac{5}{34}$

Example 1.28

A group of 40 employees at EL-AK SERIES PUBLICATIONS were each required to have interest in Teaching, Nursing and Piloting. However, when the CEO checked their interests, he found out that:

20 had interest in Teaching, 24 the Nursing and 22 the Piloting.

10 had interest in Teaching and Nursing.

12 had interest in Teaching and Piloting.
16 had interest in only two of the three professions.
If every employee had interest in at least one of the professions, find how many employees:

(i) Had interest in all the three professions
(ii) Had interest in only Nursing and Piloting

Solution

Let U be the employees of the Enterprise
T be those who had interest in Teaching
N be those who had interest in Nursing
P be those who had interest in Piloting
Thus,
$n(U) = 40,\ n(T) = 20,\ n(N) = 24,\ n(P) = 22,\ n(T \cap N) = 10,\ n(T \cap P) = 12$
Where for example, n(T) means the number of elements in T and so on.
Let the number of employees who had interests in all the professions be x.
Implies;
$n(T \cap N \cap P) = x$

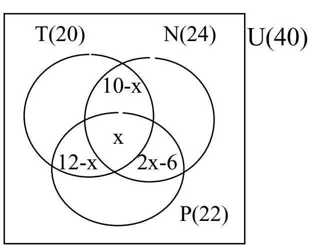
$Since\ n(T \cap N) = 10 \Rightarrow n(T \cap N \cap P^1) = 10 - x$

$Similarly,\ n(T \cap P) = 12 \Rightarrow n(T \cap P \cap N^1) = 12 - x$

But 16 employees had interest in only two of the three professions.
Therefore,
$n(N \cap P \cap T^1) + (10 - x) + (12 - x) = 16$

$\Rightarrow n(N \cap P \cap T^1) = 2x - 6$

The information is shown on the venn diagram below:



Now considering region T,

$$n(T) = n(T \cap N' \cap P') + n(T \cap N' \cap P) + n(T \cap N \cap P) + n(T \cap N \cap P')$$

$$20 = n(T \cap N' \cap P') + 12 - x + x + 10 - x$$

$$20 = n(T \cap N' \cap P') + 22 - x$$

$$\Rightarrow 20 - 22 + x = n(T \cap N' \cap P')$$

$$\Rightarrow x - 2 = n(T \cap N' \cap P')$$

Similarly from region N,

$$n(N) = n(T' \cap N \cap P') + n(T' \cap N \cap P) + n(T \cap N \cap P) + n(T \cap N \cap P')$$

$$24 = n(T' \cap N \cap P') + 2x6 + x + 10 - x$$

$$24 = n(T' \cap N \cap P') + 2x + 4$$

$$24 - 2x - 4 = n(T' \cap N \cap P')$$

$$20 - 2x = n(T' \cap N \cap P')$$

Lastly, from region P,

$$n(P) = n(T' \cap N' \cap P) + n(T' \cap N \cap P) + n(T \cap N \cap P) + n(T \cap N' \cap P)$$

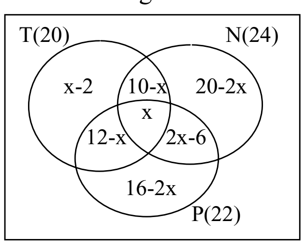
$$22 = n(T' \cap N' \cap P) + 2x - 6 + x + 12 - x$$

$$22 = n(T' \cap N' \cap P) + 2x + 6$$

$$22 - 2x - 6 = n(T' \cap N' \cap P)$$

$$16 - 2x = n(T' \cap N' \cap P)$$

The Venn diagram



U(40)

But n(TUNUP) = Summation of all the regions in the three sets (circles).

Implies, summing gives,

$n(T \cup N \cup P) =$

$(16-2x)+(2x-6)+(12-x)+x+(x-2)+(10-x)+(20-2x)$

$\Rightarrow 40 = 50-2x$

Grouping liked terms gives;

$2x = 50-40$

$\frac{2x}{2} = \frac{10}{2} \Rightarrow x = 5$

Hence,
(i) 5 employees had interest in all the three professions
(ii) Number who had interest in only Nursing and Piloting
= 2x-6 = 2(5)-6= 4

1.7 **Properties Of Set Operations**

1.7.1 **Commutative Property**

Let S and T be sets, then by the Commutative property,

(i) $S \cap T = T \cap S$ Commutative law for Intersection
(ii) $S \cup T = T \cup S$ Commutative law for Union

Example 1.29

Given that X = {2, 4, 5, 7, 10} and Y = {1, 4, 6, 7, 9, 10}
Show that: $(i)\ X \cap Y = Y \cap X \quad (ii)\ X \cup Y = Y \cup X$

Solution

$(i)\ X \cap Y = \{4,7,10\}\ \ and\ \ Y \cap X = \{4,7,10\}$

$\Rightarrow X \cap Y = Y \cap X$

$(ii)\ X \cup Y = \{1,2,4,5,6,7,9,10\}\ \ and\ \ Y \cup X = \{1,2,4,5,6,7,9,10\}$

$\Rightarrow X \cup Y = Y \cup X$

1.7.2 **Associative Property**

Let ***S***, ***T*** and ***R*** be sets, then;

(i) $S \cap (T \cap R) = (S \cap T) \cap R$ Associative law for Intersection
(ii) $S \cup (T \cup R) = (S \cup T) \cup R$ Associative law for Union

Example 1.30

If A = {2,4,6,8,10}, B = {1,2,4,5,7} and C = {1,2,3,4,7,10}
Show that: $(A \cap B)C = A \cap (B \cap C)$

Solution

$A \cap B = \{2,4\} \quad \Rightarrow (A \cap B) \cap C = \{2,4\}$
$B \cap C = \{1,2,4,7\} \quad \Rightarrow A \cap (B \cap C) = \{2,4\}$
$Hence,\ (A \cap B) \cap C = A \cap (B \cap C)$

Example 1.31

Given that: $Q = \{c, d, e, f\},\ R = \{a, c, d, e, g\}\ \ and\ \ S = \{a, b, c\}$
Show that: $Q \cup (R \cup S) = (Q \cup R) \cup S$

Solution

$R \cup S = \{a, b, c, d, e, g\} \quad \Rightarrow Q \cup (R \cup S) = \{a, b, c, d, e, f, g\}$
$Q \cup R = \{a, c, d, e, f, g\} \quad \Rightarrow (Q \cup R) \cup S = \{a, b, c, d, e, f, g\}$
$Hence,\ Q \cup (R \cup S) = (Q \cup R) \cup S$

1.7.3 **Distributive Property**

Let S, T and R be sets, then;

(i) $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$ Distributive law
(i.e. Intersection is distributive over union)
(ii) $S \cup (T \cap R) = (S \cup T) \cap (S \cup R)$ Distributive law
(i.e. Union is distributive over intersection)

Example 1.32

If A = {1, 2, 5, 8, 9}, B = {2, 5, 8, 9, 10} and C = {5, 9, 12, 13}
Verify that:

$(i)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$(ii)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution

$(i)\ B \cup C = \{2,5,8,9,10,12,13\} \Rightarrow A \cap (B \cup C) = \{2,5,8,9\}$

$Then\ \ A \cap B = \{2,5,8,9\}\ \ and\ \ A \cap C = \{5,9\}$

$\Rightarrow (A \cap B) \cup (A \cap C) = \{2,5,8,9\}$

$Hence,\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$(ii)\ B \cap C = \{5,9\} \Rightarrow A \cup (B \cap C) = \{1,2,5,8,9\}$

$Then\ \ A \cup B = \{1,2,5,8,9,10\}\ and\ \ A \cup C = \{1,2,5,8,9,12,13\}$

$Hence,\ \ (A \cup B) \cap (A \cup C) = \{1,2,5,8,9\}$

$\therefore A \cup (B \cap C) = (A \cup B) \cup (A \cup C)$

1.7.4 **The Indempotent Law**

For any given set S, then;

$S \cap S = S$ and $S \cup S = S$ satisfies the Indempotent law

Example 1.33

If S = {1, 2, 3}, verify the indempotent law on S.

Solution

$S \cap S = \{1,2,3\}\ \ and\ S \cup S = \{1,2,3\}$

$Hence,$

$S \cap S = S\ \ and\ \ S \cup S = S$

EXERCISE

QUE. A

The sets

$A = \{---,-6,-4,-2,0,2,4,6,---\}$, $B = \{x : 0 \le x \le 9\}$ *and* $C = \{x : -4 < x \le 0\}$ are subsets of Z, the set of integers. (i) Describe the members of the set A^1

(ii) Find $A^1 \cap B$ (iii) Represent the sets B *and* C on a Venn diagram

QUE. B

(i) The set $M = \{x : 3x + 4 > 10\}$ is a subset of $N = \{x : 1 \le x \le 9\}$ where x is an integer. What is M?

QUE. C

The sets $P = \{multiples\ of\ 3\}$, $Q = \{factors\ of\ 72\}$ *and* $R = \{even\ numbers\}$ are subsets of $U = \{18 \le x \le 36\}$ (a) List the elements of P, Q *and* R (b) Find $(i)\ P \cap Q\ (ii)\ Q \cap R\ (iii)\ P \cap R$ (c) State the relationship between (i) and (ii)

QUE. D

If

$P = \{x : 0 \le x < 9\}, Q = \{x : 2 < x < 5\}$ *and* $R = \{x : 4 \le x \le 6\}$,

find $P \cap (Q \cup R)$ *where* x *is an integer.*

QUE. E

Given that: $A = \{x : \frac{2x-1}{3} < \frac{x+3}{5}$ *and* $B = \{x : \frac{2x}{3} - 5 < 3x + 2\}$,

find $A \cap B$

QUE. F

In a class of 50 students, 30 offer Economics, 17 offer Government and 7 offer neither Economics nor Government. How many students offer both subjects?

QUE. G

In a class of 32 students, 18 read Chemistry, 16 read Physics and 22 read Mathematics, 6 read all the three subjects, 3 read Chemistry and Physics only and 5 read Physics only. Each student reads at least one subject. Find the number of students who read

(i) Chemistry only (ii) Only one subject (iii) exactly two subjects

QUE. H

A group of 50 girls were asked which of the three colours, red, yellow and green they liked. 5 of them said they liked all three colours, 25 liked red and 22 liked green. 15 liked red and yellow, 12 liked red and green, 4 liked only yellow and 2 liked only green.

(i) Illustrate the information on a Venn diagram
(ii) How many girls did not like any of the three colours?

QUE. I

A survey of 150 traders in a market shows that 90 of them sell cassava, 70 sell maize and 80 sell yam. Also, 26 sell cassava and maize, 30 sell cassava and yam and 40 sell yam and maize. Each of the traders sells at least one of the crops.

(i) Represent the information on a Venn diagram
(ii) Find the number of traders who sell all the three food crops
(iii) How many of the traders sell one food crop only?

QUE. J

There are 100 boys in a sports club. 65 of them play soccer, 50 play Hockey and 40 play Basket ball. 25 of them play soccer and Hockey, 20 play Hockey and Basket ball and 5 play all three games. Each boy plays at least one of the three games. Find the number of boys who play:
(i) Soccer only (ii) Basket ball only (iii) Exactly two games

QUE. K

The CEO of EL-AK SERIES MICROSYSTEMS ENTERPRISE made a survey of the languages spoken by his staff and found that:
40% speak Gonja
45% speak Hausa
60% speak Kasem
15% speak Hausa and Gonja
20% speak Kasem and Gonja
30% speak Kasem and Hausa
10% speak all the three languages
Find what percentage of the enterprise:

(i) Speak only Kasem
(ii) Speak only one of the three languages
(iii) Speak Kasem and Hausa but not Gonja
(iv) Speak exactly two of the three languages
(v) Do not speak any of the languages

QUE. L

A group of men were asked to indicate which of the Ghanaian universities, Legon, UDS or Winneba they attended. It was found out that 30 attended Legon, 23 UDS and 17 Winneba. 7 attended Legon and UDS. 12 attended two of the three universities. It was also found that of those who attended two of the universities could not attend Winneba and 3 could not attend UDS. However, everyone attended at least one of the universities.

(i) How many men were interviewed?

(ii) How many attended only one of the universities?

QUE. M

17 shortlisted candidates for the SRC post in Navrongo Campus include:
11 men; 6 of whom live in Navrongo,
12 married people; 8 of whom live in Navrongo,
12 residents of Navrongo and 9 married men.
There was only one single woman who lived outside Navrongo. How many of the shortlisted candidates were:

(i) Married men from Navrongo?
(ii) Single women?

QUE. N

The 60 Resource Persons at a Mathematics Workshop were asked to indicate which of the three meals, Eba, Amala and Pounded yam they liked. It was discovered that:
20 liked pounded yam
40 liked Eba
30 liked Amala
2 did not like any of the meals
5 liked all the three meals
How many Resource Persons liked:

(i) Exactly two of the meals (ii) Only one of the meals

QUE. O

A number of boys bought EL-AK Series, Approacher's Serie and Aki-Ola Series books. Three bought one of each book. Of the boys who bought two books, 3 did not buy
EL-AK Series, 5 not Approacher's Series and 2 not Aki-Ola Series. The same number of boys bought EL-AK Series only as bought EL-AK Series with other books. The same number bought Aki-Ola Series only as bought Approacher's only. More boys bought EL-AK Series and Aki-Ola Series but not Approacher's Series than bought

Aki-Ola Series only, but more boys bought Approacher's Series only than bought Approacher's Series and Aki-Ola Series but not EL-AK Series. How many boys were there?

QUE. P

A SHS form 3 class of EL-AK Series College SHS were asked to indicate which of the hobbies, Reading, Dancing or Gardening they liked. The results revealed that:
35 liked Gardening
50 liked Reading
65 liked Dancing
28 liked only two of the hobbies
10 did not like any of the three hobbies.
Find how many students liked:

(a) All the three hobbies
(b) Only one of the three hobbies

QUE. Q

Given that D = {3,5,7}, E = {2,4,6,8} and F = {1,2,3,4,5}
List the elements in each of the following set:
$(i) D \cup E$ $(ii) D \cup F$ $(iii) E \cup F$ $(iv)\ (D \cup E) \cap F$
$(v)\ D \cup (E \cap F)$ $(vi)\ (D \cap F) \cup (E \cap F)$ $(vii) (D \cup E) \cap (D \cup F)$

QUE. R

List the elements in each of the following sets:
$(i)\ \{x : x \text{ is an even natural number less than } 20\}$
$(ii)\ \{x | x \text{ is a natural number greater than } 6\}$
$(iii)\ \{x | x \text{ is an odd natural number greater than } 11\}$
$(iv)\ \{x : x \text{ is an odd natural number less than } 14\}$
$(v)\ \{x : x \text{ is an even natural number between } 4 \text{ and } 79\}$
$(vi)\ \{x : x \text{ is an odd natural number between } 12 \text{ and } 75\}$

QUE. S

Write each of the following sets using set-builder notation:
(i) {3, 4, 5, 6} (ii) {1, 3, 5, 7} (iii) {5, 7, 9, 11, …}
(iv) {4, 5, 6, 7, …} (v) {6, 8, 10, 12, …, 82}
(vi) {9, 11, 13, 15, …, 51}

QUE. T

Draw venn diagrams to illustrate each of the pairs of sets A and B and list $A \cup B$.

1. A = {1, 5, 7, 10} and B = {5, 8, 10, 11}
2. A = {a, b, c, d} and B = {1, 2, 3, 4}
3. A = {x, y, z} and B = {a, b, c, d}
4. A = {1, 3, 5} and B = {1, 3, 4, 5, 8}
5. A = {5, 10, 15, 20, 25} and B = {10, 20}

QUE. U

In the following, list the members of the sets A and B. Also draw Venn diagrams to illustrate A and B and list the members of $A \cup B$ *and* $A \cap B$.

1. A = { days starting with S or T} and B = {days with exactly six letters}
2. A = {prime factors of 42} and B = {factors of 54}
3. A = {x:x is an integer and 1<x<7} and
   B = {x:x is a multiple of 3 and $2 < x \leq 12\}$ }
4. A = {x:x is a prime number and $7 \leq x < 20\}$ and
   B = {x:x is an integer 0<x<7}

QUE. V

Verify that:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
For each of the following sets:
1. A = {10, 12, 15}, B = {9, 10, 11, 12, 13} and C = {8, 12, 15}
2. A = {2, 3, 5, 8}, B = {2, 5, 10, 12} and C = {2, 4, 5, 7, 10}

1. A = {a, c, e, f}, B = {a, b, c} and C = {d, e, f}
2. A = {u, w, y, z}, B = {u, v, z} and C = {w, y, z}
3. A = {a, d, e}, B = {b, d, f, g} and C = {a, b, c, d, e}

QUE. W

Of 100 farmers in a village, 65 grow tomatoes and 75 grow onions. Each farmer is known to grow at least one of the two crops. How many farmers grow both crops?

QUE. X

A survey of the reading habits of 130 students showed that 30 read both comics and novels, 10 read neither comics nor novel and twice as many students read comics as read novels.

1. How many students read comics?
2. How many read novels?
3. How many read only comics or only novels?

QUE. Y

In a group of 100 traders, 70 sell gari, 10 sell only rice and 15 sell only maize. 50 sell both gari and rice, 45 sell both rice and maize and 65 sell maize. Each trader sells at least one of the three items. Find the number of traders who sell

(a) all three items
(b) gari and maize but not rice
(c) gari and rice but not maize
(d) rice

CHAPTER 2
LOGICAL REASONING

2.1 **Introduction**

Logic is the basis of all mathematical reasoning that has applications in the design of computing machines, artificial intelligence and computer programming languages.
In short, ***Logic*** is an aspect that examines one's convincing or reasoning power.
A ***logical expression*** or a ***Proposition*** is a statement which represents a true or false condition.
A ***Statement*** is a verbal or written assertion which is either simple or compound.
A ***Simple statement*** is a complete and precise statement with meaning.

Example 2.1

1. Akos is beautiful (simple statement)
2. Abena is sick (simple statement)

A ***Compound statement*** is a statement formed when two simple statements are combined.

NB

Every compound statement ***must*** be able to separate into distinct simple statements

Example 2.2

1. It is raining and it is cold (Compound statement)
2. If Kwame is a Ghanaian then Kwame is an African (Compound statement)
3. Either Aziz is a student or Kofi is in Junior High School (Compound statement)

NB

T and ***F*** are used to denote ***true*** and ***false*** truth values respectively

2.2 **Logical Operators**

These are the connectivities between two or more simple statements. These operators are: **AND** $(\wedge)$, **OR** $(\vee)$, **IMPLICATION** $(\Rightarrow)$ and **EQUIVALENCE** $(\Leftrightarrow)$.

NB

Logical Operators are illustrated using the Truth Table.

A ***Truth Table*** is a table which gives all the truth values of a compound statement.
The general truth table we shall be using for the two simple statements ***p*** and ***q*** is:

p	q
T	T
T	F
F	T
F	F

2.2.1 **The AND** $(\wedge)$ **Operator**

This is also called a ***Conjunction.*** If ***p*** and ***q*** are any two simple statements, then
'***p*** and ***q***' written as $'p \wedge q'$ is only ***true*** when both ***p*** and ***q*** are true and ***false*** if otherwise.
That is, in filling the column for 'p **and** q' on the table, we first locate the rows that contain 'T' for the first column (first statement) and another 'T' for the second column (second statement) and record 'T' and then any other pairing which does not contain two 'Ts' in a row is recorded 'F'.

Truth Table for the '**AND**' operator

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2.2.2 The OR $(\vee)$ Operator

This is also called the ***Disjunction.*** Let ***p*** and ***q*** be two simple statements, then
'***p*** or ***q***' written as $'p \vee q'$ is ***False*** when both ***p*** and ***q*** are false and ***True*** if otherwise.
That is, in filling the column for 'p **or** q' on the table, we first locate the rows that contain F'' for the first column (first statement) and another 'F' for the second column (second statement) and record 'F' and then any other pairing which does not contain two 'Fs' in a row is recorded 'T'.
Truth Table for the '**OR**' operator:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

2.2.3 The IMPLICATION $(\Rightarrow)$ operator

Let ***p*** and ***q*** be simple statements such that '***p*** implies ***q***' written as $'p \Rightarrow q'$ and interpreted as '***If p then q***'.
Thus, $'p \Rightarrow q'$ is ***False*** only if ***p*** is true and ***q*** is false and ***True*** if otherwise.
That is, in filling the column for 'p **implies** q' on the table, we first locate the rows that contain 'T' for the first column (first statement)

and ‘F’ for the second column (second statement) and record ‘F’ and then any other pairing which does not contain a first ‘T’ and a second ‘F’ in a row is recorded ‘T’.

Truth Table for the **‘IMPLICATION’** operator:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

2.2.4 **The EQUIVALENCE** $(\Leftrightarrow)$ **operator**

If ***p*** and ***q*** are simple statements such that $'p \Rightarrow q'$ and $'q \Rightarrow p'$ then ‘***p*** is **equivalent** to ***q***’ and written as $'p \Leftrightarrow q'$.

Thus, $'p \Leftrightarrow q'$ is ***True*** if ***p*** and ***q*** are the same and ***False*** if otherwise.

That is, in filling the column for ‘p is **equivalent** to q’ on the table, we first locate the rows that contain the same truth values (either F-F or T-T) for both the first statement and the second statement and record T and then any other pairing which does not contain the same truth values in a row is recorded F.

Truth Table for **‘EQUIVALENCE’** operator:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

2.5 **Some Useful Properties**

The following are some useful properties

2.5.1 **The NOT** $(\sim)$

This is also called the ***Negation.*** Generally, we negate verbal statements by using the word "***NOT***". Let ***p*** be a statement, then the negation of ***p*** is written as $\sim p$.
Truth Table for 'negation'

p	$\sim p$
T	F
F	T

E.g. If p: The temperature ***is*** below $37°C$
$\sim p$: The temperature is ***not*** below $37°C$

2.5.2 **TAUTOLOGY**

A compound statement whose truth values in the truth table is **'*T*'** throughout is called a ***Tautology.***

Example 2.3

Show that $(p \wedge q) \Rightarrow p$ is a tautology

Solution

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since the truth values of the compound statement in the last column of the truth table are all ***Trues (T's)*** throughout, the statement $(p \wedge q) \Rightarrow p$ is a ***tautology*** ELSE a ***contradiction*** when ***False (F's)*** throughout.

2.6 **VALID ARGUMENTS OR DEDUCTIONS**

An ***argument*** is said to be ***valid*** if and only if the conclusion follows from some statements usually called ***Premises.***

NB

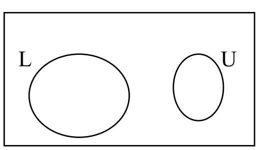
The truth of the conclusion is irrelevant when testing for the validity of an argument.
Thus, we can say that ***Tamale*** is in ***Nigeria*** and the conclusion will be true from the premises given. Meanwhile in reality, Tamale is not in Nigeria.

2.4.1 **Some Key Words**

1. We use ***No, Never, All, do not*** to mean ***disjoint sets***

Illustration

Consider the statement: S: ***No*** Lecturer wears uniform
It means the set L={Lecturers}, U={People who wear uniform} are disjoint sets.



Thus, $L \cap U = \phi$

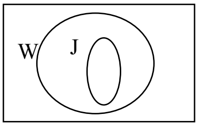
Again, if T is a statement such that T: Sick people are ***never*** active. It implies that $S = \{Sick\ people\}$, $A = \{Active\ people\}$ are disjoint sets.

Further more, if x: ***All*** *JHS* students ***do not*** write WASSCE, it means
$J = \{JHS\ students\}$, $W = \{WASSCE\ students\}$ are disjoint sets.

2. We use **All,** or **No, Not** or **If … Then** to mean ***Subsets***

Illustration

Consider the statement: q: there is ***no*** JHS Student who does ***not*** wear uniform. It means that all JHS Students wear uniforms. That is the set $J = \{JHS\ students\}$ is a subset of the set $W = \{People\ who\ wear\ uniforms\}$



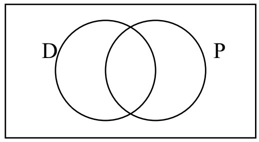
Thus, $J \subset W$

Similarly, the statement: q: ***All*** SHS Students write WASSCE. It means $S = \{SHS\ students\}$ is a subset of
$U = \{People\ who\ write\ WASSCE\}$
Also, the statement: T: ***If*** children take medicine, ***then*** they are sick. It means $C = \{Children\}$ is a subset of "People who take medicine"

3. ***Some,*** or ***Most*** or ***Not all*** means ***Intersection*** of sets

Illustration

Consider the statement: s: ***Some*** Doctors are Pastors.
It means the intersection of the sets
$D = \{Doctors\}\ and\ P = \{Pastors\}$



Thus, $D \cap P$

Similarly, the statement "r: ***Most*** Nursery Pupils speak good English" means the intersection of the sets
$N = \{Nursery\ Pupils\}\ and\ G = \{Pupils\ who\ speak\ good\ English\}$
Again, the statement: "***Not all*** dropouts are useless" means the intersection of the sets $D = \{Dropouts\}\ and\ U = \{useless\ people\}$

Example 2.4

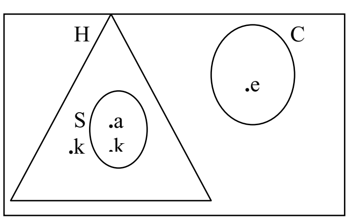
Consider the following two statements:
p: All students are hardworking
q: No hardworking person is careless

(i) Draw a Venn diagram to illustrate the above statements
(ii) Which of the following statements are valid or not valid conclusions from $p\ and\ q$?
   (α) Ama is a student $\Rightarrow$ Ama is not careless
   (β) Kwame is hardworking $\Rightarrow$ Kwame is a student
   (γ) Esi is careless $\Rightarrow$ Esi is not a student

Solution

Let
$U = \{people\},\ H = \{hardworking\ people\},\ C = \{careless\ people\},\ S = \{students\}$
and let also, Kwame = k, Ama = a and Esi = e

(i)



NB

In making valid deductions, we test for the number of regions that the first part of a particular statement can be represented in the Venn diagram drawn. Thus, when the first part of a particularly statement can be represented in more than one region in the Venn diagram, that statement is automatically said to be ***invalid***. Whiles when a statement can only be represented in one region in the Venn diagram, that statement is said to be ***valid***.

(ii) (α) Valid

NB

'Ama is a Student' can only be represented in region S in the Venn diagram and since when Ama is in region **S** she cannot also be in region **C**, we say that it follows from the conclusion that 'Ama is not careless'.

(β) Not valid

NB

'Kwame is hardworking' can be represented in two regions, first in region H but outside the region S and secondly in region S.
Thus, since we have two regions that the given statement can be represented,
it does not follow from the conclusion that 'Kwame is a student'.

(γ) Valid

NB

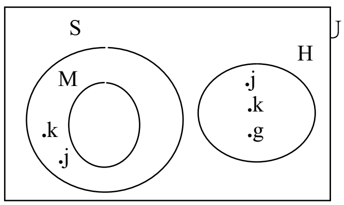
'Esi is careless' can only be represented in one region, region C.
Hence, we say it follows from the conclusion that 'Esi is not a Student'.

Example 2.5

Consider the following two statements:
s : All students with measles stay in the sick bay
t : All students in the sick bay do not do homework.
Which of the following is or are valid deductions from the two statements?
(i) Kofi does not have measles, so Kofi does his homework
(ii) George has done his homework, $\Rightarrow$ he does not stay in the bay
(iii) Jane does not have measles, so she does not stay in the sick bay

Solution

Let U = {All Students}
M = {Studdents with Measles}
S = {Students who stay in the sick bay}
H = {Students who do their homework}
Kofi = k, George = g and Jane = j



(i) Invalid

NB

'Kofi does not have measles' can be represented in two regions, first in region S but outside region M and secondly in region H.
Since the statement can be represented in more than one region, we say it does not follow from the conclusion that 'Kofi does his homework'.

(ii) Valid

NB

'George has done his homework' can only be represented in region H.
Hence, it follows from the conclusion that 'George does not stay in the sick bay'.

(iii) Invalid

NB

'Jane does not have measles' can be represented in two regions, first in region S but outside M and secondly in region H.
Hence, it does not follow from the conclusion that 'Jane does not stay in the sick bay'.

EXERCISE

QUE. A

Consider the following statements:
p : If students work hard they will pass their examinations
q : If students pass their examinations, then they do not sleep in class:
(a) Draw a Venn diagram to represent the statements $p \text{ and } q$
(b) Deduce whether the following conclusions are valid or not
(i) Nana does not sleep in class so she is hard-working
(ii) Nii does not pass his examination so he sleeps in class
(iii) Naa works hard so she does not sleep in class

QUE. B

Using Venn diagrams, indicate whether or not each of the following arguments is valid.

(1) Books borrowed for two weeks belong to Prof. Elvis. All Library books can be borrowed for two weeks. So all Library books belongs to Prof Elvis.

(2) A second degree equation in x and y has a term in xy. The equation of a circle is a second degree equation in x and y. Therefore the equation of a circle has a term in x and y.

(3) Thrice the number x is greater than 20. 20 is equal to five times x. Therefore thrice the number x is greater than five times the number x.

(4) Some numbers are multiples of 6. Some numbers are multiples of 11. Hence, some multiples of 6 are also multiples of 11.

(5) Some animals are friendly. All parrots are friendly. Hence, some animals are parrots.

(6) All students enjou going to school. Some people in Damongo are students. Therefore some people in Damongo enjoy going to school.

(7) All squares have four lines of symmetry. Some rhombuses are squares. Therefore some rhombuses have four lines of symmetry.

(8) All prime numbers are divisible by 5. Ten is a prime number. Therefore 10 is divisible by 5.

(9) All diamonds are very expensive. Some cars are very expensive. Hence, some cars are diamonds.

(10) My brothers like playing soccer. All male Brazilians like playing soccer.
Therefore my brothers are all male Brazilians.

QUE. C

Consider the following two statements:
w: Some students are hardworking
z: Some hardworking students are not careless

(a) Draw a Venn diagram to illustrate the above statements
(b) State whether the following conclusions are valid or not

valid from the statements $w \ and \ z$

(i) Jacob is careless $\Rightarrow$ Jacob is hardworking

(ii) Zenzen is hardworking $\Rightarrow$ Zenzen is careless

(iii) Owusu is not a student $\Rightarrow$ Owusu is not careless

QUE. D

The following statements are true of a certain school.

p : There is ***no*** left-handed boy in the school team

q : ***All*** the good students of Mathematics are in the football team.

If

$L = \{left - handed \ boys\}, T = \{football \ team\}, M = \{good \ Mathematics \ students\}$

(a) Draw a Venn diagram to illustrate $p \ and \ q$

(b) Using your Venn diagram, state whether or not each of the following is a valid deduction from $p \ and \ q$

(i) Fosu is not a good student of Mathematics; therefore he is not in the football team.

(ii) Opare is left-handed so he is not a good student of Mathematics

CHAPTER 3
OPERATIONS ON RATIONAL NUMBERS
(THE REAL NUMBER SYSTEM)

3.1 **Forms Of Numbers**

The following are the various categories of numbers:

3.1.1 **Natural Numbers**

The set of ***natural numbers*** is $\{1, 2, 3, ---\}$. This set of numbers are usually called the set of ***counting numbers***. Natural numbers are denoted by N.
Thus, $N = \{1, 2, 3, ---\}$.

3.1.2 **Whole Numbers**

The set of ***whole numbers*** is simply the set of natural numbers including ***zero***. Whole numbers are denoted by ***W.*** Thus, $W = \{0, 1, 2, 3, ---\}$.

3.1.3 **Integers**

The set of integers is the set of ***positive*** and ***negative*** whole numbers including zero. Integers are denoted by ***Z***. Thus,
$Z = \{---, -2, -1, 0, 1, 2, 3, ---\}$

3.2 **Integer Arithmetics**

When working with integers with negative signs, special care must be taken

3.2.1 **Addition Of Integers**

The different forms of integer addition are illustrated below:

$5 + 4 = 9$	$-5 + 1 = -4$	$7 + (-3) = 7 - 3 = 4$
$-6 + 8 = 2$	$-2 + (-4) = -2 - 4 = -6$	$3 + (-5) = 3 - 5 = -2$

3.2.2 **Subtraction Of Integers**

Similarly, the different forms of integer subtraction are illustrated below:

$5-4=1$ $\quad$ $-5-(-1)=-5+1=-4$ $\quad$ $3-5=-2$

$-6-8=-14$ $\quad$ $2-(-4)=2+4=6$

NB

We can also define addition and subtraction on the number line in that, for addition we move to the right and for subtraction we move to the left.
Thus, in adding, we move to the position of the first number on the number line and then count on to the right the number of places of the second number.
For instance, in performing: 4+2, we move to the position of four on the number line and then count on two places to the right to arrive at six.

Similarly, in subtraction we move to the position of the first number on the number line and count on to the left the number of places of the second number.
For instance, in performing: 4-2, we move to four on the number line and then count on two places to the left to arrive at 2.

3.2.3 **Multiplication Of Integers**

In integer multiplication, the product of two ***positive*** numbers is ***positive***, the product of two ***negative*** numbers is ***positive*** and the product of a ***positive*** number and a ***negative*** number is ***negative***. Thus, we have the following:

$5\times4=20$

$-5\times-4=20$

$5\times-4=-20$

$-5\times4=-20$

3.2.4 **Division Of Integers**

When dividing ***two positive*** integers the answer is ***positive***, when dividing ***two negative*** integers the answer is ***positive***, when dividing a ***negative*** integer by a ***positive*** integer the answer is ***negative*** and when dividing a ***positive*** integer by a ***negative*** integer the answer is ***negative***.

Thus, we have the following:

$\frac{9}{3} = 3$ $\frac{-4}{-2} = 2$ $\frac{-6}{2} = -3$ $\frac{8}{-4} = -2$

3.3 **Rational Numbers**

Rational numbers are numbers which can be expressed as the ratio of two integers where the denominator is not ***zero***. Thus, rational numbers are ***fractions***. The set of rational numbers is denoted by ***Q***.

NB

Every rational number can be expressed as either a terminating or recurring decimal and every terminating or recurring decimal is a rational number.

3.4 **Forms Of Fractions**

3.4.1 **Proper Fractions**

A proper fraction is a fraction in which the numerator is ***less*** than the denominator.

Examples are: $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, etc$

3.4.2 **Improper Fractions**

An improper fraction is a fraction in which the numerator is ***greater*** than the denominator. Examples are: $\frac{3}{2}, \frac{4}{3}, \frac{7}{5}, etc$

3.4.3 **Mixed Numbers**

These are numbers that contain both ***whole*** numbers and ***fractions.*** Example is: $3\frac{1}{2}$ and read as three whole number one over two.

NB

All ***Improper*** fractions can be expressed into ***mixed*** fractions and vice versa.

Thus, $\frac{5}{2} = 2\frac{1}{2}$ and read as two whole number one over two.

Likewise, $5\frac{2}{3} = \frac{17}{3}$ and read as seventeen over three.

3.4.4 **Decimal Fractions**

Common fractions can be converted to decimal fractions by dividing the denominator into the numerator.

By so doing,

1. We first write the denominator outside a root sign and then place the numerator inside the same root sign
2. If the number in the root sign is greater than that outside the root sign:
    (i) We find how many times it can divide the number outside it with or without a remainder and then write the result on top of the root sign.
    (ii) We then multiply what is written on top of the root sign by the number outside the root sign and write this new result under the number inside the root sign.
    (iii) Find the difference of the two numbers inside the root sign
    (iv) If the difference found is zero then the result on top of the root sign is taken as the final answer.
    (v) Again, if the the difference of the two numbers inside the root sign is not zero and less than the number outside the root sign, we then write it as a tenth. i.e.

multiply it by 10 and repeat the procedure until zero is obtained in the difference.

3. If the number in the root sign is less than that outside the root sign:
   (i) We write a zero with a decimal point after it on top of the root sign
   (ii) We then express the number outside the root sign as a tenth or multiply it by 10 and then find how many times it can divide the number outside it with or without a remainder and then write the result after the decimal point on top of the root sign
   (iii) We then multiply what is written after the decimal point on top of the root sign by the number outside the root sign and write this new result under the number inside the root sign.
   (iv) Find the difference of the two numbers inside the root sign and write it under
   (v) If the difference found is zero then the result on top of the root sign is taken as the final answer.
   (vi) Again, if the the difference of the two numbers inside the root sign is not zero and less than the number outside the root sign, we then write it as a tenth. i.e. multiply it by 10 and repeat the procedure until zero is obtained in the difference and the resulting value on top of the root sign is taken as the answer.

Example 3.1

Write each of the following fractions as decimals:

$(i)\ \frac{4}{5} \quad (ii)\ \frac{6}{8} \quad (iii)\ 3\tfrac{1}{2} \quad (iv)\ \frac{2}{3}$

Solution

$(i)\ \frac{4}{5} = 4 \div 5 = 0.8$

$$\Rightarrow 5\overline{)40}^{\,0.8}$$

$$\begin{array}{r} \underline{40} \\ 0 \end{array}$$

$(ii)\ \frac{6}{8} = 6 \div 8 = 0.75$

$$\Rightarrow 8\overline{)60}^{\,0.75}$$

$$\begin{array}{r} \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

$(iii)\ 3\tfrac{1}{2} = \frac{7}{2} = 3.5$

$$\Rightarrow 2\overline{)7}^{\,3.5}$$

$$\begin{array}{r} \underline{6} \\ 10 \\ \underline{10} \\ 0 \end{array}$$

$(iv)\ \frac{2}{3} = 0.666666...$

$$\Rightarrow \quad \begin{array}{r} 0.6\dot{6} \\ 3\overline{)20} \\ \underline{18} \\ 20 \\ \underline{18} \end{array}$$

3.5 **Arithmetics Of Fractions**

3.5.1 **Addition And Subtraction Of Fractions**

In ***adding*** or ***subtracting*** fractions of the same denominator, we just add or subtract their numerators and divide the result by the denominator. Again, in adding or subtracting fractions of different denominators, we find their ***LCM*** and simplify appropriately**.**

NB

All mixed fractions are changed into improper fractions before performing any addition or subtraction.

Example 3.2

Express the following as single fractions:

1. $\frac{2}{5} + \frac{1}{5}$ 2. $4\frac{1}{2} - 3\frac{4}{5}$ 3. $\frac{2}{3} + \frac{5}{6} + \frac{1}{2}$ 4. $\frac{2}{3} + \frac{2}{5} - \frac{7}{15}$
5. $\frac{x}{6} + \frac{x-y}{3} + \frac{x+y}{2}$

Solution

1. $\frac{2}{5} + \frac{1}{5} = \frac{2+1}{5} = \frac{3}{5}$

2. $4\frac{1}{2} - 3\frac{4}{5} = \frac{9}{2} - \frac{19}{5}$

$$= \frac{5\times 9 - 2\times 19}{10} = \frac{45-38}{10} = \frac{7}{10}$$

3. $\frac{2}{3} + \frac{5}{6} + \frac{1}{2} = \frac{(2\times 2)+(1\times 5)+(3\times 1)}{6} = \frac{4+5+3}{6} = \frac{12}{6} = 2$

4. $\frac{2}{3} + \frac{2}{5} - \frac{7}{15} = \frac{(5\times 2)+(3\times 2)-(1\times 7)}{15} = \frac{10+6-7}{15} = \frac{9}{15} = \frac{3}{5}$

5. $\frac{x}{6} + \frac{x-y}{3} + \frac{x+y}{2} = \frac{(1\times x)+2(x-y)+3(x+y)}{6}$

$$= \frac{x+2x-2y+3x+3y}{6} = \frac{6x+y}{6}$$

3.5.2 **Multiplication Of Fractions**

In multiplying fractions, we multiply the numerators together and then multiply the denominators together but changing all mixed fractions into improper fractions.

Example 3.3

Perform the following operations:

1. $\frac{1}{2}\times\frac{3}{5}$ 2. $3\frac{1}{7}\times 4\frac{1}{3}\times 1\frac{3}{4}$

Solution

1. $\frac{1}{2}\times\frac{3}{5} = \frac{1\times 3}{2\times 5} = \frac{3}{10}$

2. $3\frac{1}{7}\times 4\frac{1}{3}\times 1\frac{3}{4} = \frac{22}{7}\times\frac{13}{3}\times\frac{7}{4} = \frac{22\times 13\times 7}{7\times 3\times 4} = \frac{2002}{84} = 23\frac{70}{84}$

Example 3.4

Simplify the following: $\dfrac{\frac{1}{3}(2\frac{1}{6}+5\frac{1}{6})}{1\frac{7}{12}}$

Solution

$$\frac{\frac{1}{3}(2\frac{1}{6}+5\frac{1}{6})}{1\frac{7}{12}}=\frac{\frac{1}{3}(\frac{13}{6}+\frac{31}{6})}{\frac{19}{12}}=\frac{\frac{1}{3}(\frac{44}{6})}{\frac{19}{12}}=\frac{\frac{44}{18}}{\frac{19}{12}}=\frac{44}{18}\times\frac{12}{19}=\frac{88}{57}=1\frac{31}{57}$$

3.5.3 **Division Of Fractions**

In dividing a fraction by another fraction, we multiply both the numerator and the denominator by the ***reciprocal*** of the denominator.

NB

The reciprocal of a fraction is obtained by turning the fraction upside down. Thus, the numerator becomes the denominator and vice versa.

Example 3.5

Simplify the following operations:

1. $\dfrac{2}{3}\div\dfrac{1}{4}$ 2. $3\frac{1}{4}\div 2\frac{1}{3}$ 3. $\dfrac{3}{7}\div 5$ 4. $5\div\dfrac{1}{2}$ 5. $\dfrac{7}{11}\div 1\frac{2}{3}$

6. $(\frac{2}{3}\ of\ 2\frac{1}{4})\div(3\frac{1}{2}-2\frac{1}{4})$

7. $37\frac{1}{2}\div\frac{5}{9}\ of\ (\frac{4}{7}+\frac{1}{5})-80\frac{1}{3}$

Solution

1. $\dfrac{2}{3}\div\dfrac{1}{4}=\dfrac{2}{3}\times\dfrac{4}{1}=\dfrac{8}{3}=2\frac{2}{3}$

2. $3\tfrac{1}{4} \div 2\tfrac{1}{3} = \dfrac{13}{4} \div \dfrac{7}{3} = \dfrac{13}{4} \times \dfrac{3}{7} = \dfrac{39}{28} = 1\tfrac{11}{28}$

3. $\dfrac{3}{7} \div 5 = \dfrac{3}{7} \div \dfrac{5}{1} = \dfrac{3}{7} \times \dfrac{1}{5} = \dfrac{3}{35}$ $(Since\ reciprocal\ of\ 5\ is\ \dfrac{1}{5})$

4. $5 \div \dfrac{1}{2} = 5 \times 2 = 10$ $(Since\ reciprocal\ of\ \dfrac{1}{2}\ is\ 2)$

5. $\dfrac{7}{11} \div 1\tfrac{2}{3} = \dfrac{7}{11} \div \dfrac{5}{3} = \dfrac{7}{11} \times \dfrac{3}{5} = \dfrac{21}{55}$

6. $(\tfrac{2}{3}\ of\ 2\tfrac{1}{4}) \div (3\tfrac{1}{2} - 2\tfrac{1}{4})$

$NB: Change\ the\ word\ 'of'\ to\ multiplication\ (\times)$

$and\ all\ mixed\ fractions\ to\ improper\ fractions$

$Thus,$

$(\tfrac{2}{3}\ of\ 2\tfrac{1}{4}) \div (3\tfrac{1}{2} - 2\tfrac{1}{4}) = (\tfrac{2}{3} \times \tfrac{9}{4}) \div (\tfrac{7}{2} - \tfrac{9}{4}) = \tfrac{3}{2} \div \tfrac{5}{4} = \tfrac{3}{2} \times \tfrac{4}{5} = \tfrac{6}{5} = 1\tfrac{1}{5}$

7. $37\tfrac{1}{2} \div \tfrac{5}{9}\ of\ (\tfrac{4}{7} + \tfrac{1}{5}) - 80\tfrac{1}{3}$

$\tfrac{75}{2} \div \tfrac{5}{9} \times \left(\tfrac{27}{35}\right) - \tfrac{241}{3} = \tfrac{75}{2} \div \left(\tfrac{5}{9} \times \tfrac{27}{35}\right) - \tfrac{241}{3} = \tfrac{75}{2} \div \tfrac{3}{7} - \tfrac{241}{3} = \tfrac{75}{2} \times \tfrac{7}{3} - \tfrac{241}{3} = \tfrac{175}{2} - \tfrac{241}{3}$

$= \tfrac{525-482}{6} = \tfrac{43}{6} = 7\tfrac{1}{6}$

3.6 **Irrational Numbers**

Irrational numbers are numbers which cannot be written as a fraction of two integers. They are denoted by ***H.*** The square roots of all prime numbers are irrational numbers and the decimal equivalent of an irrational number is neither terminating nor recurring. Egs. $\sqrt{2}, \sqrt{3}, \pi\ etc.$

NB

The product of two irrational numbers is also an irrational number
Thus, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$

3.7 **Real Numbers**

The set of real numbers is the set of natural, whole, integers, rational and irrational numbers put together. Real numbers are denoted by the symbol ***R*.**

Example 3.6

Without using tables or calculators, evaluate the following:

1. $\dfrac{2\frac{7}{8}\times 1\frac{1}{5}}{8-2\frac{1}{4}}$

2. $\dfrac{3\frac{1}{3}-2\frac{1}{2}}{\frac{5}{12}}$

3. $1\frac{2}{3}-(1\frac{3}{4}\div 2\frac{5}{8})$

4. $(1\frac{2}{7}-\frac{1}{3})\times 1\frac{3}{4}$

5. $\dfrac{4+\frac{1}{y^2}}{\frac{x}{y}+\frac{m}{y^2}}$

6. $1\frac{1}{2}-\frac{3}{5}\div\frac{2}{3}$

7. $(3\frac{1}{4}+4\frac{1}{4})\div(\frac{5}{6}-\frac{2}{3})$

Solution

1. $\dfrac{2\frac{7}{8}\times 1\frac{1}{5}}{8-2\frac{1}{4}}=\dfrac{\frac{23}{8}\times\frac{6}{5}}{8-\frac{9}{4}}=\dfrac{\frac{138}{40}}{\frac{23}{4}}=\dfrac{138}{40}\times\dfrac{4}{23}=\dfrac{3}{5}$

2. $\dfrac{3\frac{1}{3}-2\frac{1}{2}}{\frac{5}{12}}=\dfrac{\frac{10}{3}-\frac{5}{2}}{\frac{5}{12}}=\dfrac{\frac{5}{6}}{\frac{5}{12}}=\dfrac{5}{6}\times\dfrac{12}{5}=2$

3. $1\frac{2}{3}-(1\frac{3}{4}\div 2\frac{5}{8})=\frac{5}{3}-\left(\frac{7}{4}\div\frac{21}{8}\right)=\frac{5}{3}-\left(\frac{7}{4}\times\frac{8}{21}\right)=\frac{5}{3}-\frac{2}{3}=\frac{3}{3}=1$

4. $(1\frac{2}{7}-\frac{1}{3})\times 1\frac{3}{4}=\left(\frac{9}{7}-\frac{1}{3}\right)\times\frac{7}{4}=\left(\frac{27-7}{21}\right)\times\frac{7}{4}=\frac{20}{21}\times\frac{7}{4}=\frac{5}{3}=1\frac{2}{3}$

5. $\dfrac{4+\frac{1}{y^2}}{\frac{x}{y}+\frac{m}{y^2}}=\dfrac{\frac{4}{1}+\frac{1}{y^2}}{\frac{x}{y}+\frac{m}{y^2}}=\dfrac{\frac{4y^2+1}{y^2}}{\frac{xy+m}{y^2}}$ *ie. LCM of both up & down*

$$\Rightarrow \frac{4y^2+1}{y^2}\times\frac{y^2}{xy+m}=\frac{4y^2+1}{xy+m}$$

6. $1\frac{1}{2}-\frac{3}{5}\div\frac{2}{3}=\frac{3}{2}-\frac{3}{5}\times\frac{3}{2}=\frac{3}{2}-\frac{9}{10}=\frac{15-9}{10}=\frac{6}{10}=\frac{3}{5}$ (*Applied BODMAS*)

7. $(3\frac{1}{4}+4\frac{1}{4})\div(\frac{5}{6}-\frac{2}{3})=\left(\frac{13}{4}+\frac{17}{4}\right)\div\left(\frac{5}{6}-\frac{2}{3}\right)$

$=\frac{13+17}{4}\div\frac{5-4}{6}=\frac{30}{4}\div\frac{1}{6}=\frac{30}{4}\times 6=45$

Example 3.7

Express as a single fraction: $\frac{x}{x-2}-\frac{x+1}{x+3}$

Solution

$\frac{x}{x-2}-\frac{x+1}{x+3}=\frac{x(x+3)-(x+1)(x-2)}{(x-2)(x+3)}=\frac{x^2+3x-x^2+2x-x+2}{(x-2)(x+3)}$

$=\frac{4x+2}{(x-2)(x+3)}$

Example 3.8

Simplify the following:

$(i)\ \frac{m+1}{m-1}-\frac{m-1}{m+1}+\frac{4}{m^2-1}$

$(ii)\ \frac{3m}{9m^2-1}-\frac{1}{2(3m+1)}$

Solution

NB

Find the LCM in both cases:

$$(i)\ \frac{m+1}{m-1}-\frac{m-1}{m+1}+\frac{4}{m^2-1}=\frac{(m+1)^2-(m-1)^2+4}{m^2-1}$$

$$=\frac{(m^2+2m+1)-(m^2-2m+1)+4}{m^2-1}$$

$$\frac{m^2+2m+1-m^2+2m-1+4}{m^2-1}=\frac{4m+4}{m^2-1}=\frac{4(m+1)}{(m+1)(m-1)}=\frac{4}{m-1}$$

$Since\ m^2-1=(m+1)(m-1)$

$$(ii)\ \frac{3m}{9m^2-1}-\frac{1}{2(3m+1)}=\frac{2(3m)-1(3m-1)}{2(9m^2-1)}$$

$$=\frac{6m-3m+1}{2(9m^2-1)}=\frac{3m+1}{2(3m+1)(3m-1)}=\frac{1}{2(3m-1)}$$

$Since\ 9m^2-1=(3m+1)(3m-1)$

NB

Arranging numbers in ***ascending*** order of magnitude means arranging the numbers starting from the smallest to the biggest whiles in ***descending*** order of magnitude means arranging from biggest to smallest.
In ***arranging*** fractions, we first find the ***LCM*** of the denominators and then divide the LCM by each denominator and finally multiply the result by both the numerators and denominators to have common denominators.
Now the resulting fractions are arranged comparing the magnitude of each numerator.

Example 3.9

Arrange in ascending order of magnitude: $\frac{7}{75}, \frac{2}{5}\ and\ \frac{1}{3}$

Solution

First, the fractions $\frac{7}{75}, \frac{2}{5}$ *and* $\frac{1}{3}$ have LCM of 75

$$\Rightarrow \frac{7}{75} = \frac{7\times1}{75\times1} = \frac{7}{75}, \quad \frac{2}{5} = \frac{2\times15}{5\times15} = \frac{30}{75}, \quad and \quad \frac{1}{3} = \frac{1\times25}{3\times25} = \frac{25}{75}$$

Hence, in ascending order we have: $\frac{7}{75}, \frac{1}{3}$ *and* $\frac{2}{5}$

3.8 **Word Problems**

Sometimes, a question may be given in a ***sentence*** form called ***word problem*** as in the example below:

Example 3.10

A girl spent $\frac{3}{5}$ of her pocket money and was left with ¢ $1,800.00$. How much was her pocket money?

Solution

Fraction spent $= \frac{3}{5}$
Amount left $=$ ¢ $1,800.00$
Total fraction $= 1$
Hence, fraction left $= 1 - \frac{3}{5} = \frac{2}{5}$
But from question,

$$\frac{2}{5} = ¢\ 1,800.00$$
$$\Rightarrow 1 = ¢\ x$$
$$x = 1800 \times \frac{5}{2} = 4500$$

Hence, her pocket money was ¢ 4500.00

3.9 **Changing Decimals To Fractions**

In changing decimals into fractions, we count and use the digits on the RHS of the decimal point.
If it is one digit it is a tenth, two digits a hundredth, three digits a thousandth.

Thus, for a tenth we divide the digit on RHS of decimal point by 10. Likewise, for a hundredth we divide the two digits on RHS of decimal point together by 100. Etc.
For instance,

$0.9 = 9\ tenths\ = \frac{9}{10}$

$0.05 = 5\ hundredths\ = \frac{5}{100}$

$0.23 = 23\ hundredths\ = \frac{23}{100}$

$0.057 = 57\ thousandths\ = \frac{57}{1000}$

$0.325 = 325\ thousandths = \frac{325}{1000}$

Example 3.11

Write each of the following decimal numbers in the form

$\frac{a}{b},\ a, b \in Z,\ b \neq 0$

$(i)\ 0.75 \quad (ii)\ 0.375 \quad (iii)\ 0.1875 \quad (iv)\ 0.\dot{5} \quad (v)\ 0.\dot{1}\dot{8}$

$(vi)\ 2.31\dot{6}$

Solution

$(i)\ 0.75 = \frac{75}{100} = \frac{3}{4}$

NB: Since the number of digits after decimal point is 2, we divide both digits after decimal point by 100.

$(ii)\ 0.375 = \frac{375}{1000} = \frac{3}{8}$

NB: Since the number of digits after decimal point is 3, we divided all digits after decimal point by 1000.

$(iii)\ 0.1875 = \frac{1875}{10000} = \frac{3}{16}$

NB: Since the number of digits after decimal point is 4, we divided all digits after decimal point by 10000

(iv) For 0.5,

Let $x = 0.5555------------(1)$

Multiply (1) by 10 to get;

$10x = 5.5555-----------------(2)$

Again, multiply (1) by 100;

$100x = 55.5555----------------(3)$

Further, multiply (1) by 1000;

$1000x = 555.5555----------------(4)$

NB: Here, the idea is to get two equations in which both have the same group of numbers after the decimal point. From above, all quations have the same group of numbers after the decimal point, so we pick any two of the equations. Thus, picking equations (1) and (2),

$x = 0.5555-----(1)$

$10x = 5.5555-----(2)$

Now, (2) − (1) *gives;*

$$\frac{9x}{9} = \frac{5}{9}$$

$$\Rightarrow x = \frac{5}{9}$$

$$\therefore 0.\dot{5} = \frac{5}{9}$$

$(v)\ 0.\dot{1}\dot{8}$

Let

$x = 0.181818----(1)$

$\Rightarrow 10x = 1.81818----(2)$

$100x = 18.181818----(3)$

$1000x = 181.81818----(4)$

Thus, equations (1) and (3) have the same group of numbers after the decimal point and equations (2) and (4) have the same group of numbers after the decimal point. Picking equations (2) and (4),

$10x = 1.81818----(2)$

$1000x = 181.81818----(4)$

$(4)-(2)$ *gives*;

$$\frac{990x}{990} = \frac{180}{990}$$

$$\Rightarrow x = \frac{2}{11}$$

$$\Rightarrow 0.\dot{1}\dot{8} = \frac{2}{11}$$

(vi) $2.31\dot{6}$

Here, we use only the decimal part

Let $x = 0.31666----(1)$

$\Rightarrow 10x = 3.1666----(2)$

$100x = 31.666------(3)$

$1000x = 316.666------(4)$

Thus, only equations (3) and (4) have the same group of numbers after the decimal point.
Hence, we pick (3) and (4),

$100x = 31.666----(3)$

$1000x = 316.666----(4)$

$(4)-(3)$ *gives*;

$$\frac{900x}{900} = \frac{285}{900}$$

$$\Rightarrow x = \frac{19}{60}$$

$$\therefore 2.31\dot{6} = 2\tfrac{19}{60}$$

3.10 **Multiplication And Division Of Decimals**

In ***multiplying*** decimals, we ignore the decimal points and multiply the numbers as whole numbers. After which we count and add the number of decimal places in the numbers together and then place the decimal point in the result of the product appropriately from the right hand side.

Similarly, in ***dividing*** decimals, it is convenient to always change both decimals to fractions and simplify.

Example 3.12

Evaluate the following:

1. 2.79×1.3 2. $0.46\div0.2$ 3. $0.1\times0.3\times0.5\times0.7$ 4. $0.25\div0.001$

Solution

1. 2.79×1.3

Ignoring the decimal points and multiplying, we write

$279\times13=3627$

But 2.79 has 2dp and 1.3 also has 1dp, so total number of dp will be (2+1=3dp)

Hence, we write 3627 to 3dp as 3.627

Therefore, $2.79\times1.3=3.627$

2. $0.46\div0.2$

Here, we change each decimal to fraction and work with the fractions. Thus, 0.46 is to 2dp so we divide the non-zeros which is 46 by 100 and 0.2 also in 1dp so we divide its non-zero which is 2 by 10 before performing the operations.

$i.e.\ 0.46=\frac{46}{100}\ \ and\ 0.2=\frac{2}{10}$

$$\Rightarrow 0.46\div0.2=\frac{46}{100}\div\frac{2}{10}=\frac{46}{100}\times\frac{10}{2}=\frac{460}{200}=\frac{23}{10}=2.3$$

3. $0.1\times0.3\times0.5\times0.7$

Ignoring the decimal point and multiplying, we have

$1\times3\times5\times7=105$

But sum of all decimal points is 4, so we write 105 to 4dp as 0.0105

Hence, $0.1\times0.3\times0.5\times0.7=0.0105$

4. $0.25\div0.001=\frac{25}{100}\div\frac{1}{1000}=\frac{25}{100}\times\frac{1000}{1}=25\times10=250$

NB

Alternatively, in the above questions, (1) and (3) could also be solved by first expressing each decimal as a fraction and then performing the operation appropriately.

Thus,

1. $2.79 \times 1.3 = 2\tfrac{79}{100} \times 1\tfrac{3}{10} = \dfrac{279}{100} \times \dfrac{13}{10} = \dfrac{3627}{1000} = 3.627$

3. $0.1 \times 0.3 \times 0.5 \times 0.7 = \dfrac{1}{10} \times \dfrac{3}{10} \times \dfrac{5}{10} \times \dfrac{7}{10} = \dfrac{105}{10000} = 0.0105$

EXERCISE

QUE. A

Evaluate the following:

1. $\dfrac{12\frac{1}{2} - 8\frac{1}{3}}{\frac{5}{12}}$ 2. $\dfrac{\frac{3}{4}\left(3\frac{3}{8} + 1\frac{5}{6}\right)}{2\frac{1}{8} - 1\frac{1}{2}}$ 3. $\dfrac{1\frac{1}{3} \times \frac{2}{3}}{\frac{1}{4} \div 1\frac{1}{2}}$ 4. $6\frac{2}{3} \div \left(3\frac{4}{15} - 1\frac{3}{5}\right)$

5. $\dfrac{3\frac{1}{2} - (-2\frac{3}{5})}{4\frac{1}{2}}$ 6. $\dfrac{\frac{1}{3}\left(2\frac{1}{6} + 5\frac{1}{6}\right)}{1\frac{7}{12}}$ 7. $2\frac{3}{4} + 1\frac{1}{8} \div 2\frac{1}{2}$

8. $\dfrac{2}{a - 2b} + \dfrac{1}{a + 2b}$ 9. $\dfrac{\dfrac{1}{3}\left(2\frac{1}{6} + 5\frac{1}{6}\right)}{1\frac{7}{12}}$ 10. $\dfrac{x}{x - 2} - \dfrac{x + 1}{x + 3}$

QUE. B

Arrange the following in ascending order of magnitude:
$4\frac{2}{5}, 4.03, \frac{28}{5}, 5.75$

QUE. C

A man spends $\frac{1}{9}$ of his monthly salary on rent, $\frac{1}{2}$ on food and $\frac{1}{4}$ on clothes and other items. If he has ¢ $19,500.00$ left at the end of the month, how much does he earn?

QUE. D

A Clerk spends $\frac{1}{5}, \frac{1}{3}$ *and* $\frac{1}{8}$ of his annual salary on rent, transport and entertainment respectively. If after all these expenses he had $N820.00$ left, find how much he earns per annum.

QUE. E

Write each of the following fractions as a decimal:

1. $\frac{1}{2}$ 2. $\frac{1}{4}$ 3. $\frac{1}{8}$ 4. $\frac{1}{16}$ 5. $\frac{3}{25}$ 6. $\frac{3}{8}$ 7. $\frac{5}{16}$

8. $\frac{7}{40}$ 9. $\frac{11}{50}$ 10. $\frac{23}{50}$ 11. $\frac{27}{80}$ 12. $\frac{39}{160}$ 13. $\frac{1}{3}$

14. $\frac{1}{3}$ 15. $\frac{1}{6}$ 16. $\frac{1}{7}$ 17. $\frac{1}{9}$ 18. $\frac{1}{11}$ 19. $\frac{2}{3}$ 20. $\frac{4}{9}$

QUE. F

Write each of the following decimal numbers in the form $\frac{a}{b}, a, b \in Z, b \neq 0$

1. 0.225 2. 0.74 3. 0.3875 4. 0.575 5. 0.4875 6. 0.35625

7. 0.368 8. 0.59375 9. 0.796875 10. $0.\dot{6}$ 11. $0.\dot{6}\dot{3}$

12. $0.\dot{4}8\dot{6}$ 13. $0.\dot{9}7\dot{2}$ 14. $0.\dot{2}\dot{4}$ 15. $0.\dot{8}1\dot{0}$ 16. $0.6\dot{3}$

17. $0.28\dot{3}$ 18. $0.9\dot{3}$ 19. $0.\dot{9}1\dot{8}$ 20. $0.\dot{5}\dot{4}$

QUE. G

Evaluate the following:

1. $18\frac{2}{3}+(-15\frac{3}{7})$ 2. $-16\frac{1}{4}+20\frac{5}{6}$ 3. $-4\frac{3}{5}+(-7\frac{2}{3})$ 4. $-6\frac{3}{4}+(-11\frac{1}{6})$

5. $-8\frac{1}{5}+(-10\frac{6}{7})$ 6. $7\frac{1}{2}-(-12\frac{1}{3})$ 7. $3\frac{2}{5}-(-6\frac{1}{4})$ 8. $6\frac{2}{3}-(-10\frac{3}{4})$

9. $9\frac{1}{5}-(-11\frac{4}{7})$ 10. $(-7\frac{1}{3})+4\frac{1}{2}$ 11. $-4\frac{3}{5}$ 12. $-7\frac{2}{3}+(11\frac{1}{6}+9\frac{2}{3})$

13. $-5\frac{1}{3}+(-8\frac{1}{5}+12\frac{2}{5})$ 14. $(-4\frac{3}{5}+6\frac{3}{4})+8\frac{1}{2}$ 15. $(-7\frac{2}{3}+11\frac{1}{6})+9\frac{2}{3}$

16. $(-5\frac{1}{3}+(-8\frac{1}{5}))+12\frac{2}{5}$ 17. $-3\frac{1}{2}-(-6\frac{2}{5})$ 18. $-7\frac{2}{5}-(-11\frac{2}{7})$

19. $-7\frac{1}{3}-(-9\frac{3}{5})$ 20. $-9\frac{1}{2}-(-12\frac{3}{4})$

QUE. H

Find the following:

1. $-2\frac{3}{4}\times 7\frac{2}{3}$ 2. $-3\frac{3}{5}\times 9\frac{1}{3}$ 3. $4\frac{4}{7}\times(-8\frac{3}{10})$ 4. $-7\frac{1}{4}\times(-6\frac{3}{5})$

5. -4.9×20.5 6. $4\frac{1}{8}\div 3\frac{3}{4}$ 7. $4\frac{4}{7}$ 8. $15\frac{2}{3}\div 7\frac{5}{6}$ 9. $5\frac{1}{15}\div 3\frac{3}{5}$

10. $5\frac{1}{3}\div 7\frac{1}{9}$ 11. $6\frac{5}{6}\div 5\frac{6}{7}$ 12. $12\frac{1}{3}\div 18\frac{1}{2}$ 13. $(2.6\times 3\frac{3}{4})\times 1\frac{7}{9}$

14. $(2.8\times(-2\frac{4}{7}))\times 4\frac{1}{6}$ 15. $(-2\frac{1}{8})\times(17\frac{1}{2}\times(-3.2))$ 16. $2.6\times(3\frac{3}{4}\times 1\frac{7}{9})$

17. $((-1\tfrac{2}{7})\times 2\tfrac{3}{11})\times(-5.6)$ 18. $(4\tfrac{1}{11}\times(-7\tfrac{1}{3}))+(4\tfrac{1}{11}\times 18\tfrac{1}{3})$

19. $(-8\tfrac{7}{11})\times((-36\tfrac{4}{11})+46\tfrac{4}{11}$ 20. $((-3\tfrac{4}{7})\times 13\tfrac{2}{5})+((-3\tfrac{4}{7})\times 7\tfrac{3}{5})$

CHAPTER 4
SURDS

A ***Surd,*** also called an ***irrational number*** is a number which cannot be expressed as a fraction of two integers.
E.gs. $\sqrt{2}, \sqrt{6}, \sqrt{3}, \sqrt{5}, \sqrt{7}$ Etc.

NB

The square root of ***prime numbers*** are all surds.

4.1 Properties Of Surds

1. $\sqrt{m^2} = m$ E.g. $\sqrt{2^2} = \sqrt{4} = 2$

2. $\sqrt{m} \times \sqrt{m} = (\sqrt{m})^2 = m$ E.g. $\sqrt{5} \times \sqrt{5} = (\sqrt{5})^2 = 5$

3. $\sqrt{m} \times \sqrt{n} = \sqrt{mn}$ E.g. $\sqrt{2} \times \sqrt{5} = \sqrt{2 \times 5} = \sqrt{10}$

4. $m \times \sqrt{n} = m\sqrt{n}$ E.g. $3 \times \sqrt{2} = 3\sqrt{2}$

5. $q\sqrt{m} \times \sqrt{n} = q\sqrt{mn}$ E.g. $3\sqrt{2} \times \sqrt{5} = 3\sqrt{2 \times 5} = 3\sqrt{10}$

6. $p\sqrt{m} \times q\sqrt{n} = pq\sqrt{mn}$ E.g. $5\sqrt{2} \times 3\sqrt{7} = 5 \times 3\sqrt{2 \times 7} = 15\sqrt{14}$

7. $m \times n\sqrt{q} = mn\sqrt{q}$ E.g. $3 \times 2\sqrt{5} = 6\sqrt{5}$

8. $\frac{\sqrt{m}}{\sqrt{n}} = \sqrt{\frac{m}{n}}$ E.g. $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} = \sqrt{3}$

4.2 Reducing Surds To Basic Or Simplest Forms

In reducing or simplifying a surd, we follow the steps below:
1. Look for two factors of the number in the square root sign such that at least one of the factors is a perfect square.
2. Write the factors found in step one as a product by just

placing the sign '$\times$' in between the two numbers.
3. Replace the number inside the square root sign by the result of step two found above.
4. Separate the resulting surd in step three into two distinct surds and simplify.

Example 4.1

Simplify the following:

1. $\sqrt{45}$ 2. $\sqrt{50}$ 3. $\sqrt{8}$ 4. $\sqrt{98}$ 5. $\sqrt{20}$

6. $\sqrt{128}$ 7. $\sqrt{2940}$ 8. $\sqrt{147}$ 9. $\sqrt{4000}$ 10. $\sqrt{\frac{18}{343}}$

Solution

1. $\sqrt{45}$

Step 1

The factors of 45 are '9 and 5' where 9 is the perfect square.

Step 2

Placing the sign '$\times$' in between the factors gives;

9×5

Step 3

Replacing 45 by 9×5 gives;

$\sqrt{45}=\sqrt{9\times 5}$

Step 4

Separating $\sqrt{9\times 5}$ into distinct surds becomes $\sqrt{9}\times\sqrt{5}$ and simplifying gives;

$\sqrt{45}=\sqrt{9\times 5}=\sqrt{9}\times\sqrt{5}=3\times\sqrt{5}=3\sqrt{5}$

2. $\sqrt{50}=\sqrt{25\times 2}=\sqrt{25}\times\sqrt{2}=5\times\sqrt{2}=5\sqrt{2}$

3. $\sqrt{8}=\sqrt{4\times 2}=\sqrt{4}\times\sqrt{2}=2\times\sqrt{2}=2\sqrt{2}$

4. $\sqrt{98}=\sqrt{49\times 2}=\sqrt{49}\times\sqrt{2}=7\times\sqrt{2}=7\sqrt{2}$

5. $\sqrt{20} = \sqrt{4\times5} = \sqrt{4}\times\sqrt{5} = 2\times\sqrt{5} = 2\sqrt{5}$

6. $\sqrt{128} = \sqrt{64\times2} = \sqrt{64}\times\sqrt{2} = 8\times\sqrt{2} = 8\sqrt{2}$

7. $\sqrt{2940} = \sqrt{49\times60} = \sqrt{49}\times\sqrt{60} = 7\times\sqrt{4\times15}$
$= 7\times\sqrt{4}\times\sqrt{15} = 7\times2\times\sqrt{15} = 14\sqrt{15}$

8. $\sqrt{147} = \sqrt{49\times3} = \sqrt{49}\times\sqrt{3} = 7\sqrt{3}$

9. $\sqrt{4000} = \sqrt{4\times1000} = \sqrt{4}\times\sqrt{1000} = 2\times\sqrt{1000}$
$= 2\times\sqrt{100\times10} = 2\times\sqrt{100}\times\sqrt{10}$
$= 2\times10\times\sqrt{10} = 20\sqrt{10}$

10. $\sqrt{\frac{18}{343}} = \frac{\sqrt{18}}{\sqrt{343}} = \frac{\sqrt{9\times2}}{\sqrt{49\times7}} = \frac{3\sqrt{2}}{7\sqrt{7}} = \frac{3}{7}\sqrt{\frac{2}{7}}$

4.3 **Addition Of Surds**

Two or more surds can be added if they are alike or have the same form. Thus, we perform addition by adding the numbers outside the root sign and attaching the result to the surd.

NB

Surds that needs to be reduced must be simplified before adding

Example 4.2

Perform the following operations:

1. $\sqrt{3}+\sqrt{3}$ 2. $5\sqrt{2}+2\sqrt{2}$ 3. $\sqrt{5}+3\sqrt{5}$ 4. $\sqrt{50}+\sqrt{32}+\sqrt{162}$

5. $\sqrt{147}+\sqrt{75}+\sqrt{27}$

Solution

1. $\sqrt{3}+\sqrt{3}=2\sqrt{3}$

2. $5\sqrt{2}+2\sqrt{2}=7\sqrt{2}$

3. $\sqrt{5}+3\sqrt{5}=4\sqrt{5}$

4. $\sqrt{50}+\sqrt{32}+\sqrt{162}=\sqrt{25\times 2}+\sqrt{16\times 2}+\sqrt{81\times 2}$
$$=\sqrt{25}\times\sqrt{2}+\sqrt{16}\times\sqrt{2}+\sqrt{81}\times\sqrt{2}$$
$$=5\sqrt{2}+4\sqrt{2}+9\sqrt{2}=18\sqrt{2}$$

5. $\sqrt{147}+\sqrt{75}+\sqrt{27}=\sqrt{49\times 3}+\sqrt{25\times 3}+\sqrt{9\times 3}$
$$=\sqrt{49}\times\sqrt{3}+\sqrt{25}\times\sqrt{3}+\sqrt{9}\times\sqrt{3}$$
$$=7\sqrt{3}+5\sqrt{3}+3\sqrt{3}=15\sqrt{3}$$

NB

Reducible surds must be reduced first before adding or subtracting

4.4 **Subtraction Of Surds**

We can only subtract surds which are alike or have the same form. Similarly, we perform subtraction by subtracting the numbers outside the root sign and attaching the result to the surd.

Example 4.3

Evaluate the following:

1. $\sqrt{3}-2\sqrt{3}$ 2. $4\sqrt{2}-\sqrt{2}$ 3. $5\sqrt{3}-3\sqrt{3}$ 4. $3\sqrt{200}-3\sqrt{8}-\frac{2}{3}\sqrt{162}$
5. $\sqrt{128}-\sqrt{18}-\sqrt{32}$

Solution

1. $\sqrt{3}-2\sqrt{3}=-\sqrt{3}$ $(NB: \textit{The number outside } \sqrt{3} \textit{ is } 1)$

2. $4\sqrt{2}-\sqrt{2}=3\sqrt{2}$ $(NB: The\ number\ outside\ -\sqrt{2}\ is\ -1)$

3. $5\sqrt{3}-3\sqrt{3}=2\sqrt{3}$

4. $3\sqrt{200}-3\sqrt{8}-\frac{2}{3}\sqrt{162}=3\sqrt{100\times 2}-3\sqrt{4\times 2}-\frac{2}{3}\times\sqrt{81}\times\sqrt{2}$

$=3\times 10\sqrt{2}-3\times 2\sqrt{2}-\frac{2}{3}\times 9\sqrt{2}=30\sqrt{2}-6\sqrt{2}-6\sqrt{2}=30\sqrt{2}-12\sqrt{2}$

$=18\sqrt{2}$

5. $\sqrt{128}-\sqrt{18}-\sqrt{32}=\sqrt{64\times 2}-\sqrt{9\times 2}-\sqrt{16\times 2}$

$=\sqrt{64}\times\sqrt{2}-\sqrt{9}\times\sqrt{2}\sqrt{16}\times\sqrt{2}$

$=8\sqrt{2}-3\sqrt{2}-4\sqrt{2}=8\sqrt{2}-7\sqrt{2}=\sqrt{2}$

4.5 **Multiplication Of Surds**

When multiplying surds, the properties of surds are employed. Thus, we multiply the numbers outside the root signs together and then multiply the numbers inside the root signs also together separately.

Example 4.4

Simplify the following:

1. $\sqrt{18}\times\sqrt{45}$ 2. $\sqrt{3}(2-\sqrt{3})$ 3. $(\sqrt{3}-1)(2-\sqrt{3})$ 4. $(2+2\sqrt{7})(5-\sqrt{7})$

5. $(3\sqrt{3}-2)(3\sqrt{3}+2)$ 6. $\sqrt{7}\left[3\sqrt{7}+\frac{6}{\sqrt{7}}\right]$

7. Without using four-figure tables or calculator, simplify:
$\sqrt{50}-3\sqrt{2}(2\sqrt{2}-5)-5\sqrt{32}$

Solution

1. $\sqrt{18}\times\sqrt{45}=\sqrt{9\times 2}\times\sqrt{9\times 5}=\sqrt{9}\times\sqrt{2}\times\sqrt{9}\times\sqrt{5}$

$=3\sqrt{2}\times 3\sqrt{5}=3\times 3\sqrt{2\times 5}=9\sqrt{10}$

ie. Surds were reduced before multiplying.

2. $\sqrt{3}(2-\sqrt{3}) = 2\sqrt{3} - \sqrt{3}\times\sqrt{3} = 2\sqrt{3} - 3$

3. $(\sqrt{3}-1)(2-\sqrt{3}) = \sqrt{3}(2-\sqrt{3}) - (2-\sqrt{3}) = 2\sqrt{3} - 3 - 2 + \sqrt{3} = 3\sqrt{3} - 5$

4. $(2+2\sqrt{7})(5-\sqrt{7}) = 2(5-\sqrt{7}) + 2\sqrt{7}(5-\sqrt{7}) = 10 - 2\sqrt{7} + 10\sqrt{7} - 2\times 7$
$= 10 + 8\sqrt{7} - 14 = 8\sqrt{7} - 4$

5. $\sqrt{3}-2)(3\sqrt{3}+2) = 3\sqrt{3}(3\sqrt{3}+2) - 2(3\sqrt{3}+2) = 3\times 3\times 3 + 6\sqrt{3} - 6\sqrt{3} - 4$
$= 27 - 4 = 23$

6. $\sqrt{7}\left[3\sqrt{7} + \frac{6}{\sqrt{7}}\right]$
$\sqrt{7}(3\sqrt{7}) + \sqrt{7}\left(\frac{6}{\sqrt{7}}\right) = (3\times 7) + 6 = 21 + 6 = 27$

7. $\sqrt{50} - 3\sqrt{2}(2\sqrt{2}-5) - 5\sqrt{32} = \sqrt{25\times 2} - (3\times 2)(\sqrt{2})^2 - 3\times -5\sqrt{2} - 5\sqrt{16\times 2}$
$= 5\sqrt{2} - 6(2) + 15\sqrt{2} - 5\times 4\sqrt{2} = 5\sqrt{2} - 12 + 15\sqrt{2} - 20\sqrt{2} = -12$

Example 4.5

(i) Simplify $3\sqrt{27} - 2\sqrt{3}(4\sqrt{3} - 5\sqrt{12})$ leaving your answer in surd form

Ans: $36 + 9\sqrt{3}$

(ii) Simplify $(1-\sqrt{5})\left[\frac{1}{5} + \sqrt{5}\right]$, leaving your answer in the form $p + q\sqrt{5}$ where p and q are rational numbers ***Ans:*** $-\frac{24}{5} + \frac{4}{5}\sqrt{5}$

(Work out missing steps)

4.6 **Conjugate Surds**

The conjugate surd of $(\sqrt{a} + \sqrt{b})$ is $(\sqrt{a} - \sqrt{b})$. Likewise, the conjugate of $(4-\sqrt{2})$ is $(4+\sqrt{2})$.

NB

The multiplication of a surd and its conjugate is a rational number. Thus, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b$

4.7 **Rationalizing Surds**

Only fractions involving surds in the denominator can be rationalized. When the denominator is a surd of one term, we just multiply both the numerator and denominator by the denominator. But when the denominator is a surd of two terms, we multiply both the numerator and denominator by the conjugate of the denominator.

Example 4.6

Simplify the following:

1. $\frac{1}{\sqrt{3}}$ 2. $\frac{2}{3-\sqrt{7}}$ 3. $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ 4. $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

5. $\frac{2+\sqrt{3}}{\sqrt{3}-1}$ 6. $\frac{15}{\sqrt{50}}$ 7. $\frac{\sqrt{48}+3}{\sqrt{3}}$ 8. $\frac{3\sqrt{5}+\sqrt{45}}{\sqrt{3}(2\sqrt{3})}$

9. $\frac{\sqrt{18}\times\sqrt{20}\times\sqrt{24}}{\sqrt{8}\times\sqrt{30}}$ 10. $\sqrt{3}\left(4\sqrt{3}-\frac{5}{\sqrt{3}}\right)$

Solution

1. $\frac{1}{\sqrt{3}}=\frac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{\sqrt{3}}{3}$

2. $\frac{2}{3-\sqrt{7}}=\frac{2(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})}=\frac{6+2\sqrt{7}}{9+3\sqrt{7}-3\sqrt{7}-7}=\frac{6+2\sqrt{7}}{2}$

3. $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{3+\sqrt{6}}{3+\sqrt{6}-\sqrt{6}-2}=\frac{3+\sqrt{6}}{1}=3+\sqrt{6}$

4. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\dfrac{(\sqrt{5}-\sqrt{2})(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}=\dfrac{5-\sqrt{10}-\sqrt{10}+2}{5-\sqrt{10}+\sqrt{10}-2}=\dfrac{7-2\sqrt{10}}{3}$

5. $\dfrac{2+\sqrt{3}}{\sqrt{3}-1}=\dfrac{(2+\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}=\dfrac{2\sqrt{3}+2+3+\sqrt{3}}{3+\sqrt{3}-\sqrt{3}-1}=\dfrac{3\sqrt{3}+5}{2}$

6. $\dfrac{15}{\sqrt{50}}=\dfrac{15}{\sqrt{25\times 2}}=\dfrac{15}{5\sqrt{2}}\times\dfrac{5\sqrt{2}}{5\sqrt{2}}=\dfrac{75\sqrt{2}}{50}=\frac{75}{50}\sqrt{2}=\frac{3}{2}\sqrt{2}$

7. $\dfrac{\sqrt{48}+3}{\sqrt{3}}=\dfrac{4\sqrt{3}+3}{\sqrt{3}}=\dfrac{\sqrt{3}\left(4\sqrt{3}+3\right)}{\sqrt{3}\times\sqrt{3}}=\dfrac{4\times 3+3\sqrt{3}}{3}$

$=\dfrac{12+3\sqrt{3}}{3}=\dfrac{12}{3}+\dfrac{3\sqrt{3}}{3}=4+\sqrt{3}$

8. $\dfrac{3\sqrt{5}+\sqrt{45}}{\sqrt{3}(2\sqrt{3})}=\dfrac{3\sqrt{5}+3\sqrt{5}}{2\times 3}=\dfrac{6\sqrt{5}}{6}=\sqrt{5}$

9. $\dfrac{\sqrt{18}\times\sqrt{20}\times\sqrt{24}}{\sqrt{8}\times\sqrt{30}}=\dfrac{3\sqrt{2}\times 2\sqrt{5}\times 2\sqrt{6}}{2\sqrt{2}\times\sqrt{30}}=\dfrac{12\sqrt{60}}{2\sqrt{60}}=\dfrac{12}{2}=6$

10. $\sqrt{3}\left(4\sqrt{3}-\dfrac{5}{\sqrt{3}}\right)=4\times 3-5=12-5=7$

Example 4.7

Solve the following equations:

1. $\sqrt{x+7}-\sqrt{3x-2}=1$ 2. $x+\sqrt{x+1}=5$ 3. $\sqrt{x^2-7}=3$

Solution

1. $\sqrt{x+7}-\sqrt{3x-2}=1$

Square both sides

$\left[\left(\sqrt{x+7}-\sqrt{3x-2}\right)\right]^2=1^2$

$(\sqrt{x+7})^2 - 2(\sqrt{x+7})(\sqrt{3x-2}) + (-\sqrt{3x-2})^2 = 1^2$

$x + 7 - 2\sqrt{(x+7)(3x-2)} + 3x - 2 = 1$

Group liked terms

$x + 7 + 3x - 2 - 1 = 2\sqrt{(x+7)(3x-2)}$

$4x + 4 = 2\sqrt{3x^2 + 19x - 14}$

factorise LHS

$2(2x+2) = 2\sqrt{3x^2 + 19x - 14}$

Divide through by 2

$\Rightarrow (2x+2) = \sqrt{3x^2 + 19x - 14}$

Square both sides

$4x^2 + 8x + 4 = 3x^2 + 19x - 14$

$4x^2 + 8x + 4 - 3x^2 - 19x + 14 = 0$

$x^2 - 11x + 18 = 0$

Solve quadratically;

$x^2 - 2x - 9x + 18 = 0$

$x(x-2) - 9(x-2) = 0$

$(x-2)(x-9) = 0$

Either

$x - 2 = 0$

$\Rightarrow x = 2$

Or

$x - 9 = 0$

$\Rightarrow x = 9$

$\therefore x = 2,9$

2. $x + \sqrt{x+1} = 5$

$\Rightarrow \sqrt{x+1} = (5 - x)$

Square both sides

$(\sqrt{x+1})^2 = (5-x)^2$

$x+1 = 25-10x+x^2$

$0 = 25-10x+x^2-x-1$

$0 = 24-11x+x^2$

$0 = x^2-11x+24$

$\Rightarrow x^2-11x+24 = 0$

$x^2-3x-8x+24 = 0$

$x(x-3)-8(x-3) = 0$

$(x-3)(x-8) = 0$

Either

$x-3 = 0$

$\Rightarrow x = 3$

Or

$x-8 = 0$

$\Rightarrow x = 8$

$\therefore x = 3, 8$

3. $\sqrt{x^2-7} = 3$

Square both sides

$x^2-7 = 9$

$x^2 = 9+7 = 16$

$x^2 = 16$

Square root both sides

$\sqrt{x^2} = \sqrt{16}$

$x = \pm 4$

Example 4.8

Simplify $(1-\sqrt{3})(\frac{1}{\sqrt{3}}+\sqrt{3})$ leaving your answer in the form $p+q\sqrt{3}$ and given that $\sqrt{2} = 1.41214,$ calculate without using tables or calculators, the value of $3-\frac{3}{\sqrt{2}}$ correct to five significant figures.

Solution

$$(1-\sqrt{3})(\frac{1}{\sqrt{3}}+\sqrt{3})=(\frac{1}{\sqrt{3}}+\sqrt{3})-\sqrt{3}(\frac{1}{\sqrt{3}}+\sqrt{3})$$

$$=\frac{1}{\sqrt{3}}+\sqrt{3}-1-3$$

$$=\frac{1}{\sqrt{3}}+\sqrt{3}-4$$

Rationaliz ing

$$\frac{1}{\sqrt{3}}=\frac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{\sqrt{3}}{3}=\frac{1}{3}\sqrt{3}$$

Substituti ng

$$=\frac{1}{3}\sqrt{3}+\sqrt{3}-4$$

$$\Rightarrow\left(\frac{1}{3}+1\right)\sqrt{3}-4=\frac{4}{3}\sqrt{3}-4$$

$$=-4+\frac{4}{3}\sqrt{3}$$

$$\therefore p=-4, q=\frac{4}{3}$$

For $3-\frac{3}{\sqrt{2}}$

Rationalizing,

$$\frac{3}{\sqrt{2}}=\frac{3\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{3\sqrt{2}}{2}=\frac{3}{2}\sqrt{2}$$

Subbstituting

$$3-\frac{3}{2}\sqrt{2}$$

But $\sqrt{2}=1.41414$

$$\Rightarrow 3-\frac{3}{2}\times 1.41214=3-2.11821=0.88179$$

$$\therefore 3-\frac{3}{\sqrt{2}}=0.88179$$

EXERCISE

QUE. A

Reduce the following surds:

1. $\sqrt{484}$ 2. $\sqrt{98}$ 3. $\sqrt{150}$ 4. $\sqrt{567}$ 5. $\sqrt{605}$ 6. $\sqrt{272}$
7. $\sqrt{819}$ 8. $\sqrt{725}$ 9. $\sqrt{1692}$ 10. $\sqrt{1472}$ 11. $\sqrt{2016}$ 12. $\sqrt{5445}$

QUE. B

Express each of the following as square roots:

1. $3\sqrt{2}$ 2. $2\sqrt{3}$ 3. $8\sqrt{2}$ 4. $\frac{\sqrt{2}}{2}$ 5. $\frac{\sqrt{2}}{2\sqrt{3}}$ 6. $\frac{2}{\sqrt{6}}$
7. $5\sqrt{7}$ ***(Hint:*** Equate each to $\sqrt{x}$ and find x then put result into $\sqrt{x}$ for final answer) **NB:** This is the reverse of reduction into simplest forms.

QUE. C

Simplify the following:

1. $4\sqrt{2}+7\sqrt{2}$ 2. $9\sqrt{2}-5\sqrt{2}$ 3. $7\sqrt{3}+15\sqrt{3}$ 4. $16\sqrt{3}-10\sqrt{3}$

5. $3\sqrt{8}+5\sqrt{18}$ 6. $28\sqrt{10}-7\sqrt{90}$

7. $\sqrt{500}-\sqrt{125}$

8. $\sqrt{147}-\sqrt{75}+\sqrt{27}$

9. $3\sqrt{8}+6\sqrt{72}-6\sqrt{50}$

10. $\sqrt{1500}+3\sqrt{3}\times5\sqrt{5}+2\sqrt{15}\,(\sqrt{15}-3)$

11. $\dfrac{(\sqrt{6})(2\sqrt{2})}{(\sqrt{3})^3}$

12. $\dfrac{3\sqrt{5}+\sqrt{45}}{\sqrt{3}(2\sqrt{3})}$

13. $(6+2\sqrt{5}\left(3-\dfrac{2}{\sqrt{5}}\right)$

14. $2\sqrt{75}+3\sqrt{48}$ 15. $8\sqrt{80}-6\sqrt{45}$

QUE. D

Rationalize the denominators of the following fractions:

1. $\dfrac{3\sqrt{2}}{\sqrt{3}}$ 2. $\dfrac{5\sqrt{3}}{\sqrt{5}}$ 3. $\dfrac{13\sqrt{7}}{\sqrt{13}}$ 4. $\dfrac{7\sqrt{5}}{2\sqrt{7}}$ 5. $\dfrac{11\sqrt{10}}{2\sqrt{11}}$ 6. $\dfrac{5\sqrt{2}}{3\sqrt{15}}$

7. $\dfrac{6\sqrt{3}}{\sqrt{15}}$ 8. $\dfrac{33\sqrt{11}}{5\sqrt{39}}$ 9. $4\left(\sqrt{\dfrac{3}{2}}\right)^2$ 10. $6\left(\sqrt{\dfrac{5}{3}}\right)^3$

11. $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{10}}$ 12. $\dfrac{\sqrt{6}+1}{\sqrt{6}}$

13. $\dfrac{2\sqrt{3}+3\sqrt{7}}{\sqrt{3}+\sqrt{7}}$ 14. $\dfrac{3\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}}$ 15. $\dfrac{\sqrt{3}+2}{2\sqrt{3}-2}$ 16. $\dfrac{3-\sqrt{7}}{\sqrt{7}+2}$

17. $\dfrac{\sqrt{2}+3}{\sqrt{2}+1}$ 18. $\dfrac{1}{3-\sqrt{2}}$ 19. $\dfrac{3\sqrt{7}}{5\sqrt{5}}$ 20. $\dfrac{\sqrt{5}}{\sqrt{8}}$

QUE. E

Given that $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$ and $\sqrt{7}=2.646$ evaluate:

1. $2\sqrt{3}(2-\sqrt{3})+3\sqrt{2}(\sqrt{2}-1)$

2. $5-\dfrac{5}{\sqrt{3}}$ 3. $2\sqrt{5}\,(6-2\sqrt{5})$ 4. $3\sqrt{7}(7-2\sqrt{7})$ 5. $\sqrt{0.0007}$

QUE. F

Simplify the following:

1. $\sqrt{2}\left(3\sqrt{2}+\frac{5}{\sqrt{2}}\right)$ 2. $3\sqrt{2}(4\sqrt{2}-6\sqrt{3})$ 3. $\sqrt{3}\left(5\sqrt{3}+\frac{8}{\sqrt{3}}\right)$

4. $4\sqrt{3}(7\sqrt{3}-5\sqrt{6})$ 5. $2\sqrt{7}\left(6\sqrt{7}+\frac{4}{\sqrt{7}}\right)$ 6. $5\sqrt{13}(2\sqrt{26}-\sqrt{39})$

7. $\sqrt{10}\left(3\sqrt{5}-\frac{9}{\sqrt{2}}\right)$ 8. $3\sqrt{21}\left(8\sqrt{7}+\frac{7}{\sqrt{3}}\right)$ 9. $(\sqrt{2}+\sqrt{3})(\sqrt{3}-\sqrt{2})$

10. $(2\sqrt{5}-4\sqrt{3})(3\sqrt{5}+5\sqrt{3})$ 11. $(6\sqrt{7}-4\sqrt{5})(6\sqrt{7}+4\sqrt{5})$

12. $(9\sqrt{10}+7\sqrt{6})(9\sqrt{10}-7\sqrt{6})$ 13. $(3\sqrt{2}-2\sqrt{3})(4\sqrt{2}+\sqrt{6})$

14. $(4\sqrt{7}-3\sqrt{5})(6\sqrt{7}+2\sqrt{35})$ 15. $(5\sqrt{3}-3\sqrt{5})(7\sqrt{15}+8\sqrt{6})$

16. $\frac{(\sqrt{6})(2\sqrt{2})}{(\sqrt{3})^3}$

CHAPTER 5

INDICES

When a number is written to have an exponent, we say that number is written in the ***index form.*** In this case the number is the ***base*** and the exponent is the ***index***
(**plural **indices).
Thus, an index form explains how many times the base number should be multiplied by itself. For instance, in 2^3, 3 is the index and 2 the base. By interpretation, 2^3 means 'multiply 2 by itself two times'. i.e. $2^3 = 2\times 2\times 2 = 8$

7.1 **Laws Of Indices**

1. $a^m \times a^n = a^{m+n}$ E.g. $3^2 \times 3^4 = 3^{2+4} = 3^6$ and $b^2 \times b^3 = b^{2+3} = b^5$

2. $a^m \div a^n = a^{m-n}$ E.g. $3^4 \div 3^2 = 3^{4-2} = 3^2 = 9$ and $x^5 \div x^2 = x^{5-2} = x^3$

3. $(a^m)^n = a^{mn}$ E.g. $(2^3)^2 = 2^{3\times 2} = 2^6$ and $(b^2)^4 = b^{2\times 4} = b^8$

5.2 **Properties Of Indices**

1. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ Eg. $\left(\frac{3}{4}\right)^2 = \left(\frac{3^2}{4^2}\right) = \left(\frac{9}{16}\right)$
2. Any non-zero number raised to the power zero is equal to 1
Eg. $b^0 = 1,\ 1000^0 = 1$
3. $(ab)^x = a^x b^x$ Eg. $(3\times 2)^2 = 3^2 \times 2^2 = 9\times 4 = 36$
4. $a^{-1} = \frac{1}{a}$ Eg. $2^{-1} = \frac{1}{2}$
5. $a^{-n} = \frac{1}{a^n}$ Eg. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \left(\frac{b^n}{a^n}\right), b \neq 0$ Eg. $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \left(\frac{3^2}{2^2}\right) = \frac{9}{4}$
7. $a^{\frac{1}{n}} = \sqrt[n]{a}$ Eg. $4^{\frac{1}{2}} = \sqrt{4} = 2$ and $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

8. $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ Eg. $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

9. If $a^x = a^y$, then $x = y$ Eg. If $3^2 = 3^x$ then $x = 2$

i.e. when the bases are equal, we equate the exponents

10. If $a^x = b^x$, then $a = b$ Eg. If $4^3 = x^3$, then $x = 4$

i.e. when the exponents are equal, we equate the bases

5.3 **Applications Of Properties Of Indices**

Example 5.1

Simplify the following:

1. $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$ 2. $27^{\frac{2}{3}}$ 3. $16^{\frac{3}{4}}$ 4. $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

5. $\frac{16^{\frac{1}{3}} \times 4^{\frac{1}{3}}}{8}$ 6. $(27 \times 3^{-3})(8 \times 2^{-3})$ 7. $6^{4x} = 6^{2x+4}$

8. $\sqrt[3]{2^6 \times 5^3}$

9. $\sqrt{\left(\frac{x^3 y^5}{xy^7}\right)}$, *where* $x > 0$ *and* $y > 0$.

Solution

1. $\left(\frac{27}{8}\right)^{-\frac{2}{3}} = \left(\frac{8}{27}\right)^{\frac{2}{3}} = \left[\left(\frac{8}{27}\right)^{\frac{1}{3}}\right]^2 = \left(\frac{\sqrt[3]{8}}{\sqrt[3]{27}}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$

2. $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

3. $16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$

4. $\left(\frac{16}{81}\right)^{\frac{1}{4}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$

5. $\frac{16^{\frac{1}{3}} \times 4^{\frac{1}{3}}}{8} = \frac{(4^2)^{\frac{1}{3}} \times 4^{\frac{1}{3}}}{8} = \frac{4^{\frac{2}{3}} \times 4^{\frac{1}{3}}}{8} = \frac{4^{\frac{2}{3}+\frac{1}{3}}}{8} = \frac{4^1}{8} = \frac{4}{8} = \frac{1}{2}$

$NB: We\ tried\ to\ make\ the\ terms\ have\ a\ common\ base$

6. $(27\times3^{-3})(8\times2^{-3}) = (3^3\times3^{-3})(2^3\times2^{-3}) = (3^{3+-3})(2^{3+-3}) = 3^0\times2^0 = 1\times1 = 1$

7. $6^{4x} = 6^{2x+4} \Rightarrow 4x = 2x+4\,(Since\ bases\ are\, equal)$

$$4x - 2x = 4$$

$$\frac{2x}{2} = \frac{4}{2} \Rightarrow x = 2$$

8. $\sqrt[3]{2^6\times5^3} = 2^{6\times\frac{1}{3}}\times5^{3\times\frac{1}{3}} = 2^2\times5^1 = 4\times5 = 20$

9. $\sqrt{\left(\frac{x^3y^5}{xy^7}\right)} = \left(\frac{x^3y^5}{xy^7}\right)^{\frac{1}{2}} = (x^{3-1}y^{5-7})^{\frac{1}{2}} = (x^2y^{-2})^{\frac{1}{2}} = xy^{-1} = \frac{x}{y}$

5.4 **Indicial Equations**

Indicial equations are equations which have the unknown variable or quantity as an ***index*** or ***base.***
In solving such equations, we always express both ***LHS*** and ***RHS*** to have the
same exponent or the same base then we equate exponents when bases are equal and equate bases when exponents are equal.

Example 5.2

Solve for the value of the variable in the following equations:

1. $4^{(x-3)} = 8$ 2. $27^{(x+1)} = 9$ 3. $3^{\sqrt{x}} = 9$ 4. $3^{(x-2)} = \frac{1}{81}$

5. $2^{(3x-1)} = 128\sqrt{2}$ 6. $8^x = 32$ 7. $5^{x-1} = 5^{2x-5}$ 8. $x^5 = 1024$

9. $8^x = 2\sqrt{2}$

10. $\left(\frac{1}{3}\right)^{3x} = 81^{-\frac{3}{4}}$

11. $\frac{64^n \times 2}{16^{1-n}} = 4^{2n}$

Solution

1. $4^{(x-3)} = 8$

$\Rightarrow (2^2)^{(x-3)} = (2^3)$

$2^{2(x-3)} = 2^3$

$2^{2x-6} = 2^3$

Since bases are equal equate the exponents

$\Rightarrow 2x - 6 = 3$

$2x = 3 + 6$

$\frac{2x}{2} = \frac{9}{2}$

$\therefore x = \frac{9}{2}$

2. $27^{(x+1)} = 9$

$\Rightarrow (3^3)^{(x+1)} = (3^2)$

$3^{3(x+1)} = 3^2$

$3^{(3x+3)} = 3^2$

$\Rightarrow 3x + 3 = 2$

$3x = 2 - 3$

$\frac{3x}{3} = -\frac{1}{3}$

$\therefore x = -\frac{1}{3}$

3. $3^{\sqrt{x}} = 9$

$\Rightarrow 3^{\sqrt{x}} = 3^2$

$\Rightarrow \sqrt{x} = 2$

Square both sides

$\Rightarrow x = 2^2 = 4$

$\therefore x = 4$

4. $3^{(x-2)} = \dfrac{1}{81}$

$3^{(x-2)} = 81^{-1}$

$3^{(x-2)} = (3^4)^{-1}$

$\Rightarrow 3^{(x-2)} = 3^{-4}$

$\Rightarrow x - 2 = -4$

$x = 2 - 4 = -2$

$\therefore x = -2$

5. $2^{(3x-1)} = 128\sqrt{2}$

$2^{(3x-1)} = 2^7 \times 2^{\frac{1}{2}}$

$\Rightarrow 2^{(3x-1)} = 2^{\frac{15}{2}}$

Equate bases since exponents are equal

$\Rightarrow 3x - 1 = \frac{15}{2}$ (*Multiply through by* 2)

$\Rightarrow 2(3x-1) = 15 \Rightarrow 6x - 2 = 15 \Rightarrow 6x = 17 \Rightarrow x = \dfrac{17}{6}$ *or* $2\frac{5}{6}$

6. $8^x = 32$

Making the bases equal we have;

$2^{3x} = 2^5 \Rightarrow 3x = 5 \Rightarrow x = \dfrac{5}{3}$ *or* $1\frac{2}{3}$

7. $5^{x-1} = 5^{2x-5}$

$\Rightarrow x - 1 = 2x - 5$

$-1 + 5 = 2x - x$

$4 = x$

8. $x^5 = 1024 \Rightarrow x^5 = 4^5 \Rightarrow x = 4$

9. $8^{x} = 2\sqrt{2} \Rightarrow 2^{3x} = 2^{1} \times 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$

$\Rightarrow 2^{3x} = 2^{\frac{3}{2}}$

$\Rightarrow 3x = \frac{3}{2} \Rightarrow \frac{6x}{6} = \frac{3}{6} \Rightarrow x = \frac{1}{2}$

10. $\left(\frac{1}{3}\right)^{3x} = 81^{-\frac{3}{4}} \Rightarrow 3^{-3x} = 3^{-3} \Rightarrow -3x = -3 \Rightarrow x = 1$

11. $\frac{64^{n} \times 2}{16^{1-n}} = 4^{2n}$

$\Rightarrow \frac{2^{6n} \times 2^{1}}{2^{4-4n}} = 2^{4n} \Rightarrow \frac{2^{6n+1}}{2^{4-4n}} = 2^{4n} \Rightarrow 2^{6n+1-4+4n} = 2^{4n}$

$\Rightarrow 6n + 1 - 4 + 4n = 4n \Rightarrow 10n - 3 = 4n \Rightarrow 10n - 4n = 3 \Rightarrow \frac{6n}{6} = \frac{3}{6}$

$\Rightarrow n = \frac{1}{2}$

5.5 Special Indicial Equations

5.5.1 Case One

Here, the question is in two parts, the first and second parts where the first part is usually the part that requires applications of laws and properties of indices whiles the second part is just what the question requires you to solve for. Thus, in solving such questions we:

1. Simplify the first part by applying laws and properties of indices where necessary.
2. Finally, we make what is asked of in the question the subject of the resulting sumplest form.

Example 5.3

If $2^{-n} = x$,

find 2^{n}

Solution

From first part,

$$2^{-n} = x$$

Apply properties of indices to LHS

$$\Rightarrow \frac{1}{2^n} = x$$

Now make 2^n *the subject*

$$\Rightarrow 2^n = \frac{1}{x}$$

Example 5.4

If $3^{-x} = y,\ find\ 3^x$

Solution

$$first\ part\ is\ \ 3^{-x} = y$$

$$Simplifying\ we\ get\ \frac{1}{3^x} = y$$

Making 3^x *the subject gives,*

$$3^x = \frac{1}{y}$$

5.5.2 **Case Two**

Here, an indicial equation will be given with a variable as either an index or a base where you cannot re-express it to have equal bases and equal indices.

Thus, we solve such questions by:

1. Taking the logarithm of both sides to base ten and simplify using the laws and properties of logarithms
2. Make the variable asked of in the question the subject

Example 5.5

Find x if $3^x = 8$

Solution

$$3^x = 8$$

Since we cannot express both sides to have a common base, we take logarithm to base 10 of both sides

Thus,

$$\Rightarrow \log_{10} 3^x = \log_{10} 8$$

$$\frac{x\log_{10} 3}{\log_{10} 3} = \frac{\log_{10} 8}{\log_{10} 3}$$

$$\Rightarrow x = \frac{0.9031}{0.4771} = 1.892$$

$$\therefore x = 1.892$$

5.5.3 **Case Three**

Here, a question is asked and the first part will not require any application of laws and properties of indices whiles the second part requires application of laws and properties of indices.

Thus, we solve such cases by:

1. First simplifying the second part of the question by applying appropriately the laws and properties of indices where necessary.
2. Then we substitute the first part of the question to the resulting expression in step one to obtain an expression for the second part.

Example 5.6

If $5^n = k,\ \mathit{find}\ 5^{(n+1)}$

Solution

Given that: $5^n = k$

Applying laws of indices to the second part; $5^{(n+1)}$ gives

$5^{(n+1)} = 5^n \times 5^1$

Now substituting first part into results gives;

$5^{(n+1)} = 5^n \times 5^1 = k \times 5 = 5k$

$\therefore 5^{(n+1)} = 5k$

5.5.4 **Case Four**

Here, in the question, both the first and second parts will require simplifications using laws and properties of indices.

Thus, we:

1. We apply appropriately the laws and properties where necessary to the first part making the term with the index form the subject
2. Similarly, obtain an expression for the second part by applying the laws and properties of indices to it
3. Finally, substitute the result of step one into the result of step two for the answer

Example 5.7

If $2^{-1} = x,\ \mathit{find}\ 2^{n+1}.$

Solution

Applying properties of indices to first part: $2^{-n} = x$

$$2^{-n} = \frac{1}{2^n}$$

Cross and multiplying gives;

$1 = x \times 2^n$

$$\Rightarrow \frac{1}{x} = \frac{x \times 2^n}{x}$$

$$\Rightarrow 2^n = \frac{1}{x} ----------------(1)$$

Further, we apply law 1 to second part: 2^{n+1}

$2^{n+1} = 2^n \times 2^1 --------------(2)$

Now, substitute (1) into (2)

$$2^{n+1} = \frac{1}{x} \times 2 = \frac{2}{x}$$

5.5.5 **Case Five**

Here, we are given two equations and asked to find the unknown variables.
Thus, we:

1. Simplify each equation appropriately applying laws and properties of indices where necessary
2. Then solve simultaneously the resulting simplified equations

Example 5.8

Find the values of x and y satisfying the equations:

$$x + 6y = 3$$

$$2^{(x+y)} = \frac{1}{4}$$

Solution

$$x + 6y = 3 ----- (1)$$

$$2^{(x+y)} = \frac{1}{4} ----- (2)$$

Since equation one does not require any further simplification, we go straight to equation two.
From (2),

$$2^{(x+y)} = 4^{-1} = (2^2)^{-1} = 2^{-2}$$

$$\Rightarrow 2^{(x+y)} = 2^{-2}$$

$$\Rightarrow x + y = -2 ----(3)$$

Solve (1) and (3) simultaneously
From (1),

$$x = 3 - 6y ----- (4)$$

Put (4) into (3)

$\Rightarrow 3-6y+y=-2$

$3-5y=-2$

$3+2=5y$

$\frac{5}{5}=\frac{5y}{5}$

$1=y$

$\therefore y=1$

Put $y=1$ into (4)

$\Rightarrow x=3-6(1)=3-6=-3$

$\therefore x=-3$

Example 5.9

Solve for x and y in the following equations:

$2^{(x+4y)}=1$

$2^{(x+8y)}=\frac{1}{2}$

Solution

$2^{(x+4y)}=1--------(1)$

$2^{(x+8y)}=\frac{1}{2}--------(2)$

$From\ (1),\ 2^{(x+4y)}=2^0 \Rightarrow x+4y=0--------(3)$

$From\ (2),\ 2^{(x+8y)}=2^{-1} \Rightarrow x+8y=-1-------(4)$

$(4)-(3)$

$4y=-1 \Rightarrow y=-\frac{1}{4}$

Put $y=-\frac{1}{4}$ $\text{int}\,o$ (3)

$x+4\left(-\frac{1}{4}\right)=0 \Rightarrow x-1=0 \Rightarrow x=1$

5.5.6 **Case Six**

Here, we solve by:

1. Applying the properties and laws of indices to the indicial terms in the equation appropriately.

2. Then represent the resulting common indicial expression in the result of step one by a variable
3. Replace the indicial expression by the variable chosen in step two
4. Solve quadratically for the variable
5. After which equate the indicial expression to the result of the variable found in step four and evaluate for the original variable in the initial equation given in the question

Example 5.10

Solve the equation: $3^{2x} - 30(3^x) + 81 = 0$

Solution

$3^{2x} - 30(3^x) + 81 = 0$

From $3^{2x} - 30(3^x) + 81 = 0$

Step 1

Applying laws of indices and rewriting becomes,

$(3^x)^2 - 30(3^x) + 81 = 0$

Step 2

Let $3^x = a$

Step 3

Substituting becomes,

$a^2 - 30a + 81 = 0$

Step 4

$a^2 - 30a + 81 = 0$

$sovlving\ quadratically$

$a^2 - 3a - 27a + 81 = 0$

$a(a-3) - 27(a-3) = 0$

$\Rightarrow (a-3)(a-27) = 0$

Either

$a - 3 = 0$

$\Rightarrow a = 3$

Or

$a-27=0$

$\Rightarrow a=27$

Step 5

But $3^x=a$

When $a=3$

$3^x=3^1$

$\therefore x=1$

When $a=27$

$\Rightarrow 3^x=27$

$\Rightarrow 3^x=3^3$

$\therefore x=3$

Example 5.11

Find the solution of the equation: $3^{2x}-3^{(x+2)}+8=0$

Solution

$3^{2x}-3^{(x+2)}+8=0$

Applying laws of indices appropriately gives;

$\left(3^x\right)^2 - 3^x \times 3^2 + 8 = 0$

$Now\ let\ 3^x = a$

$\Rightarrow a^2 - 9a + 8 = 0$

$Solving\ quadratically\ gives,$

$a^2 - a - 8a + 8 = 0$

$a(a-1) - 8(a-1) = 0$

$Either\ a - 1 = 0 \Rightarrow a = 1$

Or

$a - 8 = 0 \Rightarrow a = 8$

$But\ 3^x = a$

$Hence,\ when\ a = 1 \Rightarrow 3^x = 1 \Rightarrow 3^x = 3^0\ \therefore x = 0$

$Again,\ when\ a = 8$

$\Rightarrow 3^x = 8$

$Taking\ \log\ to\ base\ 10\ of\ both\ sides\ gives,$

$\log_{10} 3^x = \log_{10} 8$

$$\Rightarrow \frac{x \log_{10} 3}{\log_{10} 3} = \frac{\log_{10} 8}{\log_{10} 3} \Rightarrow x = 1.892$$

$Hence,\ x = 0,\ 1.892$

Example 5.12

Solve: $3^{2x+3} - 4(3^{(x+1)}) + 1 = 0$

Solution

$3^{2x+3} - 4(3^{(x+1)}) + 1 = 0$

Similarly, applying the laws of indices appropriately gives;

$3^{2x} \times 3^3 - 4[(3^x \times 3^1] + 1 = 0$

$\Rightarrow (3^x)^2 \times 3^2 - 4[3^x \times 3^1] + 1 = 0$

Let $3^x = a$

Substituting becomes;

$27a^2 - 12a + 1 = 0$

$27a^2 - 9a - 3a + 1 = 0$

$\Rightarrow 9a(3a - 1) - (3a - 1) = 0$

$\Rightarrow (3a - 1)(9a - 1) = 0$

Either $3a - 1 = 0 \Rightarrow a = \frac{1}{3}$

Or

$9a - 1 = 0 \Rightarrow a = \frac{1}{9}$

But $3^x = a$

Hence,

when $a = \frac{1}{9}$, $3^x = \frac{1}{9} \Rightarrow 3^x = 3^{-2} \Rightarrow x = -2$

Again, *when* $a = \frac{1}{3}$, $3^x = \frac{1}{3} \Rightarrow 3^x = 3^{-1} \Rightarrow x = -1$

Therefore, $x = -2, -1$

EXERCISE

QUE. A

Solve the following equations:

1. $9^{x} = 81$ 2. $25^{x} = 125$ 3. $16^{x} = 8$ 4. $49^{x} = 7$ 5. $8^{x} = 32$

6. $3^{2x} = 81$ 7. $5^{3x} = 5$ 8. $625^{x} = 5$ 9. $5^{x} = 1$ 10. $125^{x} = 625$

11. $729^{x} = 81$ 12. $16^{x} = \frac{1}{2}$ 13. $3^{x} = \frac{1}{81}$ 14. $49^{-x} = 7$ 15. $3^{-3x} = \frac{1}{3}$

16. $10^{-3x} = \frac{1}{100}$ 17. $x^{2} = 625$ 18. $x^{3} = 64$ 19. $x^{4} = \frac{16}{81}$ 20. $x^{5} = 243$

21. $x^{-2} = 25$ 22. $x^{-3} = 8$ 23. $x^{-4} = 256$ 24. $x^{-5} = \frac{1}{32}$ 25. $x^{\frac{1}{2}} = 2$

26. $x^{-\frac{1}{3}} = 3$ 27. $x^{\frac{3}{4}} = 3^{3}$ 28. $x^{\frac{3}{2}} = 1000$ 29. $(3x)^{\frac{1}{2}} = 5$ 30. $\left(\frac{x}{3}\right)^{-\frac{1}{2}} = 3$

31. $\left(\frac{3x}{2}\right)^{\frac{3}{2}} = 3^{\frac{1}{2}}$ 32. $\left(\frac{x}{5}\right)^{-\frac{1}{3}} = 2$ 33. $5^{x} = \frac{1}{25}$ 34. $x^{4} = \frac{1}{16}$ 35. $x^{-\frac{3}{2}} = 216$

36. $\left(\frac{1}{8}\right)^{(4x-9)} = 64^{(3x-1)}$ 37. $2^{(9x+4)} = \frac{1}{32}$ 38. $3^{(5x+3)} = 81^{(x+3)}$ 39. $4^{(5x-x^{2})} = 4^{6}$

40. $9^{x^{2}} = 3^{(3x-1)}$ 41. $\left(\frac{1}{4}\right)^{(3x^{2}+1)} = 128^{x}$ 42. $4^{2-x} \times 16^{x+1} = 64$ 43. $\frac{1}{2}(81^{n}) = \frac{1}{54}$

44. $3^{2x} - 4(3^{x}) + 3 = 0$ 45. $3^{x} = 243$

QUE. B

Find the values of the following:

1. $\dfrac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}}$ 2. $\left(\dfrac{256}{81}\right)^{-\frac{3}{4}}$ 3. $(0.04)^{-\frac{3}{2}}$ 4. $\dfrac{x^{-\frac{2}{3}} \times x^{\frac{1}{4}}}{x^{\frac{1}{6}}}$

5. $\dfrac{8^{\frac{1}{6}} \times 4^{\frac{1}{3}}}{32^{\frac{1}{6}} \times 16^{\frac{1}{12}}}$ 6. $\left(\dfrac{27}{8}\right)^{-\frac{4}{3}}$ 7. $\dfrac{(x^{\frac{3}{2}} + x^{\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})}{(x^{\frac{3}{2}} - x^{\frac{1}{2}})^{2}}$

8 . $(\sqrt{5})^{-2} \times 75^{\frac{1}{2}} \times 12^{-\frac{1}{2}}$

NB:

$$\left(\sqrt{5}\right)^{-2} \times 75^{\frac{1}{2}} \times 12^{-\frac{1}{2}} = (5^{\frac{1}{2}})^{-2} \times (25^{\frac{1}{2}} \times 3^{\frac{1}{2}}) \times (4^{-\frac{1}{2}} \times 3^{-\frac{1}{2}}) = 5^{-1} \times 5^{1} \times 3^{\frac{1}{2}} \times 3^{-\frac{1}{2}} \times 4^{-\frac{1}{2}}$$
$$= 5^{0} \times 3^{0} \times \tfrac{1}{2} = \tfrac{1}{2}$$

9 . $\dfrac{27^{-\frac{1}{2}} \times 81^{\frac{3}{4}}}{9^{\frac{1}{2}}}$

10. $10000^{-\frac{1}{4}}$ 11. $81^{\frac{1}{4}}$ 12. $27^{\frac{2}{3}}$ 13. $125^{-\frac{1}{3}}$ 14. $\left(\dfrac{1}{16}\right)^{-\frac{3}{4}}$ 15. $\left(\dfrac{1}{216}\right)^{\frac{1}{3}}$

16. $32^{\frac{2}{5}}$ 17. $36^{\frac{5}{2}}$ 18. $49^{-\frac{1}{2}}$ 19. $64^{\frac{2}{3}}$ 20. $625^{-\frac{3}{4}}$

QUE. C

Solve the following equations simultaneously:

1. $\left(\dfrac{1}{3}\right)^{2x} = \left(\dfrac{1}{9}\right)^{(y+2)}$ and $3^{(x-1)} = 27^{(y-1)}$
2. $3^m \times 3^n = 243$ and $3^m \div 3^{2n} = 9$
3. $2^{(x-y)} = 8$ and $2^{(3x-y)} = 128$ 4. $2^{x+y} = 8$ *and* $2x + y = 4$

QUE. D

If $(3n-1)\frac{1}{3} = 2,\ find\ n$

CHAPTER 6

LOGARITHMS

The ***logarithm*** of any number ***x*** to the base ***b*** is the exponent to which ***b*** must be raised to obtain ***x.*** In calculations, the word 'logarithm' is shortened as ***'log'*** or ***'lg'***. From the definition above,

$y = \log_{10} x$ is equivalent to $x = 10^y$

and

$y = \log_e x$ is equivalent to $x = e^y$

NB

1. A logarithm is an index
2. The logarithm of a negative number does not exist
3. Generally, whenever the base of a log is not written, it is taken to be in base 10

6.1 **Changing Logarithmic Forms To Exponential Forms**

By the definition of logarithms above, a logarithm can be changed into an exponential form.
This is done by simply simply leaving the number on the LHS and sending the base across to the RHS and then put it down after which we put the original number already on that side on top of it. In other words leave the log number and take its base across and make it the new base to the number already on that side.

Example 6.1

Change each logarithmic form to an equivalent exponential form

1. $\log_2 8 = 3$ 2. $\log_{25} 5 = \frac{1}{2}$ 3. $\log_2 (\frac{1}{4}) = -2$ 4. $\log_3 9 = 2$

5. $\log_4 2 = \frac{1}{2}$

Solution

1. *From* $\log_2 8 = 3$, 8 *is the number and* 2 *is the base. So we leave the number* 8 *on the LHS and take the base* 2 *to the RHS to become the base of that side.*

Thus, $\log_2 8 = 3$ *is equivalent to* $8 = 2^3$.

2. $\log_{25} 5 = \frac{1}{2}$ is equivalent to $5 = 25^{\frac{1}{2}}$

3. $\log_2 (\frac{1}{4}) = -2$ is equivalent to $\frac{1}{4} = 2^{-2}$

4. $\log_3 9 = 2$ is equivalent to $9 = 3^2$

5. $\log_4 2 = \frac{1}{2}$ is equivalent to $2 = 4^{\frac{1}{2}}$

6.2 **Changing Exponential Forms To Logarithmic Forms**

By the definition of logarithms, an index can be changed into a logarithmic form.

Thus, we make the index the only number on that side and then bring the base across to the other side to become the base of the log.

Example 6.2

Change each exponential form to an equivalent logarithmic form

1. $49 = 7^2$ 2. $3 = \sqrt{9}$ 3. $\frac{1}{5} = 5^{-1}$ 4. $81 = \left(\frac{1}{3}\right)^{-4}$ 5. $8^{-\frac{2}{3}} = \frac{1}{4}$

Solution

1. $49 = 7^2$ is equivalent to $\log_7 49 = 2$

2. $3 = \sqrt{9}$ is equivalent to $\log_9 3 = \frac{1}{2}$

3. $\frac{1}{5} = 5^{-1}$ is equivalent to $\log_5 \frac{1}{5} = -1$

4. $81 = \left(\frac{1}{3}\right)^{-4}$ *is equivalent to* $\log_{\frac{1}{3}} 81 = -4$

5. $8^{-\frac{2}{3}} = \frac{1}{4}$ *is equivalent to* $-\frac{2}{3} = \log_8 \frac{1}{4}$

6.3 **Solutions Of The Equation** $y = \log_b x$

We evaluate $\log_b x$ by first equating it to a variable and applying the definition of logarithm and properties and laws of indices appropriately.

Example 6.3

Using the definition of logarithm, find the following:
1. $\log_4 8$ 2. $\log_3 x = -2$ 3. $\log_9 27$ 4. $\log_8 64$ 5. $\log_2 0.25$

Solution

1. $\log_4 8$
Let $\log_4 8 = y$

$\Rightarrow 8 = 4^y$

$\Rightarrow 2^3 = (2^2)^y$

$\Rightarrow 2^3 = 2^{2y}$

$\frac{3}{2} = \frac{2y}{2}$

$\therefore y = \frac{3}{2}$

$\log_4 8 = \frac{3}{2}$

2. $\log_3 x = -2$

$\Rightarrow x = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$\therefore x = \frac{1}{9}$

$\log_3\left(\frac{1}{9}\right) = -2$

3. $\log_9 27$

Let $\log_9 27 = y$

$\Rightarrow 27 = 9^y$

$3^3 = (3^2)^y$

$3^3 = 3^{2y}$

$\frac{3}{2} = \frac{2y}{2}$

$\frac{3}{2} = y$

$y = \frac{3}{2}$

$\therefore \log_9 27 = \frac{3}{2}$

4. $\log 64$

Let $\log_8 64 = t$

$\Rightarrow 64 = 8^t \Rightarrow 8^2 = 8^t \Rightarrow 2 = t$

Hence, $\log_8 64 = 2$

5. $\log_2 0.25$

Let $\log_2 0.25 = x$

$\Rightarrow 0.25 = 2^x \Rightarrow \frac{25}{100} = 2^x \Rightarrow \frac{1}{4} = 2^x \Rightarrow 2^{-2} = 2^x \Rightarrow -2 = x$

Hence, $\log_2 0.25 = -2$

6.4 **Laws Of Logarithms**

1. $\log_a x + \log_a y = \log_a(xy)$ Eg. $\log_3 4 + \log_3 5 = \log_3(4 \times 5) = \log_3 20$

2. $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ Eg. $\log_2 15 - \log_2 3 = \log_2\left(\frac{15}{3}\right) = \log_2 5$

3. $\log_a x^n = n\log_a x$ Eg. $\log_2 5^3 = 3\log_2 5$

6.5 **Properties Of Logarithms**

1. $\log_b 1 = 0$ Eg. $\log_{10} 1 = 0$ ie. The logarithm of 1 to any base is 0.
2. $\log_b b = 1$ Eg. $\log_{10} 10 = 1$ ie. The logarithm of any number to the same base is 1.
3. $\log_b b^x = x$ Eg. $\log_{10} 10^2 = 2\log_{10} 10 = 2\times 1 = 2$
4. $b^{\log_b x} = x, x > 0$ Eg. $3^{\log_3 2} = 2$

6.6 **Applications Of Properties Of Logarithms**

Example 6.4

Simplify the following:

1. $\log_e 1$ 2. $\log_e e^{(2x+1)}$ 3. $\log_{10} 0.01$ 4. $10^{\log_{10} 7}$ 5. $e^{\log_e x^2}$

Solution

1. $\log_e 1 = 0$
2. $\log_e e^{(2x+1)} = (2x+1)\log_e e = 2x+1$
3. $\log_{10} 0.01 = \log_{10} \frac{1}{100} = \log_{10} 100^{-1} = -1\log_{10} 100 = -1\log_{10} 10^2$
$= -1\times 2\log_{10} 10 = -1\times 2\times 1 = -2$
4. $10^{\log_{10} 7} = 7$
5. $e^{\log_e x^2} = x^2$

Example 6.5

Evaluate the following:

1. $3\log_{10} x + \log_{10} 3 = \log_{10} 81$

2. $\log_3(2x-1) - \log_3(2x+1) = 1$

3. $\log_b x = \frac{2}{3}\log_b 27 + 2\log_b 2 - \log_b 3$

4. $\log_b x = \frac{2}{3}\log_b 8 + \frac{1}{2}\log_b 9 - \log_b 6$

5. $\log k - \log(k-2) = \log 5$ 6. $\log 25 + \log 32 - \log 8$

7. $\log(5x-4) = \log(x+1) + \log 4$

8. $\frac{1}{2}\log_{10} \frac{25}{4} - 2\log_{10} \frac{4}{5} + \log_{10} \frac{320}{125}$

9. $3\log_{10} 2 + \log_{10} 20 - \log_{10} 1.6$

Solution

1. $3\log_{10} x + \log_{10} 3 = \log_{10} 81$

$\log_{10} x^3 + \log_{10} 3 = \log_{10} 81$ (*third law of* $\log s$ *is applied to first term*)

$\Rightarrow \log_{10} 3x^2 = \log_{10} 81$ (*First law is applied to LHS*)

Take antilog of both sides becomes,

$3x^2 = 81$

$\Rightarrow \frac{3x^3}{3} = \frac{81}{3}$

$\Rightarrow x^3 = 27$

Cube root both sides to get,

$\therefore x = 3$

2. $\log_3(2x-1) - \log_3(2x+1) = 1$

Applying law 2 to LHS becomes,

$\Rightarrow \log_3\left(\frac{2x-1}{2x+1}\right) = 1$

By definition of logs,

$\left(\frac{2x-1}{2x+1}\right)=3$ $(ie\ anti\log\ of\ both\ sides)$

$\frac{2x-1}{2x+1}=3$

$\Rightarrow 2x-1=3(2x+1)$

$2x-1=6x+3$

$-1-3=6x-2x$

$\frac{-4}{4}=\frac{4x}{4}$

$-1=x$

$\Rightarrow x=-1$

3. $\log_b x=\frac{2}{3}\log_b 27+2\log_b 2-\log_b 3$

$=\frac{2}{3}\log_b 3^3+2\log_b 2-\log_b 3$

$=3\times\frac{2}{3}\log_b 3+2\log_b 2-\log_b 3$

$=2\log_b 3+2\log_b 2-\log_b 3$

$=\log_b 3^2+\log_b 2^2-\log_b 3$

$=\log_b 9+\log_b 4-\log_b 3$

$=\log_b(9\times 4)-\log_b 3$

$=\log_b\left(\frac{9\times 4}{3}\right)=\log_b(3\times 4)=\log_b 12$

$\Rightarrow \log_b x=\log_b 12$

Take antilog of both sides

$\Rightarrow x=12$

4. $\log_b x = \frac{2}{3}\log_b 8 + \frac{1}{2}\log_b 9 - \log_b 6$

$(Apply\ third\ law\ to\ first\ two\ terms)$

$= \log_b 8^{\frac{2}{3}} + \log_b 9^{\frac{1}{2}} - \log_b 6$

$(Apply\ law 1\ to\ first\ two\ terms)$

$= \log_b (8^{\frac{2}{3}} \times 9^{\frac{1}{2}}) - \log_b 6$

$(Apply\ properties\ to\ first\ term)$

$= \log_b \left[\left(\sqrt[3]{8}\right)^2 \times \sqrt{9} \right] - \log_b 6$

Simplify first term

Implies,

$= \log_b (2^2 \times 3) - \log_b 6 = \log_b 12 - \log_b 6$

Applying law 2 gives;

$$= \log_b \tfrac{12}{6} = \log_b 2$$

$\Rightarrow \log_b x = \log_b 2\ (Taking\ anti\log\ of\ both\ sides\ becomes)$

$x = 2$

5. $\log k - \log(k-2) = \log 5\ (Apply\ second\ law\ to\ LHS)$

$\Rightarrow \log(\frac{k}{k-2}) = \log 5\ (Taking\ anti\log\ gives)$

$\Rightarrow \frac{k}{k-2} = 5$

Simplifying we have,

$k = 5(k-2) \Rightarrow k = 5k - 10 \Rightarrow 10 = 5k - k \Rightarrow 10 = 4k \Rightarrow \frac{10}{4} = \frac{4k}{4}$

$Hence,\ k = \frac{5}{2}$

6. $\log 25 + \log 32 - \log 8\ (Apply\ first\ and\ second\ laws\ here)$

$\Rightarrow \log \frac{25 \times 32}{8} = \log 100 = \log 10^2 = 2\log 10 = 2$

$\therefore \log 25 + \log 32 - \log 8 = 2$

7. $\log(5x-4) = \log(x+1) + \log 4\ (Apply\ first\ law\ to\ RHS)$

$\Rightarrow \log(5x-4) = \log 4(x+1)\ (Take\ anti\log)$

$\Rightarrow 5x - 4 = 4(x+1) = 4x + 4 \Rightarrow 5x - 4x = 4 + 4 \Rightarrow x = 8$

8. $\frac{1}{2}\log_{10}\frac{25}{4}-2\log_{10}\frac{4}{5}+\log_{10}\frac{320}{125}=\log\left(\frac{25}{4}\right)^{\frac{1}{2}}-\log\left(\frac{4}{5}\right)^{2}+\log\frac{64}{25}$

$$=\log\frac{\sqrt{25}}{\sqrt{4}}-\log\frac{4^2}{5^2}+\log\frac{64}{25}=\log\frac{5}{2}-\log\frac{16}{25}+\log\frac{64}{25}$$

$$=\log\left(\frac{5}{2}\div\frac{16}{25}\right)+\log\frac{64}{25}=\log\left(\frac{5}{2}\times\frac{25}{16}\right)+\log\frac{64}{25}$$

$$=\log\frac{5}{2}\times\frac{25}{16}\times\frac{64}{25}=\log 10=1$$

9. $3\log_{10}2+\log_{10}20-\log_{10}1.6=3\log_{10}2+\log_{10}(10\times 2)-[\log_{10}(16\times 10^{-1})]$

$$=3\log_{10}2+\log_{10}10+\log_{10}2-[\log_{10}16+\log_{10}10^{-1}]$$

$$=3\log_{10}2+1+\log_{10}2-[\log_{10}2^4+\log_{10}10^{-1}]=3\log_{10}2+1+\log_{10}2-4\log_{10}2+1\log_{10}10$$

$$=(3+1-4)\log_{10}2+1+1=0\log_{10}2+2=2$$

Example 6.6

Given that $\log_5 2=0.431$ and $\log_5 3=0.682$, find the values of:

1. $\log_5 10$ 2. $\log_5\left(\dfrac{3}{2}\right)$ 3. $\log_5 12$ 4. $\log_5\left(\dfrac{3}{8}\right)+2\log_5\left(\dfrac{4}{5}\right)-\log_5\left(\dfrac{2}{5}\right)$

Solution

NB

Here it is advisable to first simplify the log by expressing the number as either a product or ratio and applying the laws of indices before substituting

1. $\log_5 10=\log_5(2\times 5)=\log_5 2+\log_5 5=0.431+1=1.431$

2. $\log_5\left(\dfrac{3}{2}\right)=\log_5 3-\log_5 2=0.682-0.431=\ 0.251$

3. $\log_5 12=\log_5(3\times 4)=\log_5 3+\log_5 4=\log_5 3+\log_5 2^2=\log_5 3+2\log_5 2$

$$=0.682+2\times 0.431=0.682+0.862=1.544$$

4. $\log_5\left(\frac{3}{8}\right)+2\log_5\left(\frac{4}{5}\right)-\log_5\left(\frac{2}{5}\right)=\log_5 3-\log_5 8+2[\log_5 4-\log_5 5]-[\log_5 2-\log_5 5]$

$$=\log_5 3-\log_5 2^3+2[\log_5 2^2-1]-[\log_5 2-1]$$
$$=\log_5 3-3\log_5 2+2[2\log_5 2-1]-\log_5 2+1$$
$$=\log_5 3-3\log_5 2+4\log_5 2-2-\log_5 2+1$$
$$=\log_5 3-3\log_5 2+4\log_5 2-\log_5 2-1$$
$$=0.682-3(0.431)+4(0.431)-0.431-1$$
$$=0.682-1.293+1.724-0.431-1$$
$$=0.682+1.724-(1.293+0.431)-1$$
$$=0.682+1.724-1.724-1=0.682-1=0.682+\bar{1}$$
$$=\bar{1}.682$$

Example 6.7

If $\log_{10}3=a$ *and* $\log_{10}5=b$, find in terms of a *and* b the value of $\log_{10}75$

Solution

$$\log_{10}75=\log_{10}(3\times 25)=\log_{10}3+\log_{10}25=\log_{10}3+\log_{10}5^2$$
$$=\log_{10}3+2\log_{10}5$$
$$=a+2b$$

Example 6.8

Given that: $\log_{10}6=0.778,$ find without using tables or calculators the value of $\log_{10}600$

Solution

$$\log_{10}600=\log_{10}(6\times 100)=\log_{10}6+\log_{10}100=\log_{10}6+\log_{10}10^2$$
$$=\log_{10}6+2\log_{10}10=\log_{10}6+2$$
$$=0.778+2=2.778$$
$$\therefore \log 600_{10}=2.778$$

Example 6.9

If $\log_{10} 9 = 0.9542,$ find the value of $\log_{10} 0.009$

Solution

$$\log_{10} 0.009 = \log_{10} \frac{9}{1000} = \log_{10} 9 - \log_{10} 1000 = \log_{10} 9 - \log_{10} 10^3$$

$$= \log_{10} 9 - 3\log_{10} 10 = \log_{10} 9 + \bar{3} = 0.9542 + \bar{3} = \bar{3}.9542$$

$$\therefore \log_{10} 0.009 = \bar{3}.9542$$

Example 6.10

Given that: $\log_{10} 7 = 0.8451$ *and* $\log_{10} 3 = 0.4771,$ find without using calculators, $\log\left(\frac{9}{7}\right)$

Solution

$$\log\left(\frac{9}{7}\right) = \log 9 - \log 7 = \log 3^2 - \log 7 = 2\log 3 - \log 7$$

$$= 2 \times 0.4771 - 0.8451 = 0.9542 - 0.8451 = 0.1091$$

Example 6.11

Solve simultaneously;

$$xy = 80$$

$$\log x - 2\log y = 1$$

Solution

$$xy = 80 - - - - - (1)$$

$$\log x - 2\log y = 1 - - - - - (2)$$

From $(2),$

$$\log x - \log y^2 = 1$$

$$\Rightarrow \log\left(\frac{x}{y^2}\right) = 1$$

Take antilog of both sides

$\Rightarrow \frac{x}{y^2} = 10$

$\Rightarrow x = 10y^2 - - - - - (3)$

Put (3) into (1)

$\Rightarrow y(10y^2) = 80$

$\frac{10y^3}{10} = \frac{80}{10}$

$\Rightarrow y^3 = 8$

Cube root both sides

$\sqrt[3]{y^3} = \sqrt[3]{8}$

$\Rightarrow y = 2$

Put $y = 2$ into (3)

$\Rightarrow x = 10(2)^2 = 10 \times 4 = 40$

$\therefore x = 40$

6.7 **Characteristic And Mantissa Of Common Logarithms**

In general, for a positive number n, we can write it in standard form as:

$m = k \times 10^n$ *where* $1 \leq k < 10$ *and* n *is an* int*eger. Then*,

$\log_{10} m = \log_{10} k + \log_{10} 10^n$

$= \log_{10} k + n$

We note that n *is an* int*eger and* $\log_{10} k$ *is a decimal between* 0 *and* 1.

We call the int*eger* n *the characteristic and the decimal* $\log_{10} k$ *the mantissa of* $\log_{10} m$.

NB

The mantissa of any log is always positive whiles the characteristic can either be negative or positive.

Example 6.12

Find the characteristic and mantissa of the common logarithms of the following numbers:

1. 2.5 2. 250 3. 25000 4. 0.0025

Solution

1. 2.5

Writing in standard form becomes;

$2.5 = 2.5 \times 10^0$

$\Rightarrow \log_{10} 2.5 = \log_{10}(2.5 \times 10^0)$ *Apply law* 1 *of* $\log s$

$$= \log_{10} 2.5 + \log_{10} 10^0 = \log_{10} 2.5 + 0 = 0.4663 + 0$$

Therefore, characteristic = 0 and mantissa = 0.4663

2. 250

Writing in standard form gives;

$250 = 2.5 \times 10^2$

$\Rightarrow \log_{10} 250 = \log_{10} 2.5 + 2 = 0.4663 + 2$

Hence, characteristic $= 2$ *and mantissa* $= 0.4663$

3. $25000 = 2.5 \times 10^4$

$\Rightarrow \log_{10} 25000 = \log_{10} 2.5 + 4 = 0.4663 + 4$

Hence, characteristic $= 4$ *and mantissa* $= 0.4663$

4. $0.0025 = 2.5 \times 10^{-3}$

$\Rightarrow \log_{10} 0.0025 = \log_{10} 2.5 + (-3) = 0.4663 + (-3)$

Hence, characteristic $= -3$ *and mantissa* $= 0.4663$

6.8 **Tables Of Common Loagrithms**

A table of common logarithms of numbers between 1 and 10 has been compiled for easy computation of the mantissa to four places of decimals. This tables are often called

four-figure tables of logarithms.

A ***table of logarithms*** has three main columns:

The first column provides the first two digits of the given number. The second column gives the third digit of the number and the third column (titled ***differences***) is the fourth digit.

6.9 **How To Use The Logarithm Table**

Here, we shall consider two forms of numbers:

1. Numbers between 1 and 10
2. Numbers greater than 10 or less than 1

6.9.1 **Finding The Logarithm of Numbers Between 1 And 10**

To find the common logarithm of a number between 1 and 10 we follow the steps below:

1. Look down the first column to the row containing the first two digits of the number
2. Move along the same row as in step 1 to the value under the column headed the third digit of the number and read it.
3. Move along the same row as before to the differences column headed the fourth digit of the number and read its value
4. Add the result of step 3 to the result of step 2
5. When the result of step 4 is a four digit number, we place a decimal point before the number for the final answer. But when the result of step 4 is in five digits, we place the decimal point after the first digit to obtain the answer.

Example 6.13

Use the table of logarithms to evaluate the following:

1. $\log_{10} 5.642$ 2. $\log_{10} 2.7$ 3. $\log_{10} 8.53$ 4. 9.999

Solution

1. $\log_{10} 5.642$

From the table, we look down the first column to the row

containing **56** (ie. First two digits of the number).
Next, we move along the same row to the column headed **4** (ie. The third digit of the number)where we read **7513**. We then look along the same row to the differences column headed **2** and read **2**. Finally, we add the last reading **2** to the first reading **7513** to obtain **7515**.

Thus, $\log_{10} 5.642 = 0.7515$

NB

In the tables of logarithms, decimal points are omitted. Hence, for the final answer we place a decimal point before each mantissa.

2. $\log_{10} 2.7$

Similarly, to find the logarithm of **2.7**, we look down the first column to the row containing **27** and then move along the same row to the column headed **0** since there is no third digit we assume it to be **zero** and read **4314**. Finally, since there is no fourth digit we stop at **4314**.

Hence, $\log_{10} 2.7 = 0.4314$

3. $\log_{10} 8.53$

Further, to find the logarithm of **8.53**, we look down the first column to the row containing **85** and then move along the same row to the column headed **3** and read **9309**. We stop here since there is no fourth digit.

Hence, $\log_{10} 8.53 = 0.9309$

4. $\log_{10} 9.999$

Finally, to compute the logarithm of **9.999**, we look down the first column to the row containing **99** and then move along the same row to the column headed **9** and read **9996**. We now continue along the same row to the differences column headed **9** and read **4**. Then add this **4** to **9996** to obtain **1.0000**.

Hence, $\log_{10} 2.7 = 1.0000$

6.9.2 **Finding The Logarithm Of Numbers Greater Than 10 Or Less Than 1**

Here, we simply write the number in standard form to obtain the characteristic and then read the mantissa from the table.

Example 6.14

Use the table of common logarithms to obtain the common logarithms of the following numbers:

1. 27.53   2. 5697   3. 0.009539   4. 0.01978

Solution

1. 27.53

Writing in standard form,

$27.53 = 2.753 \times 10^1$

$\Rightarrow \log_{10}(2.753 \times 10^1) = \log_{10} 2.753 + 1\log_{10} 10 = \log_{10} 2.753 + 1$

Find the mantissa $\log_{10} 2.753$ *from table*

$\Rightarrow$ *look along the first column to* 27 *and move along the same row to the column headed* 5 *and read the value* 4393*. Then continue along to the differences column headed* 3 *and read the value* 5*. Now add* 5 *to* 4393 *obtain* 4398*.*

Thus, $\log_{10} 2.753 + 1 = 0.4398 + 1 = 1.4398$

Hence, $\log_{10} 27.53 = 1.4398$

2. 5697

$5697 = 5.697 \times 10^3$

Similarly, $\log_{10} 5.697 = 0.7556$ *from table*

$\therefore \log_{10} 5697 = 3.7556$

3. $0.009539 = 9.539 \times 10^{-3}$

Similarly, $\log_{10} 9.539 = 0.9795$ *from the table*

$\therefore \log_{10} 0.009539 = \bar{3}.9795$

4. $0.01978 = 1.978 \times 10^{-2}$

$Similarly,\ from\ the\ table,\ \log_{10} 1.978 = 0.2962$

$\Rightarrow \log_{10} 0.01978 = \bar{2}.2962$

6.10 **Use Of Logarithms**

The main use of logarithms is in calculations involving multiplication or division.
Here, we shall make use of only the ***common logarithm*** (logs to the base ten).
In using logarithms to evaluate expressions, we take ***log*** of the expression and then apply appropriately the laws and properties of logarithms. After which we take the ***antilog*** for the final answer.

Example 6.15

Use logarithms to evaluate the following:

1. 3.75×8.725 2. $6.725 \div 4.897$ 3. $\dfrac{82.51 \times 5.752}{87.35}$ 4. $\dfrac{73.89 \times 152.6}{41.65 \times 8.975}$

5. $\sqrt{55.5 \times 21.91}$ 6. $\sqrt[3]{76.95}$ 7. $\sqrt{0.9567}$ 8. $0.5921^{\frac{1}{3}}$

9. $\sqrt[3]{0.4276}$

10. $\dfrac{86.19 \times (0.0462)^2}{\sqrt{0.846}}$

Solution

1. 3.75×8.725

$Taking\ \ \log s\ we\ get;$

$\log(3.75 \times 8.725)$

$Apply\ first\ law\ of\ \log s$

$\Rightarrow \log 3.75 + \log 8.725 = 1.5147$

$Now\ take\ anti\log$

$\Rightarrow Anti\log 1.5147 = 32.71$

$\therefore 3.75 \times 8.725 = 32.71$

2. $6.725 \div 4.897$

$NB: 6.725 \div 4.897 = \dfrac{6.725}{4.897}$

Taking log *gives*;

$\log\left(\dfrac{6.725}{4.897}\right)$

Applying law two gives;

$\log 6.725 - \log 4.897 = 0.8277 - 0.6899 = 0.1378$

Take anti log *to get*;

$Anti\log 0.1378 = 1.374$

$\therefore 6.725 \div 4.897 = 1.374$

3. $\dfrac{82.51 \times 5.752}{87.35}$

Taking log *becomes*;

$\log \dfrac{82.51 \times 5.752}{87.35}$

Applying first and second laws;

$\log 82.51 + \log 5.752 - \log 87.35 = 1.9166 + 0.7599 - 1.9412 = 0.7353$

Take anti log *to get*;

$Anti\log 0.7353 = 5.437$

$\therefore \dfrac{82.51 \times 5.752}{87.35} = 5.437$

4. $\dfrac{73.89 \times 152.6}{41.65 \times 8.975}$

$$\Rightarrow \log\left(\frac{73.89 \times 152.6}{41.65 \times 8.975}\right) = (\log 73.89 + \log 152.6) - (\log 41.65 + \log 8.975)$$

$$= (1.8686 + 2.1835) - (1.6196 + 0.9530)$$

$$= 4.0521 - 2.5726 = 1.4795$$

$Anti\log 1.4795 = 30.16 \quad \Rightarrow \dfrac{73.89 \times 152.6}{41.65 \times 8.975} = 30.16$

5. $\sqrt{55.5\times 21.91} = (55.5\times 21.91)^{\frac{1}{2}}$

Take log

$\Rightarrow \log(55.5\times 21.91)^{\frac{1}{2}}$

Apply law three

$= \frac{1}{2}\log(55.5\times 21.91)$

Apply law one

$\Rightarrow \frac{1}{2}(\log 55.5 + \log 21.91) = \frac{1}{2}(1.7443 + 1.3406) = \frac{1}{2}(3.0849) = 1.5425$

$Anti\log 1.5425 = 34.87$

$\therefore \sqrt{55.5\times 21.91} = 34.87$

6. $\sqrt[3]{76.95} = (76.95)^{\frac{1}{3}}$

$\Rightarrow \log(76.95)^{\frac{1}{3}} = \frac{1}{3}\log 76.95 = \frac{1}{3}\times 1.8862 = 0.6287$

$Anti\log 0.6287 = 4.253$

$\therefore \sqrt[3]{76.95} = 4.253$

7. $\sqrt{0.9567} = (0.9567)^{\frac{1}{2}}$

$\log(0.9567)^{\frac{1}{2}} = \frac{1}{2}\log 0.9567 = \frac{1}{2}\times \bar{1}.9808$

NB : *How to evaluate* $\frac{1}{2}\times \bar{1}.9808$

First we rewrite $\bar{1}.9808$ *as a sum of two numbers of which* -2 *is one the denominator here is* 2.

$$ie.\ \bar{1}.9808 = -1 + 0.9808$$
$$= -2 + 1.9808$$

$\Rightarrow \frac{1}{2}\times \bar{1}.9808 = \frac{1}{2}\times(-2 + 1.9808) = \frac{1}{2}\times -2 + \frac{1}{2}\times 1.9808 = -1 + 0.9904$

$= \bar{1}.9904$ (*The negative only affects the whole number part*)

$Anti\log \bar{1}.9904 = 0.9958$

$\therefore \sqrt{0.9567} = 0.9958$

8. $0.5921^{\frac{1}{3}}$

$\Rightarrow \log 0.5921^{\frac{1}{3}} = \frac{1}{3}\log 0.5921 = \frac{1}{3}\times\bar{1}.7724$

We now express $\bar{1}.7724$ *as a sum of two number of which* -3 *is one since denominator here is* 3

$\Rightarrow \frac{1}{3}\times\bar{1}.7724 = \frac{1}{3}\times(-1+0.7724) = \frac{1}{3}\times(-3+2.7724)$

$= \frac{1}{3}\times -3 + \frac{1}{3}\times 2.7724$

$= -1 + 0.9243 = \bar{1}.9243$

Anti $\log \bar{1}.9243 = 0.8401$

$\therefore 0.5921^{\frac{1}{3}} = 0.8401$

9. $\sqrt[3]{0.4276}$

$\sqrt[3]{0.4276} = (0.4276)^{\frac{1}{3}}$

Take logarithm

$= \log(0.4276)^{\frac{1}{3}} = \dfrac{1}{3}\log(0.4276)$

But $\log(0.4276) = \bar{1}.6310 = \bar{3}+2.6310$

$\Rightarrow \dfrac{1}{3}\log(0.4276) = \dfrac{1}{3}\times\bar{1}.6310 = \bar{1}.6310\div 3 = \dfrac{\bar{3}+2.6310}{3} = \dfrac{\bar{3}}{3}+\dfrac{2.6310}{3}$

$= \bar{1}+0.8770 = \bar{1}.8770$

Take anti log

$= anti\log\bar{1}.8770 = 0.7534$

Hence, $\sqrt[3]{0.4276} = 0.7534$

10. $\dfrac{86.19\times(0.0462)^2}{\sqrt{0.846}}$

$$\frac{86.19\times(0.0462)^2}{\sqrt{0.846}}=\frac{86.19\times(0.0462)^2}{(0.846)^{\frac{1}{2}}}$$

Take logarithm

$$=\frac{\log 86.19+\log(0.0462)^2}{\log(0.846)^{\frac{1}{2}}}=\log 86.19+2\log(0.0462)-\frac{1}{2}\log 0.846$$

$$=1.9355+2\times\bar{2}.6646-\frac{1}{2}\times\bar{1}.9274$$

NB : For $2\times\bar{2}.6646$ *we find* $2\times-2=-4$ *and then* $2\times 0.6646=1.3292$

Then workout whole numbers ie. $-4+1=-3\equiv\bar{3}$ *and attach decimal to get* $\bar{3}.3292$

Thus, $\quad=1.9355+\bar{3}.3292-\bar{1}.9637$

NB:

$$1.9355+\bar{3}.3292=(1+-3)+(0.9355+0.3292)=-2+1.2647=(-2+1)+0.2647$$
$$=-1+0.2647=\bar{1}.2647$$

$$\Rightarrow 1.9355+\bar{3}.3292-\bar{1}.9637=1.2647-\bar{1}.9637=\bar{1}.3010$$

Similarly, $1.2647-\bar{1}.9637=[1-(-1)]+(0.2647-0.9634)=0+\bar{1}.3010=\bar{1}.3010$

Take antilog

$\Rightarrow \textit{anti}\log\bar{1}.3010=0.2$

Therefore, $\dfrac{86.19\times(0.0462)^2}{\sqrt{0.846}}=0.2$

6.11 **Powers And Roots**

The power and root of a number can be found by the following procedures:

6.11.1 **Powers**

Steps

1. Find the logarithm of the number to base 10. i.e. the log of the base
2. Multiply the result of step 1 by the power
3. Find the antilogarithm of the result of step 2

Example 6.16

Use logarithm to evaluate:

1. 153^2 2. 1.05^{100}

Solution

1. 153^2

Step 1

$\log 153 = 2.184691431$

Step 2

$2.184691431 \times 2 = 4.369382862$

Step 3

Antilog $4.369382862 = 23409$

Therefore $153^2 = 23409$

2. 1.05^{100}

$\log 1.05 = 0.021189299$

$\Rightarrow 0.0212 \times 100 = 2.118929907$

Antilog $2.118929907 = 131.5$

$\therefore 1.05^{100} = 131.5$

6.11.2 **Roots**

Steps

1. Find the logarithm of the number to base 10. i.e. the log of the number inside the square root sign
2. Divide the result of step 1 by the root. i.e. step 1 result by the number outside the root sign
3. Find the antilog of the result of step 2

Example 6.17

Use logarithms to evaluate the following:

1. $\sqrt{7}$ 2. $\sqrt[10]{1024}$

Solution

1. $\sqrt{7}$

Step 1

$\log_{10} 7 = 0.84509804$

Step 2

$0.84509804 \div 2 = 0.42254902$

Step 3

Antilog $0.4255 = 2.64575$

Therefore, $\sqrt{7} = 2.64575$

2. $\sqrt[10]{1024}$

Step 1

$\log 1024 = 3.010299957$

Step 2

$3.010299957 \div 10 = 0.3010299957$

Step 3

$Anti\log 0.3010299957 = 2$

Therefore, $\sqrt[10]{1024} = 2$

EXERCISE

QUE. A

Express the following in index notations:

1. $p = \log_q r$ 2. $2 = \log_5 25$ 3. $\log_{27} 3 = \frac{1}{3}$ 4. $\log_{\frac{1}{3}} 27 = -3$

5. $\log_{10} 0.001 = -3$ 6. $7 = \log_2 128$

7. $\log_2 2 = 1$ 8. $\log_2 8 = 3$ 9. $\log_2 \frac{1}{16} = -4$ 10. $\log_2 64 = 6$

11. $\log_2 \frac{1}{128} = -7$ 12. $\log_2 256 = 8$

QUE. B

Express the following in logarithmic notations:

1. $2^4 = 16$ 2. $81 = \left(\frac{1}{3}\right)^{-4}$ 3. $9^{\frac{3}{2}} = 27$ 4. $64 = 16^{\frac{3}{2}}$

5. $0.01 = 10^{-2}$ 6. $\left(\frac{1}{8}\right)^0 = 1$ 7. $16 = 2^4$ 8. $1000 = 10^3$

9. $3 = 9^{\frac{1}{2}}$ 10. $n = 3^2$ 11. $n = 25^{-1}$ 12. $n = 16^3$ 13. $m = a^2$

14. $m = 2^a$ 15. $x = 2^r$ 16. $n = a^{0.3}$ 17. $n = (\frac{1}{2})^{-3}$ 18. $n = a^x$

19. $625 = 5^4$ 20. $\frac{1}{8} = 2^{-3}$ 21. $64 = 8^2$ 22. $n = 5^x$

QUE. C

Solve for x in the following:

1. $\log_2 16 = x$ 2. $\log_2 64 = x$ 3. $\log_3 \frac{1}{27} = x$ 4. $\log_4 x = -1$

5. $\log_{10} x = 2$ 6. $\log_5 x = -3$ 7. $\log_x 8 = 3$ 8. $\log_x 100 = 2$

9. $\log_x 64 = 3$ 10. $\log_{\frac{1}{3}} x = 2 = -2$ 11. $\log_{\frac{1}{2}} 8 = x$ 12. $\log_3 x = -4$

13. $\log_5 x = 3$ 14. $\log_4 256 = x$ 15. $\log_{100} x = \frac{1}{2}$ 16. $\log_{10} x = 0$

17. $\log_2 8 = x$ 18. $\log_{\frac{1}{2}} 4 = x$ 19. $\log_x 16 = 2$ 20. $\log_x (\frac{1}{9}) = -2$

QUE. D

Use logarithms to evaluate the following:

1. 6.52×7.25 2. 8.96×9.587 3. 4.197×7.859 4. 8.976×5.357

5. 2.319×5.695 6. 9.698×8.679 7. $9.56 \div 4.56$ 8. 8.376×5.76

9. 6.517×5.62 10. $5.217 \div 3.575$ 11. $7.859 \div 9.376$ 12. $6.512 \div 7.289$

13. $\frac{36.75 \times 29.28}{27.86}$ 14. $\frac{65.89 \times 1.365}{59.57}$ 15. $\frac{185.7 \times 9788}{575.6}$ 16. $\frac{7.359 \times 6.783}{1.567 \times 5.395}$

17. $\frac{875.3 \times 96.53}{62.87 \times 125.6}$ 18. $\frac{35.67 \times 10.51}{225.5 \times 0.0057}$ 19. $\sqrt[3]{6975}$ 20. $\sqrt[4]{128.9}$

21. $\sqrt{18.95 \times 72.27}$ 22. $\left(\frac{23.7 \times 15.6}{62.5}\right)^{\frac{1}{3}}$ 23. $\sqrt{\frac{55.5}{18.9}}$ 24. $\left(\frac{83.75}{19.56}\right)^3$

QUE. E

Use the table of common logarithms to obtain the logarithms of the following:

1. 8.53 2.9.05 3. 4.905 4. 1.059 5. 8.754 6. 3.725

7. 3.567

QUE. F

Find the characteristic of the common logarithm of each of the following numbers:

1. 2.75 2. 97.2 3. 0.059 4. 0.0056 5. 0.0000125

6. 95760 7. 1 8. 8509000

QUE. G

If $\log_{10} 2 = 0.3010, \log_{10} 5 = 0.6990, \log_{10} 6 = 0.7782, \log_{10} 7 = 0.8485$
and $\log_{10} 9 = 0.9542$

Find the common logarithms of the following numbers.

1. 12 2. 24 3. 72 4. 3 5. $\frac{6}{4}$ 6. 18 7. 30 8. 60

9. 15 10. 48 11. 36 12. 144 13. 120 14. 9

15. . $\frac{1}{3}$ 16. 3 17. $\frac{3}{2}$ 18. $6^{\frac{1}{3}}$ 19. $9^{\frac{1}{4}}$ 20. $6^{-0.5}$ 21. $\sqrt{60}$

22. $\sqrt{90}$ 23. 54^2 24. 15^2 25. 9^{-1} 26. 70.55 27. 7055

28. 0.7055 29. 0.0007055 30. 70550 31. 7055000 32. 0.07055

33. 70550000 34. 125 35. $5^{\frac{1}{2}}$ 36. 625 37. $25^{\frac{1}{3}}$

QUE. H

Simplify the following:

1. $-\log_{10}\left(\frac{1}{1000}\right)$

2. $\frac{\log 729}{\log 9}$

3. $\log_{10} 2.25 + 4\log_{10} 2 - 2\log_{10} 0.6$

4. $\log_{10} 6 + \log_{10} 45 - \log_{10} 27$

5. $2 - 3\log_{10} 2$

6. $\log 3 + 3\log 2 - 3\log 4$

QUE. I

Solve the following for x

1. $2^{3x-2} = 5$ 2. $\log x^2 = (\log x)^2$ 3. $\log_2 (x+5) = 2\log_2 3$

4. $\log_e (x+8) - \log_e x = 3\log_e 2$ 5. $\log_4 x + \log_4 (x+2) = \frac{1}{2}\log_4 9$

6. $3\log_b 2 + \frac{1}{2}\log_b 25 - \log_b 20 = \log_b x$

7. $\frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + 2\log_b 2 = \log_b x$

8. $\log_{10}(8x+1) - \log_{10}(2x+1) = \log_{10}(x+2)$

9. $\log_4 16 = \log_x 36$ 10. $2\log\frac{2}{3} = \frac{1}{2}\log x - \log 18 + \log 16$

CHAPTER 7

CHANGE OF SUBJECT AND SUBSTITUTION INTO FORMULAE

7.1 **Change Of Subject**

Change of subject is simply an aspect that isolates a single ***variable*** on one side of an equation usually referred to as ***making a variable the subject*** of an equation. The variable isolated is called the ***subject*** of the equation. Thus, in the relation, $y = 2x - 7$, y is the subject.

NB

Every variable in an equation can be made the subject.

Example 7.1

Make x the subject of the following relations.

$(i)\ a = 2x \qquad (ii)\ d = 3y + x \qquad (iii)\ y = mx + c$

$(iv)\ u = \frac{x}{3} + d \qquad (v)\ z = mx + cx + d \qquad (vi)\ p(x - a)k$

Solution

$(i)\ a = 2x \quad \textit{Divide through by } 2$

$\Rightarrow \frac{a}{2} = \frac{2x}{2}$

$\Rightarrow \frac{a}{2} = x \quad \text{Re}\textit{arrange to get } x \textit{ on LHS}$

$\therefore x = \frac{a}{2}$

$(ii)\ d = 3y + x \quad \textit{Subtract } 3y \textit{ from both sides}$

$\Rightarrow d - 3y = 3y + x - 3y$

$\Rightarrow d - 3y = x \quad \text{Re}\textit{arrange to get } x \textit{ on RHS}$

$(iii)\ y = mx + c \quad \textit{Subtract } c \textit{ from both sides}$

$\Rightarrow y - c = mx + c - c$

$y-c=mx \quad$ *Divide through by m*

$\Rightarrow \dfrac{y-c}{m}=\dfrac{mx}{m} \Rightarrow \dfrac{y-c}{m}=x$

$\therefore x=\dfrac{y-c}{m} \quad$ *Rearrange to get x on LHS*

$(iv)\ u=\dfrac{x}{3}+d \quad$ *Subtract d from both sides*

$\Rightarrow u-d=\dfrac{x}{3}+d-d$

$\Rightarrow u-d=\dfrac{x}{3} \quad$ *Multiply through by 3*

$\Rightarrow 3(u-d)=3\times\dfrac{x}{3}$

$\Rightarrow 3(u-d)=x$

$\Rightarrow x=3(u-d)$

$(v)\ z=mx+cx+d \quad$ *Factorise x on RHS and subtract d from both sides*

$\Rightarrow z-d=(m+c)x+d-d$

$\Rightarrow z-d=(m+c)x \quad$ *Divide through by m + c*

$\Rightarrow \dfrac{z-d}{m+c}=\dfrac{(m+c)x}{m+c} \Rightarrow \dfrac{z-d}{m+c}=x \quad$ *Rearrange to get x on LHS*

$\Rightarrow x=\dfrac{z-d}{m+c}$

$(vi)\ p=(x-a)k \quad$ *Divide through by k*

$\Rightarrow \dfrac{p}{k}=\dfrac{(x-a)k}{k}$

$\Rightarrow \dfrac{p}{k}=x-a \quad$ *Add a to both sides*

$\Rightarrow \dfrac{p}{k}+a=x-a+a$

$\Rightarrow \dfrac{p}{k}+a=x \quad$ *Rearrange to get x on LHS*

$\therefore x=\dfrac{p}{k}+a$

Example 7.2

Make c the subject of the relation: $\frac{ac+d}{e} = p$

Solution

$$\frac{ac+d}{e} = p$$
$$\Rightarrow ac + d = ep$$
$$ac = ep - d$$
$$c = \frac{ep-d}{a}$$

Example 7.3

If $t = \frac{a-m}{1+am}$, express a in terms of t and m.

Solution

$$t = \frac{a-m}{1+am}$$

$$\Rightarrow t(1+am) = a - m$$
$$t + amt = a - m$$
$$t + m = a - amt$$
$$t + m = a(1 - mt)$$
$$\Rightarrow a = \frac{t+m}{1-mt}$$

Example 7.4

Make y the subject of the relation $x = \frac{ay}{b} + c$

Solution

$x = \frac{ay}{b} + c$

(Multiply through by LCM)

$bx = ay + bc$

$bx - bc = ay$

$\Rightarrow \frac{bx - bc}{a} = \frac{b(x - c)}{a}$

Example 7.5

If $a = \frac{2}{b} - \frac{1}{c}$ find an expression for c in terms of a and b.

Solution

$a = \frac{2}{b} - \frac{1}{c}$

Multiply through by bc

i.e. $abc = 2c - b$

$b = 2c - abc$

$b = c(2 - ab)$

$c = \frac{b}{2 - ab}$

Example 7.6

Make y the subject of the following relations.

$(i)\ a = b + \frac{1}{y} \qquad (ii)\ b = a + dy^2 \qquad (iii)\ d = bc + y^{\frac{1}{2}} \qquad (iv)\ g = \frac{b}{y^3}$

Solution

$(i)\ a = b + \frac{1}{y}$ *Subtract b from both sides*

$\Rightarrow a - b = b + \frac{1}{y} - b$

$\Rightarrow a - b = \frac{1}{y}$ *Multiply through by y*

$\Rightarrow y(a-b) = 1$ *Divide through by $(a-b)$*

$\Rightarrow \frac{y(a-b)}{a-b} = \frac{1}{a-b}$

$$\Rightarrow y = \frac{1}{a-b}$$

$(ii)\ b = a + dy^2$ *Subtract a from both sides*

$\Rightarrow b - a = a + dy^2 - a$

$\Rightarrow b - a = dy^2$ *Divide through by d*

$\Rightarrow \frac{b-a}{d} = y^2$ *Take the square root of both sides*

$\pm\sqrt{\frac{b-a}{d}} = y$

$\therefore y = \pm\sqrt{\frac{b-a}{d}}$

$(iii)\ d = bc + y^{\frac{1}{2}}$ *Subtract bc from both sides*

$\Rightarrow d - bc = bc + y^{\frac{1}{2}} - bc$

$\Rightarrow d - bc = y^{\frac{1}{2}}$ *Square both sides to get rid of fractional index*

$\Rightarrow y = (d - bc)^2$

$(iv)\ g = \frac{b}{y^3} - c$ *Add c to both sides*

$\Rightarrow g + c = \frac{b}{y^3} - c + c$

$\Rightarrow g + c = \frac{b}{y^3}$ *Multiply through by* y^3

$\Rightarrow y^3(g + c) = b$ *Divide through by* $(g + c)$

$\Rightarrow y^3 = \frac{b}{g+c}$ *Take cube root of both sides*

$\Rightarrow y = \sqrt[3]{\frac{b}{g+c}}$

Example 7.7

Make h the subject of the relation: $V = \frac{1}{3}\pi r^2 h$

Solution

$V = \frac{1}{3}\pi r^2 h$

Multiply through by 3

$\Rightarrow 3V = \pi r^2 h$

Divide through by πr^2

$\frac{3V}{\pi r^2} = h$

Example 7.8

Make x the subject of the relation $\frac{x-1}{a} + \frac{y}{b} = 1$

Solution

$\frac{x-1}{a} + \frac{y}{b} = 1$

$b(x-1) + ay = ab$

$bx - b + ay = ab$

$bx = ab + b - ay$

$x = \frac{ab + b - ay}{b} = \frac{a(b-y)+b}{b}$

Example 7.9

Make v the subject of this relation.

$a(3+v) = b(v+1)$

Solution

$a(3+v) = b(v+1)$ *Expand the brackets*

$\Rightarrow 3a + av = bv + b$ *Group liked terms*

$\Rightarrow av - bv = b - 3a \quad \textit{Factorise out } v$

$\Rightarrow (a-b)v = b - 3a \quad \textit{Divide through by } (a-b)$

$\Rightarrow v = \dfrac{b-3a}{a-b}$

Example 7.10

Make t the subject of the relation $r = m\sqrt{\left(\frac{t}{n}\right)}$

Solution

$r = m\sqrt{\left(\frac{t}{n}\right)}$

Divide through by m

$\Rightarrow \dfrac{r}{m} = \sqrt{\left(\frac{t}{n}\right)}$

Square both sides

$\dfrac{r^2}{m^2} = \dfrac{t}{n}$

$\therefore t = \dfrac{nr^2}{m^2}$

Example 7.11

Given that $\dfrac{a}{b} = \dfrac{c}{d}$, *find an expression for* $\dfrac{a}{a+b}$

Solution

$\dfrac{a}{b} = \dfrac{c}{d}$

$ad = bc$

$a = \dfrac{bc}{d}$

Put $a = \dfrac{bc}{d}$ into $\dfrac{a}{a+b}$

$\dfrac{bc}{d} \div \left(\dfrac{bc}{d} + b\right) = \dfrac{bc}{d} \div \dfrac{bc+bd}{d} = \dfrac{bc}{d} \times \dfrac{d}{bc+bd} = \dfrac{bc}{b(c+d)} = \dfrac{c}{c+d}$

Example 7.12

Make z the subject of the following relations.

$(i)\ a = \dfrac{2z+1}{5+z} \qquad (ii)\ a = \dfrac{bz+c}{d+3z}$

Solution

$(i)\ a = \dfrac{2z+1}{5+z}$ *Multiply through by* $(5+z)$

$\Rightarrow a(5+z) = 2z+1$ *Expand the bracket*

$\Rightarrow 5a + az = 2z+1$ *Group liked terms*

$5a - 1 = 2z - az$ *Factorise out* z

$\Rightarrow 5a - 1 = z(2-a)$ *Divide through by* $(2-a)$

$\Rightarrow \dfrac{5a-1}{2-a} = z$

$\therefore z = \dfrac{5a-1}{2-a}$

$(ii)\ a = \dfrac{bz+c}{d+3z}$ *Multiply through by* $(d+3z)$

$a(d+3z) = bz+c$ *Expand bracket*

$\Rightarrow ad + 3az = bz + c$ *Group liked terms*

$\Rightarrow 3az - bz = c - ad$ *Factorise* z *out*

$\Rightarrow z(3a-b) = c - ad$ *Divide through by* $(3a-b)$

$\Rightarrow a = \dfrac{c-ad}{3a-b}$

7.2 **Substitution Into Formulae**

The act of replacing unknown variables with numbers in an expression or relation is called ***substitution***.

Example 7.13

Find the value of $x^3 + xy^2 + 3y^3 \quad when \ x = -2 \ \ and \ \ y = 1$.

Solution

$$x^3 + xy^2 + 3y^3 = (-2)^3 + (-2)(1)^2 + 3(1)^3 = -8 - 2 + 3 = -7$$

Example 7.14

Evaluate $ab - b^2(a - b), \ if \ \ a = -2 \ \ and \ \ b = 3$

Solution

$$(-2)(3) - (3)^2(-2 - 3) = -6 - 9(-5) = -6 + 45 = 39$$

Example 7.15

Without using a calculator, evaluate

$$\frac{2x - y}{z} + \frac{z + 2y}{x} \quad when \ \ x = 2, y = -3, \ z = 4$$

Solution

$$\frac{2x-y}{z} + \frac{z+2y}{x} = \frac{2(2)-(-3)}{4} + \frac{4+2(-3)}{2} = \frac{4+3}{4} + \frac{4-6}{2} = \frac{7}{4} + \frac{-2}{2} = \frac{7-4}{4} = \frac{3}{4}$$

Example 7.16

(i) Make l the subject of the relation $A = \pi rl + \pi r^2$

(ii) Find l when $r = 3, A = 176 \ \ and \ \ \pi = \frac{22}{7}$

Solution

$(i)\ \ A = \pi rl + \pi r^2$

$$\frac{A - \pi r^2}{\pi r} = l$$

(ii)

$$l = \frac{A - \pi r^2}{\pi r} = \frac{176 - \frac{22}{7}(3)^2}{\frac{22}{7}(3)} = \frac{176 - \frac{198}{7}}{\frac{66}{7}} = \frac{1034}{7} \div \frac{66}{7} = 15.67$$

Example 7.17

If $\frac{1}{m} + \frac{1}{n} = \frac{1}{p}$

(i) Express m in terms of n and p

(ii) Calculate m correct to one decimal place when $n = 17.24 \ \ and \ \ p = 16.41$

Solution

(i)

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{p}$$

$np + mp = mn$

$np = mn - mp$

$np = m(n - p)$

$$\frac{np}{n - p} = m$$

(ii) $m = \frac{np}{n - p} = \frac{17.24(16.41)}{17.24 - 16.41} = \frac{282.9084}{0.83} = 340.9$

Example 7.18

If $t = \sqrt{\frac{p - r}{p + r}},\ find$

(a) r in terms of p and t

(b) The value of r when $t = 3,\ p = 10$

Solution

(a)

$$t = \sqrt{\frac{p-r}{p+r}}$$

Square both sides

$$t^2 = \frac{p-r}{p+r}$$

$$t^2(p+r) = p-r$$

$$t^2 p + t^2 r = p - r$$

$$t^2 r + r = p - t^2 p$$

$$r(t^2+1) = p(1-t^2)$$

$$r = \frac{p(1-t^2)}{t^2+1}$$

(b) $r = \dfrac{p(1-t^2)}{t^2+1} = \dfrac{10(1-(3)^2)}{3^2+1} = -8$

Example 7.19

Given that $u = -2,\ v = 3\ and\ w = 5,\ evaluate\ u^3 + \dfrac{v}{2w}$

Solution

Substituting we have,

$$u^3 + \frac{v}{2w} = (-2)^3 + \frac{3}{2(5)} = -8 + \frac{3}{10} = \frac{-80+3}{10} = -\frac{77}{10} = -7.7$$

Example 7.20

For a lens, the focal length f, the object distance u and the image distance v are related by the formula:

$\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}.\ Find\ f\ when\ u = 50cm\ and\ v = 30cm$

Solution

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$$

Substituting gives

$$\frac{1}{f}=\frac{1}{50}+\frac{1}{30}=\frac{30+50}{1500}=\frac{80}{1500}$$

$$\Rightarrow \frac{1}{f}=\frac{80}{1500}\Rightarrow \frac{80f}{80}=\frac{1500}{80}\Rightarrow f=18.75cm$$

Example 7.21

The area, A of a triangle is given by the relation

$A=\sqrt{s(s-a)(s-b)(s-c)}$

where a, b and c are lengths of the sides of the triangle and

$s=\dfrac{a+b+c}{2}$ *is the semiperimeter. If a triangle has sides of length* $4cm$, $5cm$ *and* $7cm$, *find its area.*

Solution

Given that a = 4, b = 5 and c = 7;

$$\Rightarrow s=\frac{4+5+7}{2}=\frac{16}{2}=8cm$$

$$\Rightarrow A=\sqrt{8(8-4)(8-5(8-7)}=\sqrt{8\times4\times3\times1}=\sqrt{96}=9.80cm^2$$

EXERCISE

QUE. A

Given that $T=\dfrac{37V}{6V-5.4M}$,

(a) Make V the subject of the relation

(b) If $T=4.5$ *and* $M=7\times10^7$, calculate the value of V leaving your answer in standard form.

QUE. B

(i) If $\frac{3m-n}{5m-n}=\frac{p}{q}$, express m in terms of p, q and n.

(ii) Make r the subject of the relation: $v=\frac{1}{6}\pi h(3r^2+h^2)$

(iii) If $p=kq \;\; and \;\; r=\frac{mk}{eq}$ express r in terms of m, p, e and q

QUE. C

(i) Find the value of $a^3-3ab-b^2, \; when \;\; a=-2 \;\; and \;\; b=2$

(ii) Evaluate $ut+\frac{1}{2}at^2, \; given \; that \; u=2, t=3 \; and \; a=-9.8$

(iii) Find the value of the expression
$a^3+2ab^2+b^3, \; when \;\; a=2 \;\; and \;\; b=-1$

QUE. D

(i) If $t=p^2-3q, \; calculate \;\; the \; value \; of \;\; t \; where \;\; p=4 \; and \;\; q=9$

(ii) If $2p=q+\sqrt{q^2+r}, \; find \; an \; expression \;\; for \; r \; in \; terms \; of \;\; p \; and \; q.$

(iii) Given that $u=-2, v=3 \; and \;\; w=5, \; evaluate \;\; u^3+\frac{v}{2w}$

QUE. E

(i) $S=\frac{n}{2}\{2a+(n-1)d\}, \; where \; n>0 \; Find \; n \; when \; a=3, \; d=4 \; and \; S=210$

(ii) If $\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \; and \;\; d=u+v,$

(a) Express d in terms of f and u (b) Find the value of d, when f = 6 and u = 6.5

QUE. F

Make x the subject in each of the following relations.

1. $a = 3x$ 2. $b = 5x + 7$ 3. $c = \frac{1}{2}x + 5$ 4. $d = ux + vx$ 5. $e = m(x - c)$

6. $f = \frac{mx + c}{d}$ 7. $g = \frac{m}{a}(2x + d)$ 8. $h = \frac{a}{2}\left(b - \frac{x}{d}\right)$ 9. $l = b\{3c + (x + 2)d\}$

10. $m = a\{by + (d - x)e\}$ 11. $d = \frac{1}{2}\{x + (m - 1)d\}$ 12. $p = \frac{1}{3q}(mx + c)$

13. $y = \frac{mnx}{b} + k$ 14. $z = \frac{mx + c}{a + b}$ 15. $w = \frac{1}{m - n}\left(\frac{ax}{b} + c\right)$

QUE. G

Make y the subject in each of the following relations.

1. $a = b\sqrt{\frac{y}{3}}$ 2. $b = \sqrt{\frac{2}{y}}$ 3. $c = a + by^3$ 4. $d = y^2 + 3e$ 5. $e = \left(\frac{a}{y} - b\right)^{\frac{1}{4}}$

6. $f = \frac{am}{y^3} - b$ 7. $g = \frac{a}{x} + \frac{b}{y}$ 8. $h = a\sqrt{\frac{b}{cy}}$ 9. $a = \frac{1}{b} + \frac{1}{y}$ 10. $f = am + by^3$

11. $h = \left(\frac{a}{y} - b\right)^{\frac{1}{2}}$ 12. $k = \frac{a}{b - y^2}$ 13. $l = \frac{m}{n - ay^3}$ 14. $n = a + 3by^2$

15. $m = \frac{1}{2}ay^2 + \frac{1}{3}bx$ 16. $p = a(b + cy^3)^{\frac{1}{2}}$ 17. $q = a\left(c + \frac{1}{y}\right)^{\frac{1}{3}}$ 18. $r = \frac{a - \sqrt{y}}{b}$

19. $s = \frac{ay^2 - b}{3c}$ 20. $u = \frac{\sqrt{a}}{5y - 3}$ 21. $v = \frac{2a}{b} + \frac{5c}{dy^2}$ 22. $w = b + \frac{c}{3 - y^2}$

23. $z = \frac{a}{5} - \frac{b}{\sqrt{y}}$ 24. $x = \frac{\sqrt{h(yd)^5}}{kl}$

QUE. H

Make v the subject in each of the following relations.

1. $b(v-1)=7(v-5)$ 2. $v+5=k+\frac{v}{3}$ 3. $d(v-7)=e-5v$

4. $2v-5=\frac{3}{5}(v-f)$ 5. $v^2+k=(v-2)(v-3)$ 6. $\frac{3v^2+av+5}{3}$

7. $v+a=3+\frac{v-b}{5}$ 8. $\frac{a+b}{v-c}=\frac{v+c}{a}$ 9. $\frac{p}{v-q}=\frac{v+q}{p+r}$

10. $3a-2v=\sqrt{(4v^2+r^2)}$

QUE. I

Make u the subject of each of the following relations:

1. $k=\frac{3+5u}{u-2}$ 2. $a=\frac{3x+u}{2u+c}$ 3. $c=\frac{my+u}{u}$ 4. $b=\frac{au+e}{c+du}$

5. $d=\frac{ax^2+bu+c}{mu-k}$ 6. $e=\sqrt{\frac{u-a}{u-b}}$ 7. $f=\frac{au+1}{bu+1}$ 8. $h=2\pi\sqrt{\frac{l-u}{g-u}}$

9. $m=a\sqrt{\frac{u^2}{p+u^2}}$ 10. $n=a\left(\frac{1+u}{3u+1}\right)^{\frac{1}{3}}$

QUE. J

1. The speed, v, of a particle moving with a constant acceleration, a, for a time, t, is given by v = u+at, where u is the initial speed. If u=50m/s, v=90m/s and t=4seconds, find a in m/s^2

2. The ideal gas equation is given as pV = kT, where p is the pressure, V is the volume, T is the temperature and k is a constant. Calculate the value of this constant when T=300K, V=50cm^3 and p=750mmHg.

3. The resistance, R of a wire of length, L is given by R=pL/A, where p is the resistivity of the material of the wire and A is the area of cross-section of the wire. Find p if R=125ohms, L=13m and A=0.05×10^{-6}.

4. For a quadratic equation, $ax^2 + bx + c = 0$, the expression $\Delta = b^2 - 4ac$ is called the **discriminant.** If $\Delta = 81,\ a = 6\ and\ c = 6,\ find\ b.$

5. If $m \oplus t = 2\pi\sqrt{\frac{k^2 + h^2}{gh}}$, calculate t if $k = 4.0 \times 10^{-2},\ h = 3.0 \times 10^{-2}\ and$ g=10. (take $\pi = \frac{22}{7}$ and leave your answer in standard form)

6. The current I amperes from a battery of n cells each with an emf of E volts and an internal resistance r ohms is given by $I = \frac{nE}{R + nr}$ where R ohms is the external resistance in the circuit. Find n when E=1.6volts, R=50 ohms, r=0.5ohms and $I = \frac{3}{4} amperes$

7. The sum of the first n terms of a geometric progression is given by $S_n = a\frac{1 - r^n}{1 - r}$, where a is the first term and r is the common ratio. Find the sum of the first 10 terms of a geometric progression if the first term is 120 and the common ratio is $-\frac{1}{2}$

8. The sum, S_n, of the first n terms of an arithmetic progression is given by $S_n = \frac{n}{2}\{2a + (n-1)d\}$, where a is the first term and d is the common difference. Find d if S_n =2744, n=30 and a=3.

9. The solution of a quadratic equation $ax^2 + bx + c = 0$ can be found by using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Find the values of x given that a=2, b=9 and c=10.

10. A solid figure is formed by a cylinder with a hemisphere on top. The radius, r, of the cylinder is equal to the radius of the hemisphere. Hence the volume, v, is given by $v = \pi r^2 h + \frac{2}{3}\pi r^3$, where h is the height of the cylinder.
   (a) Make h the subject of the relation
   (b) Calculate the height, h, of the cylinder, if the volume of the solid figure is $270cm^3$ and the radius is 3cm.

11. The compound interest formula is given by $A = P\left(1 + \frac{r}{100}\right)$ where A is the amount, P is the Principal, r% is the rate of interest per annum and n is the number of years. Given that A=¢400000.00, P=¢200000.00 and n=2years, calculate the value of r correct to the nearest whole number.

CHAPTER 8

LINEAR EQUATIONS

8.1 **Introduction**

An ***equation*** is a sentence that expresses the equality of two algebraic expressions.
In otherwords, an ***equation*** is a mathematical expression (usually numerical) involving the use of the equal sign (=).
A ***linear equation*** is an equation which has one (1) as the highest power of the unknown existing variable. Examples are:
$x+2=6 \quad or \quad 3x-1=10 \ etc.$

NB

In working for the ***unknown variable***, a question could be asked to ***solve*** or ***evaluate*** the equation and the result of solving the equation becomes the ***root or solution*** of the equation.
Generally, in ***solving linear equations***, we:

1. Change all mixed fractions into improper fractions
2. Get rid of all fractions by multiplying through by LCM
3. Open any bracket by expanding all brackets
4. Group liked terms, preferably variable terms on LHS and simplify for the variable

Example 8.1

Solve the following for x

(i) $2x=10$
(ii) $2x=3-4x$
(iii) $2x-2=10$
(iv) $2x-2=10+x$
(v) $2(x-4)+5x=34$
(vi) $\frac{x}{2}-\frac{1}{3}=\frac{x}{3}+\frac{5}{6}$
(vii) $x-0.1x=0.75x+4.5$

(viii) $8-3(x-5)+7=3-(x-5)+2(x-11)$

(ix) $\frac{y}{2}-\frac{y-4}{5}=\frac{23}{10}$

Solution

$(i)\ \ 2x=10$

Divide through by 2

$\Rightarrow\ \frac{2x}{2}=\frac{10}{2}$

$\therefore\ x=5$

$(ii)\ \ 2x=3-4x$

Group liked terms

$\Rightarrow\ 2x+4x=3$

$\frac{6x}{6}=\frac{3}{6}$

$x=\frac{1}{2}\ \ or\ \ 0.5$

$(iii)\ \ 2x-2=10$

Group liked terms

$2x=10+2$

$\frac{2x}{2}=\frac{12}{2}$

$\therefore\ x=6$

$(iv)\ \ 2x-2=10+x$

Group liked terms

$2x-x=10+2$

$x=12$

$(v)\ 2(x-4)+5x=34$ *Expand bracket*

$\Rightarrow 2x-8+5x=34$

$\Rightarrow 7x-8=34$ *Add 8 to both sides*

$\Rightarrow 7x-8+8=34+8$

$\Rightarrow 7x=42$ *Divide through by 7*

$\Rightarrow \frac{7x}{7}=\frac{42}{7}\Rightarrow x=6$

$(vi)\ \frac{x}{2} - \frac{1}{3} = \frac{x}{3} + \frac{5}{6}$

To solve this equation, we multiply each side by 6, the LCM for 2, 3 and 6:

$\Rightarrow 6\left(\frac{x}{2} - \frac{1}{3}\right) = 6\left(\frac{x}{3} + \frac{5}{6}\right)$

$\Rightarrow \frac{6x}{2} - \frac{6}{3} = \frac{6x}{3} + \frac{30}{6}$ *Divide to eliminate the fractions*

$3x - 2 = 2x + 5$ *Group liked terms*

$\Rightarrow 3x - 2x = 5 + 2$

$x = 7$

$(vii)\ x - 0.1x = 0.75x + 4.5$

Since the number with the most decimal places in this equation is 0.75 (75 hundreds), multiplying each side by 100 will eliminate all decimals.

$x - 0.1x = 0.75x + 4.5$ *Multiply each side by* 100

$\Rightarrow 100(x - 0.1x) = 100(0.75x + 4.5)$ *Expand brackets*

$\Rightarrow 100x - 10x = 75x + 450$

$90x = 75x + 450$ *Group liked terms*

$90x - 75x = 450$

$15x = 450$ *Divide each side by* 15

$\Rightarrow \frac{15x}{15} = \frac{450}{15} \Rightarrow x = 30$

$(viii)\ 8 - 3(x-5) + 7 = 3 - (x-5) + 2(x-11)$ *Expand*

$\Rightarrow 8 - 3x + 15 + 7 = 3 - x + 5 + 2x - 22$

$\Rightarrow 30 - 3x = x - 14$

$30 + 14 = x + 3x$

$\frac{44}{4} = \frac{4x}{4} \Rightarrow x = 11$

$(ix)\ \frac{y}{2} - \frac{y-4}{5} = \frac{23}{10}$

First, multiply each side by 10, the LCM of 2, 5 and 10

$$\Rightarrow 10\left(\frac{y}{2}\right)-10\left(\frac{y-4}{5}\right)=10\left(\frac{23}{10}\right)$$

$$\Rightarrow \frac{10y}{2}-\frac{10(y-4)}{5}=\frac{230}{10}$$

$5y-2y+8=23$

$3y+8=23$

$\Rightarrow 3y=23-8$

$3y=15$ *Divide each side by* 3

$$\Rightarrow \frac{3y}{3}=\frac{15}{3}\Rightarrow y=5$$

Example 8.2

(i) Solve $3(6+7y)+2(1-5y)=42$

(ii) *If* $3x-4=2(y-2)$, *find* $\frac{x}{y}$, *where* $y\neq 0$

(iii) Solve the equation: $5(x-2)-8(x-1)=3x-10$

Solution

(i) $3(6+7y)+2(1-5y)=42$

Expand brackets

$\Rightarrow 18+21y+2-10y=42$

$20+11y=42$

Group liked terms

$11y=42-20$

$$\frac{11y}{11}=\frac{22}{11}$$

$\Rightarrow y=2$

(ii) $3x-4=2(y-2)$

$3x-4=2y-4$

$\Rightarrow 3x=2y$

Divide through by 3y

$$\frac{3x}{3y}=\frac{2y}{3y}$$

$$\frac{x}{y} = \frac{2}{3}$$

$(iii)\ 5(x-2) - 8(x-1) = 3x - 10$

$\Rightarrow 5x - 10 - 8x + 8 = 3x - 10$

$-3x - 2 = 3x - 10$

$\Rightarrow 10 - 2 = 3x + 3x$

$\frac{8}{6} = \frac{6x}{6} \Rightarrow x = \frac{4}{3}$

Example 8.3

Solve the following equations:

(i) $\frac{2x}{5} = 2$

(ii) $\frac{8}{6-y} = -2$

(iii) $12 + \frac{6}{x} = 9$

(iv) $\frac{x}{5} + 3 = \frac{x}{3} + 5$

(v) $\frac{y-2}{2} + \frac{y+1}{4} = 3$

(vi) $\left(1\tfrac{1}{2}\right)a - 4\tfrac{1}{2} = 1\tfrac{1}{2} + \left(\tfrac{3}{4}\right)a$

Solution

$(i)\ \frac{2x}{5} = 2$

Multiply through by 5 to get rid of the only fraction

$\Rightarrow 5 \times \frac{2x}{5} = 5 \times 2$

$\frac{2x}{2} = \frac{10}{2}$

$x = 5$

$(ii)\ \frac{8}{6-y} = -2$

Multiply through by $(6-y)$

$\Rightarrow 8=-2(6-y)$

$8=-12+2y$

$8+12=2y$

$\frac{20}{2}=\frac{2y}{2}$

$\therefore 10=y \;\; or \;\; y=10$

$(iii)\;\; 12+\frac{6}{x}=9$

Multiply through by x to get rid of fraction

$12\times x+\frac{6}{x}\times x=9\times x$

$12x+6=9x$

$12x-9x=-6$

$\frac{3x}{3}=\frac{-6}{3}$

$x=-2$

$(iv)\;\; \frac{x}{5}+3=\frac{x}{3}+5$

Multiply each term by 15 (LCM)

$\frac{x}{5}\times 15+3\times 15=\frac{x}{3}\times 15+5\times 15$

$3x+45=5x+75$

$45-75=5x-3x$

$\frac{-30}{2}=\frac{2x}{2}$

$\Rightarrow x=-15$

$(v)\;\; \frac{y-2}{2}+\frac{y+1}{4}=3$

Multiply through by 4

$4\times\frac{y-2}{2}+4\times\frac{y+1}{4}=4\times 3$

$2(y-2)+y+1=12$

$2y-4+y+1=12$

$3y-3=12$

$$\frac{3y}{3} = \frac{15}{3}$$
$$\therefore\ y = 5$$

$(vi)\ \left(1\tfrac{1}{2}\right)a - 4\tfrac{1}{2} = 1\tfrac{1}{2} + \left(\tfrac{3}{4}\right)a$

Change all mixed fractions into improper fractions

$$\left(\frac{3}{2}\right)a - \frac{9}{2} = \frac{3}{2} + \left(\frac{3}{4}\right)a$$

$$\left(\frac{3a}{2}\right) - \frac{9}{2} = \frac{3}{2} + \left(\frac{3a}{4}\right)$$

Multiply through by 4

$$4\left(\frac{3a}{2}\right) - 4\times\frac{9}{2} = 4\times\frac{3}{2} + 4\left(\frac{3a}{4}\right)$$
$$6a - 18 = 6 + 3a$$
$$6a - 3a = 6 + 18$$
$$\frac{3a}{3} = \frac{24}{3}$$
$$a = 8$$

8.2 **Word Problems**

Mathematical expressions written in sentences are referred to as ***word problems***.

Here, we often use algebra to solve problems by translating them into algebraic equations by setting up a new equation that represents or the problem.

Generally, in ***solving*** word problems, we:

1. Carefully read to understand the question
2. Represent one of the unknown quantities by a variable and express all other variables in terms of this variable
3. Form an equation with the two expressions by placing an equal sign where the word 'is' appears in the question
4. Solve the resulting equation for the variable

NB

We use 'plus' for 'more than' or 'greater than' and we use 'minus' for 'less than' or 'smaller than'.

Likewise, we use ‘times’ for ‘multiplication’.
Also in context, in dealing with ‘years ago’ we subtract whiles with ‘years to come’ we add.

8.2.1 **Translating Words Into Algebra**

Operation	**Verbal Phrase**	**Algebraic Expression**
Addition	The sum of a number and 8 Five is added to a number Two more than a number A number is increased by 3	x+8 x+5 x+2 x+3
Subtraction	Four is subtracted from a number Three less than a number The difference between 7 and a number Some number decreased by 2 A number less 5	x-4 x-3 7-x x-2 x-5
Multiplication	The product of 5 and a number Seven times a number Twice a number One-half of a number	5x 7x 2x $\frac{1}{2}x$ or $\frac{x}{2}$
Division	The ratio of a number to 6 The quotient of 5 and a number Three divided by some number	$\frac{x}{6}$ $\frac{5}{x}$ $\frac{3}{x}$

Example 8.4

The result of adding 10 to a certain number is the same as multiplying the square of the number by $\frac{1}{2}$. Which of the following expressions represents this statement?

Solution

Let the number be x

Adding 10 to the number becomes $x+10$

Multiplying the square of the number by $\frac{1}{2}$ becomes $\frac{1}{2}x^2$

Implies, $x+10=\frac{1}{2}x^2$

$$2x+20=x^2$$
$$x^2-2x-20=0$$

Example 8.5

The sum of two-third of a number and four-fifth of the same number is 22. Find the number.

Solution

Let the number be x

Two-third of x is $\frac{2}{3}x$

Four-fifth of x is $\frac{4}{5}x$

Sum of two-third and four-fifth of the number is 22

becomes $\frac{2}{3}x+\frac{4}{5}x=22$

Implies, $\frac{2}{3}x+\frac{4}{5}x=22$

$$10x+12x=330$$
$$\frac{22x}{22}=\frac{330}{22}$$
$$x=15$$

NB

In solving problems involving ages of two or more persons, we first represent the younger person's age by a variable and express the older one's age in terms of the younger one's age.

Example 8.6

Offei is x years old now. Five years ago he was half as old as he is now. How old is he now?

Solution

Offei's age now is x years

Five years ago Offei was $(x-5)$ *years old*

If five years ago he was half his age now becomes $\frac{x}{2} = x - 5$

Implies,

$\Rightarrow$ *From* $\frac{x}{2} = x - 5$

$x = 2x - 10$

$10 = 2x - x$

$10 = x$

Hence, Offei is 10 years old now.

Example 8.7

Elvis is three times as old as Akos. In ten years time, Elvis will be twice as old as Akos. How old are they?

Solution

Let Akos age be x years old now

$\Rightarrow$ *Elvis is* $3x$ *years now.*

In ten years time,

$\Rightarrow$ *Akos will be* $(x+10)$ *years*

and

Elvis will be $3x+10$ *years*

$\Rightarrow 3x + 10 = 2(x+10)$

$3x + 10 = 2x + 20$

$\Rightarrow 3x - 2x = 20 - 10$

$\Rightarrow x = 10$

Hence, Akos is 10 years old and Elvis is 30 years old

NB

Considering the three consecutive integers 3, 4, 5, we note that each integer is 1 larger than the previous integer. Thus, to represent three unknown consecutive integers, we let

$x = the\ first\ integer$

$x + 1 = the\ second\ integer$

$x + 2 = the\ third\ integer$

Again, considering the three consecutive even integers 4, 6, 8 we note that each even integer is 2 larger than the previous even integer. To represent three unknown consecutive even integers, we let

$x = the\ first\ even\ integer$

$x + 2 = the\ second\ even\ integer$

$x + 4 = the\ third\ even\ integer$

Note that consecutive odd integers as well as consecutive even integers differ by 2,
So the same expression is used in either case.

Example 8.8

The sum of three consecutive integers is 228. Find the integers.

Solution

Let

$x = the\ first\ integer$

$x + 1 = the\ second\ integer$

$x + 2 = the\ third\ integer$

Since the sum of these three expressions for the consecutive integers is 228, we have;

$x+(x+1)+(x+2)=228 \quad ie.\ the\ sum\ of\ the\ integers\ is\ 228$

$\Rightarrow 3x+3=228$

$3x=228-3$

$3x=225 \quad Divide\ through\ by\ 3$

$\frac{3x}{3}=\frac{225}{3}$

$\Rightarrow x=75$

Hence,

$x+1=75+1=76$

and

$x+2=75+2=77$

Therefore, the three consecutive integers that have a sum of 228 are 75, 76 and 77.

Example 8.9

Find three consecutive odd integers such that the sum of the last two is 15 less than 5 times the first.

Solution

Let first odd number be x
Implies, second odd number will be x + 2 (Since consecutive odd numbers differ from each other by 2)
Third odd number will be x + 4
Sum of last two is sum of second and third numbers
= x + 2 + x + 4 = 2x + 6
Implies, 2x + 6 = 5x -15 (i.e. sum of last two is 15 less than 5 times the first)
Implies, 6 + 15 = 5x – 2x   Implies, 21 = 3x
Therefore, x = 7 Hence, first number is 7, second is 9 and third is 11

Example 8.10

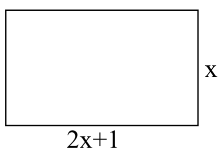
The length of a rectangular piece of property is 1 foot more than twice the width. If the perimeter is 302 feet, find the length and width.

Solution

We let;

x = the width

2x + 1 = the length



The perimeter of a rectangle is modeled by the equation:

$2L + 2W = P$

$2L + 2W = P$

From question we have;

$2L + 2W = P \quad \text{Re}\,place\ L\ by\ 2x+1\ \ and\ \ W\ by\ x$

$\Rightarrow 2(2x+1) + 2(x) = 302 \quad \text{Re}\,move\ brackets$

$4x + 2 + 2x = 302$

$6x + 2 = 302$

$6x = 302 - 2$

$6x = 300 \quad Divide\ through\ by\ 6$

$\Rightarrow \dfrac{6x}{6} = \dfrac{300}{6}$

$\Rightarrow x = 50\ and\ 2x+1 = 2(50)+1 = 100+1 = 101$

$Hence,\ the\ length\ is\ 101 feet\ and\ the\ width\ is\ 50 feet.$

EXERCISE

QUE. A

Solve the following equations:

1. $\frac{y-5}{y}=\frac{5}{3y}-\frac{1}{5}$
2. $\frac{x}{2}+\frac{2x}{3}=14$
3. $\frac{4x}{7}-2=\frac{x-1}{3}$
4. $\frac{3x}{5}+2=\frac{x-2}{3}$
5. $3=\frac{42}{t}-18$

QUE. B

(i) If $5p-2q=3(p+q),\ \textit{find}\ \frac{p}{q}\ \textit{where}\ q\neq 0$

(ii) $4(a-2)=3b-8,\ \textit{find}\ \frac{a}{b},\ \textit{where}\ b\neq 0$

QUE. C

Kofi had 36 mangoes and kwame had 48. How many mangoes should Kofi give to Kwame so that Kwame would have three times as many mangoes as Kofi?
Hint: Use $48+x=3(36-x)$

QUE. D

The sum ofthree consecutive integers is 84. Find the integers

QUE. E

Two consecutive odd integers have a sum of 128.
What are the integers?

QUE. F

Kwame, Adjoa and Kwesi divided ¢ 450.00 among themselves. Adjoa received twice as much as Kwesi and Kwame received three times as much as Adjoa. How much did each receive?

QUE. G

The sum of a number, three-fourth of the number and five-eight of the number is 19. Find the number.

QUE. H

Find three consecutive even integers whose sum is 252.

QUE. I

The sum of three numbers is 81. The second number is twice the first, and the third number is six more than the second. Find the numbers.

QUE. J

Kofi is now five times as old as Kweku. In ten years time Kofi will be three times as old as Kweku. How old are they?

QUE. K

The sum of two numbers is 54. One exceeds the other by 14. Find the numbers.

QUE. L

Prof Elvis bought 6 plates and 10 drinking cups from a shop. A plate cost him ¢ 20.00 more than a drinking cup. If he spent ¢ 1080.00 altogether, how much did a plate and a drinking cup cost?

QUE. M

Solve each of the following:

1. $-72-x=15$ 2. $-3x-19=5-2x$ 3. $-12x-15=21$

4. $5x+10(x+2)=110$ 5. $26=4x+16$ 6. $\frac{x}{3}+\frac{1}{2}=\frac{7}{6}$

7. $-3(x-16)=12-x$ 8. $\frac{1}{2}x+\frac{1}{4}=\frac{1}{4}(x-6)$ 9. $2(x+9)-x=36$

10. $2+3(x-1)=x-1$ 11. $-\frac{5}{7}x-1=3$ 12. $1.3-0.2(6-3x)=0.1(0.2x+3)$

13. $\frac{2}{3}x+5=-\frac{1}{3}x+17$ 14. $8-\frac{x-2}{2}=\frac{x}{4}$ 15. $\frac{y-3}{3}-\frac{y-2}{2}=-1$

16. $\frac{1}{2}\left(y-\frac{1}{6}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{1}{2}-3y\right)$ 17. $4-6(2x-3)+1=3+2(5-x)$

18. $2(0.5x+1.5)-3.5=3(0.5x+0.5)$ 19. $2x-3=0$

20. $8-\frac{2}{3}(60x-900)=\frac{1}{2}(400x+6)$ 21. $\frac{a-3}{4}-\frac{2a-5}{2}=\frac{a+1}{3}-\frac{1}{6}$

CHAPTER 9

SIMULTANEOUS EQUATIONS IN TWO VARIABLES

Two or more equations that can be solved together at the same time are called ***simultaneous equations***. Thus, solving at the same time means solving ***'simultaneously'***. Here, we shall consider ***two variable*** simultaneous equations.

9.1 **Methods Of Solving Simultaneous Equations**

Basically, we will be looking at four methods of solving simultaneous equations. Namely:

1. Elimination method
2. Substitution method
3. Equality method
4. Graphical method

9.1.1 **The Elimination Method**

Steps

1. Ensure that at least the coefficients of one of the variables of a corresponding term of the two equations are the same. When this is satisfied and the signs of those terms are also the same (say both are positives or both are negatives), we subtract either equation from the other, term wisely to eliminate that variable in those terms. Again, if the signs are different (say one is negative and the other positive), we add the two equations to eliminate that variable in those terms. Further, when there happens to be no equal corresponding coefficients, we manipulate either of the equations by multiplying it by a constant to make the coefficients of one corresponding term in the equations equal before adding or subtracting
2. Solve the resulting equation for one of the variables

3. Substitute the value of the variable found in step 2 into any of the two equations and solve for the other variable (the eliminated variable)
4. For proof of your answers, substitute values of the variables found into the two equations to see whether the equations will be satisfied

Example 9.1

Find the values of x and y that satisfy the equations
$x+6y=3 \;\; and \;\; x+y=-2$

Solution

$x+6y=3 ----------- (1)$

$x+y=-2 -----------(2)$

Eqn (1) – eqn (2) (Since coefficients of first terms are equal and have same sign)

$\Rightarrow 6y-y=3-(-2)$

$\frac{5y}{5}=\frac{5}{5}$

$\Rightarrow y=1$

Put $y=1$ int$o$ (1)

$x+6(1)=3$

$x=3-6=-3$

$\therefore x=-3 \;\; and \;\; y=1$

Example 9.2

Solve the simultaneous equations: $2x+y=7 \;\; and \;\; x+3y=1$

Solution

$2x+y=7 ---------(1)$

$x+3y=1 ---------(2)$

Multiply eqn (2) by 2 (to make coefficients of first terms equal)

$\Rightarrow 2x+6y=2------(3)$

Eqn (3) – eqn (1)

$\frac{5y}{5} = \frac{-5}{5}$

$\Rightarrow y = -1$

Put $y = -1$ into $eqn(2)$

$\Rightarrow x + 3(-1) = 1$

$x = 1 + 3 = 4$

9.1.2 **Substitution Method**

Steps

1. Make one of the variables the subject of any of the equations
2. Put the result of step 1 into the other equation and solve for the unknown variable
3. Put back the result of the variable found in step 2 into the expression in step 1 for the value of the other variable

Example 9.3

Solve the simultaneous equations: $2x - 5y = 1$ *and* $x - 2y = 3$

Solution

$2x - 5y = 1 ------(1)$

$x - 2y = 3 ------(2)$

Make x the subject of eqn (2)

$x = 3 + 2y ------(3)$

Put eqn (3) into (1)

$\Rightarrow 2(3 + 2y) - 5y = 1$

$6 + 4y - 5y = 1$

$6 - y = 1$

$\Rightarrow 6 - 1 = y$

$\therefore y = 5$

Put $y = 5$ into $eqn(3)$

$x = 3 + 2(5) = 3 + 10 = 13$

Example 9.4

Solve the simultaneous equations: $3x+5y=21$ *and* $7x-2y=8$

Solution

$3x+5y=21-------(1)$

$7x-2y=8--------(2)$

Make x the subject of eqn (1)

$3x=21-5y$

$x=\frac{21-5y}{3}-----(3)$

Put eqn (3) into eqn (2)

$7\left\{\frac{21-5y}{3}\right\}-2y=8$

$147-35y-6y=24$

$\frac{-41y}{-41}=\frac{-123}{-41}$

$y=3$

Put $y=3$ into (3)

$x=\frac{21-5(3)}{3}=\frac{21-15}{3}=\frac{6}{3}=2$

Example 9.5

If $4x-3y=-1$ *and* $x+2y=8$, *find the value of* $(x+y)$

Solution

$4x - 3y = -1 - - - - - - (1)$
$x + 2y = 8 - - - - - - - - (2)$
$From\ (2),$
$x = 8 - 2y - - - - - - - - (3)$
$Put\ (3)\ \text{int}\,o\ (1)$
$\Rightarrow 4(8 - 2y) - 3y = -1$
$32 - 8y - 3y = -1 \Rightarrow 32 - 11y = -1$
$\Rightarrow 32 + 1 = 11y$
$\frac{33}{11} = \frac{11y}{11} \Rightarrow y = 3$
$Put\ \ y = 3\ \text{int}\,o\ (3)$
$\Rightarrow x = 8 - 2(3) = 8 - 6 = 2$
$\therefore x + y = 2 + 3 = 5$

9.1.3 **Equality method**

Steps

1. Make ***either*** of the variables in both equations the subjects resulting to two additional equations
2. Equate the resulting equations in step 1
3. Solve for the resulting variable in step 2
4. Put back the value of the variable found in step 3 into any one of the equations formed in step 1 and solve for the remaining variable

Example 9.6

Find the value of x in the simultaneous equations:
$2x + y = 4\ \ and\ \ x - 2y = 2$

Solution

$2x + y = 4 - - - - - - (1)$
$x - 2y = 2 - - - - - - (2)$

Make x the subject of both equations

From eqn (1)

$x = \dfrac{4-y}{2} ------(3)$

From eqn (2)

$x = 2 + 2y-----(4)$

Equate eqn (3) and eqn (4)

$\dfrac{4-y}{2} = 2 + 2y$

$4 - y = 4 + 4y$

$\Rightarrow 0 = 5y$

$y = 0$

Put $y = 0$ int$o$ eqn (4)

$x = 2 + 2(0) = 2$

Therefore, $x = 2$

Example 9.7

Find the truth set of the simultaneous equations:

$\frac{5}{6}x - \frac{3}{4}y = 2;\ \frac{1}{2}x - \frac{2}{3}y = \frac{5}{2}$

Solution

$\frac{5}{6}x - \frac{3}{4}y = 2---------(1)$

$\frac{1}{2}x - \frac{2}{3}y = \frac{5}{2}--------(2)$

Get rid of the fractions in each equation

Multiply eqn (1) by 12

$\Rightarrow 10x - 9y = 24------(3)$

Multiply eqn (2) by 6

$\Rightarrow 3x - 4y = 15------(4)$

Make x the subject of eqn (3)

$x = \dfrac{24+9y}{10} ------(5)$

Make x the subject of eqn (4)

$$x = \frac{15+4y}{3} - - - - - - (6)$$

Equate eqn (5) and eqn (6)

$$\frac{24+9y}{10} = \frac{15+4y}{3}$$

Cross and multiply

$$3(24+9y) = 10(15+4y)$$

$$72 + 27y = 150 + 40y$$

$$72 - 150 = 40y - 27y$$

$$\frac{-78}{13} = \frac{13y}{13}$$

$$y = -6$$

Put $y = -6$ into $eqn(5)$

$$x = \frac{24+9(-6)}{10} = \frac{24-54}{10} = \frac{-30}{10} = -3$$

Truth set $= \{x, y : x, y = -3, -6\}$

9.1.4 **Graphical method**

Steps

1. Find the x and y intercepts of equation (1)
2. Plot and join these two points by a straight line on a graph sheet using appropriate scale
3. Find the x and y intercepts of equation (2)
4. Plot and join these two points by a straight line on the same graph sheet
5. Trace the point of intersection of both graphs to the x and y axis for the solution of the equations. The corresponding value on x axis gives the x value and that on the y axis gives the y value

Example 9.8

Using a scale of 2cm to 1unit on the x axis and 2cm to 2units on the y axis, draw the graph for the straight lines $y + 2x = 1$ *and* $y - 3x = 11$ on the same graph sheet. From your graphs, find the co-ordinates of the point of intersection

Solution

From eqn (1)

$y+2x=1------(1)$

For x intercept, put $y=0$

$\Rightarrow x=0.5$

Implies, x intercept is $(0.5,0)$

For y intercept, put $x=0$

$\Rightarrow y=1$

Implies, y intercept is $(0,1)$

Plot joining these two intercepts with a straight line on a graph sheet.

From eqn (2)

$y-3x=11$

For x intercept put $y=0$

$\Rightarrow x=-3.7$

Implies, x intercept is $(-3.7,0)$

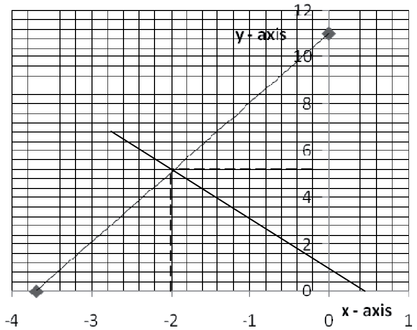
For y intercept, put $x=0$

$\Rightarrow y=11$

Implies, y intercept is $(0,11)$

Plot joining these two intercepts with a straight line on the same graph sheet

Hence, tracing the intersection to x and y axes, the coordinates are (-2, 5)



9.2 **Word Problems**

When a mathematical expression is written in sentence form, we call it a ***word problem.*** Here, we interpret the statement into two simultaneous equations and solve using any of the four methods applied above.

Example 9.9

The sum of two numbers is 8 and their product is -33. Find the numbers.

Solution

Let x and y be the numbers
Implies,
$x + y = 8 - - - - - (1)$
$xy = -33 - - - - - (2)$
From eqn (1)
$x = 8 - y - - - - (3)$
Put eqn (3) into eqn (2)
$(8 - y)y = -33$
$8y - y^2 = -33$
$y^2 - 8y - 33 = 0$
Solving quadratically gives $y = -3 \; or \; y = 11$
Putting $y = -3 \; and \; y = 11$ into eqn (3) gives $x = -3 \; or \; x = 11$
Therefore, the numbers are $-3 \; and \; 11$

Example 9.10

The cost of a packet of sugar is x cedis and the cost of a tin of milk is y cedis. If 3 packets of sugar and 4 tins of milk cost ¢ 635.00 and 4 packets of sugar and 3 tins of milk cost ¢ 695.00, write two equations connecting x and y. Hence, find x and y.

Solution

Cost of a packet of sugar = x cedis
Cost of a tin of milk = y cedis
Cost of 3 packets of sugar = 3x cedis
Cost of 4 tins of milk = 4y cedis
Now, 3 packets of sugar and 4 tins of milk cost ¢ 635.00 becomes:
$3x + 4y = 635 - - - - - (1)$
Again, 4 packets of sugar and 3 tins of milk cost ¢ 695.00 becomes:
$4x + 3y = 695 - - - - - (2)$
Solving equations (1) and (2) simultaneous gives $x = ¢\ 125$ and $y = ¢\ 65$

Example 9.11

In an examination, the sum of the marks obtained by Kofi and Ama was 81 and the difference was 17. If Kofi had a larger mark, find Ama's mark.

Solution

Let the mark obtained by Kofi be x and that of Ama be y.
Implies, we have the following equations from question

$x + y = 81 - - - - - - (1)$

$x - y = 17 - - - - - - (2)$

$(1) + (2)$ *gives*;

$\frac{2x}{2} = \frac{98}{2} \Rightarrow x = 49$

Put $x = 49$ int*o* (1)

$\Rightarrow 49 + y = 81$

$y = 81 - 49 = 32$

Hence, Kofi is 49*years old and Ama is* 32*years old*

EXERCISE

QUE. A

Find the solution set of the following simultaneous equations:

(i) $2p-3q=4$ *and* $3p+2q=19$

(ii) $3x+y=12$ *and* $x-2y=11$

(iii) $y+2x=3$ *and* $3y-4x=4$

(iv) $2p+q=1$ *and* $p-2q=8$

QUE. B

Kofi bought six books and ten pencils from a store. Ama bought three books and twenty-two pencils of the same kind from that store. If each of them paid ¢ 17,000 for the items, find
the cost of (i) Each pencil (ii) Each book
(iii) Two books and four pencils

QUE. C

The total cost of 60 apples and 100 eggs is ¢ 108,000.00. The cost of 72 apples is the same as that of 30 eggs. Find how much 12 apples and 20 eggs will cost.

QUE. D

(i) Find the value of $(t-s)$ from the two relations:
$4s-t+1=0$ *and* $2t-5t+1=0$

(ii) Solve the following equations simultaneously
(α) $x+y=5$ *and* $x^2+y^2=13$ (β) $2^{x+y}=\frac{1}{4}$ *and* $x+6y=3$

QUE. E

Solve the following pairs of simultaneous equations.

1. $5x+3y=11$
$x-3y=5$

2. $4x+y=9$
$4x-3y=5$

3. $3y+4x=7$
$3y+5x=8$

4. $x+3y=2$
$2x+5y=10$

5. $2x+5y=20$
$3x-2y=-46$

6. $8a+10b=-8$
$3a-5b=5$

7. $5+3y=29$
$3x+7y=7$

8. $\frac{m}{2}+\frac{n}{7}=9$
$\frac{m}{7}-\frac{n}{2}=-5$

9. $y-6x=1$
$5x+y=12$

10. $\frac{3}{5}p+\frac{2}{3}q=12$
$\frac{1}{2}p+q=14$

11. $2s-t=1$
$s+4t=23$

12. $m-2n=-13$
$3m+n=-4$

13. $6x+3y=4$
$10x-6y=3$

14. $2y-8z=6$
$y+4z=9$

15. $6a+15b+6=0$
$a+14b-22=0$

16. $6m+4n=2$
$3m+3n=-2\frac{1}{4}$

17. $5p+2q=10$
$3p+2q=6$

18. $2x+5y=13$
$3x-7y=5$

19. $3a-7b=11$
$7a+2b=-11$

20. $5m-2n=3$
$7m-3n=5$

QUE. F

Solve the following pairs of simultaneous equations using the substitution method.

1. $5x+2y=11$, $y=2x+1$
2. $2a+3b=12$, $a=b+1$
3. $4s+10t=19$, $2t=3s$
4. $5m+7n=50$, $3m+n=14$
5. $3x-y=0$, $12x-5y=-6$
6. $\frac{1}{2}k-2l=1$, $k=4l+2$
7. $2a-b=5$, $5a-2b=14$
8. $x=1+6y$, $3y+2x=-13$
9. $8a+2b=13$, $16a+b=14$
10. $3x-4y=12$, $x+5y=-15$
11. $3p-2q=4$, $5p+3q=13$
12. $3s+4t=-5$, $7s+3t=1$
13. $a=3b+2$, $b=3a+2$
14. $\frac{s}{4}-\frac{t}{7}=2$, $3s-5t=-45$
15. $3x-6y=5$, $15x+12y=11$
16. $\frac{m-5n}{2}=m-3$, $5m-10n=16$
17. $\frac{x}{2}-\frac{y}{3}=0$, $\frac{y}{2}-\frac{x}{4}=2$
18. $3k-6=2l$, $\frac{k}{4}-\frac{l}{9}=1$
19. $x+y=\frac{11}{6}$, $2y-3x=8$
20. $3p+2q=15$, $2p+3q=15$
21. $5x-2y=1$, $2y-3x=-3$
22. $7a-3b=7$, $5b=6a+11$
23. $\frac{5m}{6}+\frac{n}{4}=7$, $\frac{2m}{3}-\frac{n}{8}=3$
24. $\frac{x+1}{3}=\frac{2-y}{2}-\frac{5}{6}$, $3x-\frac{1}{2}=\frac{y-3}{3}-\frac{1}{4}$
25. $\frac{p}{2}-\frac{q}{3}=1$, $\frac{p}{4}-\frac{q}{9}=1$
26. $\frac{x+y+1}{x-y-1}=2$, $\frac{x+y-1}{x-y-1}=-1$

QUE. G

1. The sum of the ages of a father and his son is 60years. The father's age is three times that of the son. Find their ages

2. If a certain number with two digits is divided by the sum of the digits, the quotient is 6 and the remainder is 5. The difference between the given number and the number formed by reversing the digits is 18. Find the given number.

3. Two thirds of one number is two more than one half of another number. The sum of the numbers is 129. Find the numbers.

4. The value of a fraction expressed as $\frac{p}{q}$ is $\frac{2}{3}$. If 3 is subtracted from the numerator and added to the denominator, its value becomes $\frac{3}{7}$. Find the values of p and q.

5. Two numbers are in the ratio of 4 to 7. The sum of three-quarters of the smaller number and one-fifth of the larger number is 22. Find the numbers.

6. The perimeter of a rectangle is 80m. the length of the rectangle is 10m more than its width. Find the length and breadth of the rectangle.

7. The sum of the masses of two heavyweight boxers is 190kg. four-fifths of the mass of the heavier boxer is 10kg less than the mass of the lighter boxer. Find their masses.

8. A man gave his two children Kodwo and Esi a total of ¢9000.00 as a present.Kodwo immediately paid Esi ¢1000.00 to repay a loan. They then found that the ratio of their monies was 5 to 4. How much did each receive?

9. One day, flying with the wind, a bird was able to reach 240kg per hour. Another day, when the speed of the wind was only half of its previous value, flying against the wind, thebird could reach only 48km per hour. Find the speeds of the wind and the bird's rate of flying when there was no wind.

10. Three oranges and two pineapples cos ¢360.00. five oranges and three pineapples cost ¢550.00. find the cost of (a) one orange (b) one pineapple.

11. Five times a larger number exceeds four times a smaller number by 55. The sum of the numbers is 38. Find the two numbers.

12. Two trains leave two different stations which are 300km apart. The first train leaves at 10.00am and the second train

leaves at 10.30am. they approach each other on parallel tracks and pass at 1.00pm. each train travels at a constant speed; the one leaving at 10.30am moves at 10km/h faster than the other. Find their speeds.

13. A boy walked for 3 hours and cycled for 4 hours, covering a total distance of 87km. A week later he walked for 2 hours and cycled for 5 hours, covering 100km. what were his average speed of walking and his average speed of riding if his walking speed and his cycling speed were constant on both occasions?

14. There are five more boys than there are girls in a class. If there were one more girl in the class, the ratio of boys to girls would be 5 to 4. How many boys and how many girls are there in the class?

15. The total mass of a mixture of two liquids has mass 3g and $1cm^3$ of the other liquid has mass 4g, what volume of each liquid is present? What mass of each liquid is present?

16. Four years ago a man was six times as old as his son, but in five years time he will be only three times as old as his son. What are their ages now?

17. The sum of two numbers is37. Five times the smaller number exceeds four times the larger by 5. Find the numbers.

18. If 3 is added to both the numerator and the denominator of a certain fraction, the result is $\frac{3}{5}$. If 1 is subtracted from both the numerator and denominator the result is $\frac{1}{2}$. What is the fraction?

CHAPTER 10

POLYGONS

A ***Polygon*** is a closed plane figure with straight sides. A polygon is said to be ***Regular*** if it has equal interior angles and equal sides. E.gs. The ***equilateral triangle*** and the ***square.*** Here we shall be considering regular polygons unless otherwise mentioned in a question. Below is a table of some polygons and their number of sides.

Polygon	Number of sides
Triangle	3
Quadrilateral	4
Pentagon	5
Hexagon	6
Heptagon	7
Octagon	8
Nonagon	9
Decagon	10

10.1 The Sum Of Interior Angles Of A Polygon

The sum of the interior angles of a polygon is given by:
$S_I = 180(n-2)$. Where n is the number of sides of the polygon.

Example 10.1

Find the sum of interior angles of a regular polygon with 7 sides.

Solution

From question, $n = 15$
Using $S_I = 180(n-2)$
$\Rightarrow S_I = 180(15-2) = 180(13) = 2340^o$

Example 10.2

Find the sum of the interior angles of a 15-sided polygon.

Solution

From question,
Using $S_I = 180(n-2)$
$S_I = 180(7-2) = 180 \times 5 = 900^o$

Example 10.3

The sum of the interior angles of a convex polygon is 1260^o. How many sides has the polygon?

Solution

From question, $S_I = 1260^o$ and using $S_I = 180(n-2)$,

$$\Rightarrow 1260 = 180(n-2)$$
$$1260 = 180n - 360$$
$$1260 + 360 = 180n$$
$$\frac{1620}{180} = \frac{180n}{180}$$
$$\Rightarrow n = 9$$

Therefore, the polygon has 9 sides

Example 10.4

Four angles of a hexagon are $130^o, 160^o, 112^o$ and 80^o. If the remaining two angles are equal, find one of them

Solution

$n = 6$ and let $x^o =$ one of the remaining angles of the hexagon
First find the sum of interior angles of a hexagon
i.e. $S_I = 180(6-2) = 180 \times 4 = 720^o$

Now sum the angles of the hexagon and equate result to the angle found above;

i.e. $130^o + 160^o + 112^o + 80^o + x + x = 720^o$

$\Rightarrow 482 + 2x = 720$

$2x = 720 - 482$

$\frac{2x}{2} = \frac{238}{2}$

$\therefore x = 119^o$

Hence, the size of each remaining angle is 119^o.

10.2 **The Interior Angles Of A Polygon**

The interior angle of a polygon is given by: $I = \frac{180(n-2)}{n}$

Where n is the number of sides.

Example 10.5

Find the interior angle of a regular polygon with 6 sides

Solution

$n = 6$ and using $I = \frac{180(n-2)}{n}$,

$I = \frac{180(6-2)}{6} = \frac{180 \times 4}{6} = \frac{720}{6} = 120^o$

Therefore, the interior angle of the regular polygon is 120^o

Example 10.6

The interior angle of a regular polygon is 108^o. How many sides has the polygon?

Solution

From question, $I = 108^o$

$$\Rightarrow 108 = \frac{180(n-2)}{n}$$

$$\Rightarrow 108n = 180n - 360$$

$$360 = 180n - 108n$$

$$\frac{360}{72} = \frac{72n}{72}$$

$$\therefore n = 5$$

Hence the polygon has 5 sides

Example 10.7

How many sides has a regular polygon whose interior angle is 156^o?

Solution

$$I = 156^o$$

$$\Rightarrow 156 = \frac{180\,(n-2)}{n}$$

$$156n = 180n - 360$$

$$360 = 180n - 156n$$

$$\frac{360}{24} = \frac{24n}{24}$$

$$\therefore n = 15$$

Hence, the polygon has 15 sides

Example 10.8

Each interior angle of a regular polygon is Ø, where Ø is the mean of 130° and 140°. find the number of sides of the polygon.

Solution

$$\theta = \frac{130+14}{2} = \frac{270}{2} = 135$$

$$But\ \theta = \frac{180(n-2)}{n}\ where\ n\ is\ number\ of\ sides\ of\ polygon$$

$$\Rightarrow 135 = \frac{180(n-2)}{n}$$

$$135n = 180n - 360$$

$$\Rightarrow 360 = 180n - 135n$$

$$\frac{360}{45} = \frac{45n}{45} \Rightarrow n = 8$$

10.3 **The Exterior Angles Of A Polygon**

The sum of exterior angles of a regular polygon is 360^o.
Again, the sum of an exterior angle and its corresponding interior angle is 180^o

i.e. $Exterior\ angle + Interior\ angle = 180^o$

Thus, **Exterior angle,** $E = \frac{360}{n}$ Where n is the number of sides of the polygon.

Example 10.9

What is the size of the exterior angle of a regular polygon of 10 sides?

Solution

$n = 10$

$$E = \frac{360}{n} = \frac{360}{10} = 36^o$$

Hence, the exterior angle of the polygon is 36^o

Example 10.10

The exterior angle of a regular polygon is 40^o. Find the sum of its interior angles.

Solution

$$E = 40^o$$
$$\Rightarrow 40 = \frac{360}{n}$$
$$\frac{40n}{40} = \frac{360}{40}$$
$$n = 9$$

But $S_I = 180(n-2)$

$$\Rightarrow S_I = 180(9-2) = 180 \times 7 = 1260^o$$

Hence, the sum of interior angles is 1260^o

Example 10.11

In a regular polygon, the interior angle is 108^o greater than the exterior angle. Calculate the number of sides of the polygon.

Solution

$$\frac{180(n-2)}{n} = \frac{360}{n} + 108$$
$$\Rightarrow 180n - 360 = 360 + 108n$$
$$180n - 108n = 720$$
$$72n = 720$$
$$\therefore n = 10$$

Example 10.12

An irregular polygon has its external angles as:
$28^o, 40^o, 120^o, 142^o$ *and* x^o.
Find the value of x.

Solution

We sum all the external angles and equate result to 360^o

$\Rightarrow 28+40+120+142+x=360 \quad \Rightarrow 330+x=360 \;\therefore x=30^o$

Example 10.13

The exterior angles of a pentagon are: $x^o, 2x^o, 3x^o, 4x^o \textit{ and } 2x^o$.
What is the size of the largest angle?

Solution

Since sum of exterior angles equals 360^o,

$x+2x+3x+4x+2x=360 \quad \Rightarrow 12x=360 \therefore x=30^o \;\; \textit{Largest angle}=4\times 30=120^o$

Example 10.14

The sum of all the interior and exterior angles of a regular polygon is 1080^o. Find the number of sides.

Solution

$\textit{Sum of Interior angles}=180(n-2) \textit{ and Sum of Exterior angles}=360$

$\textit{Hence}, 180(n-2)+360=1080$

$\Rightarrow 180n-360+360=1080$

$\frac{180n}{180}=\frac{1080}{180} \Rightarrow n=6$

Example 10.15

Find the number of sides of a regular polygon whose interior angle is thrice the exterior angle.

Solution

$$Interior\ angles,\ I = \frac{180(n-2)}{n}\ \ and\ \ Exterior\ angle,\ n = \frac{360}{n}$$

From question,

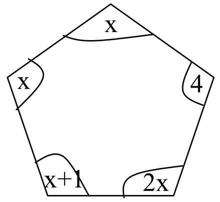
$$\frac{180(n-2)}{n} = 3\left(\frac{360}{n}\right) \Rightarrow 180(n-2) = 3(360)$$

$$\Rightarrow 180n - 360 = 1080 \Rightarrow n = 8$$

10.4 **Special Cases**

Sometimes, a diagram is given and will be required to find the value of a variable in the diagram. Thus, we first use the relation for ***sum of interior angles*** of a polygon to calculate the sum of angles for the given figure. Then equate this result to the summation of all interior angles illustrated in the figure and solve for the variable.

Example 10.16



From the diagram above, find the value of x.

Solution

S_I=180(n-2)

From figure, n = 5

Implies, S_I = 180(5-2) = 180(3) = 540

Now, summing and equating all angles to 540 gives;

x + 4 + x + 2x + x + 1 = 540

5x = 540 – 5

5x = 535 Implies, x = 107

EXERCISE

QUE. A

Find the number of sides of a polygon whose sum of interior angles is the f.f:

(i) 900^o (ii) 4320^o (iii) 2520^o

QUE. B

Calculate the sum of the interior angles of the polygons having the following number of sides:

(i) 15 (ii) 18 (iii) 20 (iv)25

QUE. C

Determine the number of sides of these regular polygons, if each interior angle is of the following size.

(i) 135^o (ii) 108^o (iii) 160^o (iv) 165^o
(v) 120^o (vi) 157.5^o (vii) 162^o (viii) 170^o

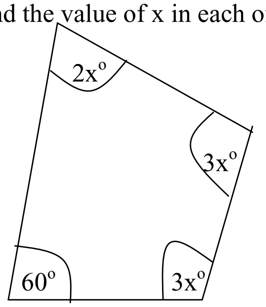
QUE. D

Three angles of an irregular octagon are 100^o, 120^o and 140^o. The remaining angles are congruent. Find the size of each of the remaining angles.

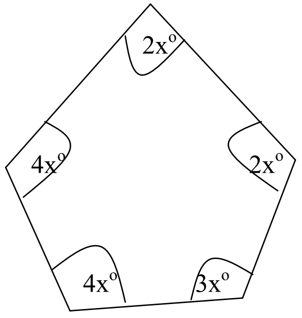
QUE. E

Find the value of x in each of the following figures.

(a)



(b)



CHAPTER 11

COORDINATE GEOMETRY

The pair (a,b) of real numbers is called a ***Coordinate.*** Where a and b represents the x *and* y coordinates respectively. We measure the x coordinate along the x axis and the y coordinate along the y axis.

11.1 **Finding The Distance Between Two Given Points**

Finding the distance between two given points is the same as finding the ***length*** of a line joining two points. Likewise, finding the ***magnitude*** of the given points. Assume that $P(x,y)$ *and* $Q(a,b)$ are two points, then the magnitude of PQ denoted by $|PQ|$ is given by:
$|PQ| = \sqrt{(x-a)^2 + (y-b)^2}$ units

Example 11.1

Find the distance between the points $A(2,3)$ *and* $B(4,5)$

Solution

From $|PQ| = \sqrt{(x-a)^2 + (y-b)^2}$,
$|AB| = \sqrt{(4-2)^2 + (5-3)^2} = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ units

Example 11.2

Find the length of the line joining the points: $P(4,9)$ *and* $Q(-1,-3)$

Solution

$|PQ| = \sqrt{(-1-4)^2 + (-3-9)^2} = \sqrt{25+144} = \sqrt{169} = 13$ units

Example 11.3

Two points $P(-5,2)$ *and* $Q(3,-7)$ are in the same plane. Find $|PQ|$

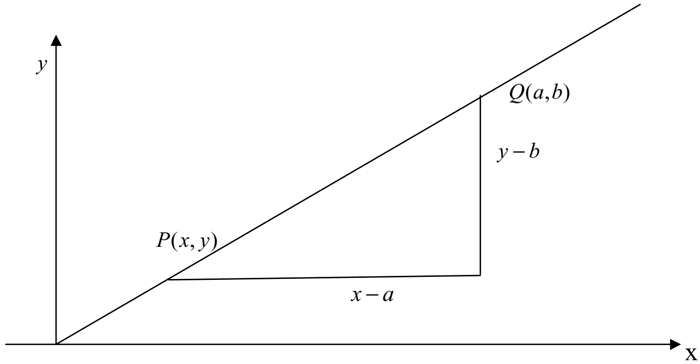
Solution

$|PQ| = \sqrt{(-5-3)^2 + (2-(-7))^2} = \sqrt{64+81} = \sqrt{145}$ units

11.2 **Finding The Gradient Of A Line Joining Two Points**

If $P(x, y)$ *and* $Q(a,b)$ are two points, then the ***gradient*** also called ***slope*** and denoted by ***m*** of the line joining P *and* Q is given by: $m = \dfrac{y-b}{x-a}$. i.e. Gradient is simply the ratio of the ***difference*** in $y-coordinates$ to the ***difference*** in the $x-coordinates$.

Illustration



Example 11.4

What is the gradient of the line joining the points $P(5,6)$ *and* $Q(6,4)$?

Solution

From $m = \dfrac{y-b}{x-a}$,

$$m = \frac{4-6}{6-5} = \frac{-2}{1} = -2$$

Example 11.5

Find the gradient of the line which passes through the points:
$(i)\ (1,3)\ \ and\ \ (6,2) \qquad (ii)\ (7,8)\ \ and\ \ (10,0)$

Solution

$$(i)\ m = \frac{2-3}{6-1} = \frac{-1}{5} = -\frac{1}{5}$$

$$(ii)\ m = \frac{8-0}{7-10} = \frac{8}{-3} = -\frac{8}{3} = -2\tfrac{2}{3}$$

11.3 **Finding The Gradient Of A Straight Line**

A ***straight line*** is a ***linear equation*** usually expressed in the form: $y = mx + c$ where m is the gradient of the line and c is the intercept on the $y-axis.$

In finding the gradient of a straight line, we make y the subject and the co-efficient of x becomes the gradient.

Example 11.6

Find the gradient of the straight line: $3x + 12y = 4$

Solution

$$3x + 12y = 4$$
$$12y = 4 - 3x$$
$$\Rightarrow \frac{12y}{12} = \frac{4}{12} - \frac{3x}{12}$$
$$y = \frac{1}{3} - \frac{1}{4}x$$
$$\therefore m = -\frac{1}{4}$$

Example 11.7

If the equation of a straight line is $2x - 4y = 3$, find its gradient.

Solution

$$2x - 4y = 3$$
$$2x - 3 = 4y$$
$$\frac{2}{4}x - \frac{3}{4} = \frac{4}{4}y$$
$$\frac{1}{2}x - \frac{3}{4} = y$$
$$\Rightarrow m = \frac{1}{2}$$

11.4 **The Mid - Point Of A Line Segment**

If $P(x, y)$ *and* $Q(a,b)$ are two points, then the co-ordinates of the ***midpoint*** of the line PQ is given by: $\left[\frac{1}{2}(x+a), \frac{1}{2}(y+b)\right]$

Example 11.8

Find the mid-point of $P(-5,7)$ *and* $Q(1,3)$

Solution

$$Midpoint = \left[\frac{1}{2}(-5+1), \frac{1}{2}(7+3)\right] = \left[\frac{1}{2}(-4), \frac{1}{2}(10)\right] = (-2,5)$$

11.5 **The Intercept On X And Y Axes**

11.5.1 **Intercept On X - Axis**

In finding the intercept on the $x-axis$ of a straight line, we substitute $y=0$ into the equation and solve for $x.$ Then put the result of x *and* $y=0$ together to form a point as the intercept

Example 11.9

Find the intercept on the $x-axis$ of the equation: $2x+4y-3=0$

Solution

Put $y=0$ into $2x+4y-3=0$
Thus, $2x+4(0)-3=0$

$\Rightarrow 2x-3=0$

$2x=3$

$\therefore x=\frac{3}{2}$

Hence, intercept is $\left(\frac{3}{2},0\right)$

11.5.2 **Intercept On Y - Axis**

In finding the intercept on the $y-axis$ of a straight line, we substitute $x=0$ into the equation and solve for $y.$ Then put the result of y *and* $x=0$ together to form a point as the intercept

Example 11.10

Find the intercept on the $y-axis$ of the equation: $2x+4y-3=0$

Solution

Put $x=0$ into $2x+4y-3=0$
Thus, $2(0)+4y-3=0$
$\Rightarrow 4y-3=0$
$4y=3$
$\therefore y=\frac{3}{4}$
Hence, intercept point is $\left(0,\frac{3}{4}\right)$

11.6 **Finding The Equation Of A Line Given The Gradient And A Point On The Line**

Generally, the equation of a straight line is given by:
$y-y_1=m(x-x_1)$ Where m is the gradient given and (x_1,y_1) is the given point

Example 11.11

What is the equation of the straight line which passes through the point $(2,3)$ and has gradient 2?

Solution

From question, $m=2$ *and* $(x_1,y_1)=(2,3)$
Using, $y-y_1=m(x-x_1)$,

$$y-3=2(x-2)$$
$$y-3=2x-4$$
$$y-3-2x+4=0$$
$$y-2x+1=0$$
$$OR$$
$$y=2x-1$$

Example 11.12

Find the equation of the line whose gradient is $-\frac{1}{2}$ which passes through the point $(7,-2)$

Solution

$$y-(-2)=-\frac{1}{2}(x-7)$$
$$y+2=-\frac{1}{2}(x-7)$$
$$2y+4=-x+7$$
$$2y+x+4-7=0$$
$$2y+x-3=0$$
$$OR$$
$$y=\frac{-x+3}{2}$$

Example 11.13

What is the equation of the line whose gradient is zero and intersects the y-axis at -2.

Solution

From question, $m=0$ and since line intersects y-axis, it means for y-intercept, $y=-2$ *and* $x=0$ hence the point $(0,-2)$

$$\Rightarrow y-(-2)=0(x-0)$$
$$y+2=0$$
$$or$$
$$y=-2$$

11.7 **Finding The Equation Of A Line Given Two Points On The Line**

Here, we use the two points to find the gradient and then any of the two given points is chosen arbitrarily as the point in the general equation of a line for the equation.

Example 11.14

Find the equation of the line passing through the points $P(2,3)$ *and* $Q(3,6)$

Solution

$$m=\frac{6-3}{3-2}=3$$

Using the point $P(2,3)$ *and* $m=3$,

$$y-3=3(x-2)$$
$$y-3=3x-6$$
$$y-3x+3=0 \quad OR \quad y=3x-3$$

Example 11.15

A straight line passes through the points (1, 4) and (3, 0). Find the :

(i) gradient of the line
(ii) equation of the line

Solution

$$(i)\ gradient,\ m=\frac{Change\ in\ y-coordinate}{Change\ in\ x-coordinate}=\frac{y_2-y_1}{x_2-x_1}=\frac{4-0}{1-3}=\frac{4}{-2}=-2$$

(ii) Using: $y-y_1=m(x-x_1)\Rightarrow y-0=-2(x-3)\Rightarrow y=-2x+6$

$\Rightarrow y + 2x - 6 = 0$

11.8 **Parallel And Perpendicular Lines**

11.8.1 **Parallel Lines**

Two lines are said to be parallel if they have the ***same*** gradient.
Thus, $m_1 = m_2$

Example 11.16

What is the equation of the line which is parallel to the line $y + 3x - 2 = 0$ and which passes through the point $(-1, 2)$?

Solution

From $y + 3x - 2 = 0$, $\quad y = 2 - 3x \quad \therefore m = -3$
Using $m = -3$ *and* $(-1, 2)$,
$y - 2 = -3(x + 1)$
$y + 3x + 1 = 0 \quad or \quad y = -3x - 1$

Example 11.17

Write the equation of the line parallel to the straight line:
$y = \frac{3}{2}x + 5$ and passing through the point $(2, 3)$

Solution

From $y = \frac{3}{2}x + 5$, $\quad m = \frac{3}{2}$
$\Rightarrow y - 3 = \frac{3}{2}(x - 2)$
$2y - 6 = 3x - 6$
$\Rightarrow 2y - 3x = 0$

Example 11.18

Find the equation of the line which passes through the point $(1,3)$ and is parallel to the line $y=\frac{2}{3}x+1$

Solution

$$m=\frac{2}{3}$$
$$\Rightarrow y-3=\frac{2}{3}(x-1)$$
$$3y-9=2x-2$$
$$\Rightarrow 3y-2x-7=0$$

11.8.2 **Perpendicular Lines**

Two lines are said to be perpendicular if the gradient of one is the negative reciprocal of the other. Thus, $m_2=-\frac{1}{m_1}$

Example 11.19

What is the equation of the line which is perpendicular to the line $y+3x-2=0$ and passes through the point $(-1,2)$?

Solution

$$m_1=-3$$
$$\therefore m_2=-\frac{1}{m_1}=-\frac{1}{-3}=\frac{1}{3}$$
$$\Rightarrow y-2=\frac{1}{3}(x+1)$$
$$3y-6=x+1$$
$$3y-x-7=0$$

Example 11.20

Find the equation of the line which is perpendicular to the line y = 2x-1 and passes through the point (2, 5).

Solution

$Equation: y = 2x - 1 \; and \; gradient, \; m = 2$

$Butgradient \; of \; a \; perpendicular \; line = -\frac{1}{m} = -\frac{1}{2}$

$U\sin g \;\; y - y_1 = m(x - x_1) \; and \; at \; po\text{int} \; (2,5)$

$\Rightarrow y - 5 = -\frac{1}{2}(x - 2) \Rightarrow 2y - 5 = -x + 2$

$\Rightarrow 2y + x - 12 = 0$

11.9 **Intersection Of Lines**

The point of intersection of two lines is the values of $x \; and \; y$ found by solving the two equations simultaneously.

Example 11.21

Find the point of intersection of the two lines
$x + y = 7 \;\; and \;\; x - y = 1$

Solution

$x + y = 7 ----(1)$

$x - y = 1-----(2)$

From eqn (2), $x = 1 + y----(3)$

Put eqn (3) into eqn (1)

$\Rightarrow 1 + y + y = 7$

$1 + 2y = 7$

$2y = 7 - 1$

$\frac{2y}{2} = \frac{6}{2}$

$y = 3$

Put $y = 3$ into eqn (3)

$\Rightarrow x = 1 + 3 = 4$

Hence the point of intersection is $(4,3)$

EXERCISE

QUE. A

Given that $A(1,3),\ B(-2,-1)\ and\ C(2,3m)$ where m is a constant, find $|AB|$ and the given value of m if $\vec{BC} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$

QUE. B

Find the gradient of the lines joining the following pairs of points:

(i) (-6, 1) and (6, 3) (ii) (-14, 2) and (-10, 10)

QUE. C

Find the equation of the lines with

(i) gradient -1 and passing through the point $(3,1)$

(ii) gradient -2 and passing through the point $(3,-2)$

(iii) gradient 5 and passing through the point $(0,4)$

QUE. D

Find the gradients and the intercepts of the following equations.

1. $y = 3x + 2$
2. $y = \frac{1}{2}x + \frac{4}{3}$
3. $y = \frac{5}{3}x + \frac{3}{5}$
4. $3y = 6x + 4$
5. $3y + 3x = 4$
6. $y + 5x - 1 = 0$
7. $\frac{x}{2} + \frac{y}{4} = 1$
8. $\frac{x}{-3} + \frac{y}{-4} = 1$
9. $\frac{x}{5} + \frac{y}{-1} = 1$
10. $\frac{x}{9} + \frac{y}{4} = 1$

QUE. E

Write the gradient-intercept form of the equation of a line with the given gradient and passing through the given point.

1. $-\frac{3}{4}, \left(1, -\frac{1}{2}\right)$ 2. $-2, (3, 0)$ 3. $\frac{5}{6}, (1, 2)$ 4. $\frac{-1}{3}, (2, -6)$

5. $\frac{7}{8}, (-3, -2)$ 6. $\frac{2}{9}, \left(\frac{1}{4}, \frac{2}{5}\right)$ 7. $\frac{3}{25}, \left(-\frac{2}{3}, 2\right)$ 8. $-\frac{2}{9}, \left(-\frac{1}{7}, -\frac{2}{5}\right)$

9. $\frac{9}{11}, \left(\frac{11}{3}, 4\right)$

QUE. F

Find the equation of the lines joining the following pairs of points:

1. $\left(1, \frac{8}{3}\right), (3, 4)$ 2. $(-2, -5), (4, 8)$ 3. $(3, -3), (-9, 4)$

4. $\left(-2, \frac{5}{6}\right), \left(-4, \frac{8}{5}\right)$ 5. $(-3, 2), (4, -5)$ 6. $(6, 7), (-4, -8)$

7. $\left(-\frac{5}{2}, \frac{1}{3}\right), \left(-\frac{1}{2}, -\frac{2}{3}\right)$ 8. $\left(-\frac{4}{7}, \frac{5}{6}\right), \left(\frac{8}{6}, \frac{-4}{5}\right)$ 9. $\left(-\frac{4}{7}, -2\right), \left(\frac{3}{5}, -6\right)$

10. $\left(\frac{9}{14}, \frac{-3}{7}\right), \left(\frac{3}{7}, \frac{9}{14}\right)$

CHAPTER 12

MODULO ARITHMETICS

12.1 **Forms Of Numbers**

Modulo arithmetic makes use of the ***remainder theory***. We shall be looking at ***three types*** of ***numbers.***

12.1.1 **Case One**

The ***first case*** is when the number is ***greater*** than the modulo. Here, in finding a number in any modulo which is greater than the modulo, we ***divide*** the ***number*** by the ***modulo*** and the ***remainder*** of the division becomes the answer.

Example 12.1

Find the following:

$(i)\ 8 \bmod 5 \quad (ii)\ 11 \bmod 2 \quad (iii)\ 9 \bmod 3$

Solution

$$(i)\ 8 \bmod 5 = \frac{8}{5} = 1 \text{ Re}m\, 3 \qquad (ii)\ 11 \bmod 2 = \frac{11}{2} = 5 \text{ Re}m\, 1 \qquad (iii)\ 9 \bmod 3 = \frac{9}{3} = 3 \text{ Re}m\, 0$$

$$\therefore 8 \bmod 5 = 3 \qquad \therefore 11 \bmod 2 = 1 \qquad \therefore 9 \bmod 3 = 0$$

12.1.2 **Case Two**

Here, the number is ***less*** than the modulo. Hence, the number becomes the ***answer*** since when divided by the modulo gives a decimal.

Example 12.2

Compute the following:

$(i)\ 4 \bmod 5 \quad (ii)\ 3 \bmod 7 \quad (iii)\ 9 \bmod 12$

Solution

$(i)\ 4 \bmod 5 = 4 \quad (ii)\ 3 \bmod 7 = 3 \quad (iii)\ 9 \bmod 12 = 9$

12.1.3 **Case Three**

This is the modulo of a ***negative number*** obtained by adding the modulo to the number until a positive number is obtained (the first positive number). This positive number obtained becomes the answer.

Example 12.3

Evaluate the following:
$(i)\ -5 \bmod 2 \quad (ii)\ -3 \bmod 5 \quad (iii)\ -7 \bmod 12$

Solution

$(i)\ -5 \bmod 2 = -5 + 2 = -3 + 2 = -1 + 2 = 1 \qquad (ii)\ -3 \bmod 5 = -3 + 5 = 2$
$(iii)\ -7 \bmod 12 = -7 + 12 = 5$

12.2 **Modulo Addition**

The sum of numbers in a given modulo is obtained by first adding the numbers and converting the result to the given modulo.

Example 12.4

Find the sum: $3 + 4 + 1 + 2 + 3 \pmod 5$

Solution

$3 + 4 + 1 + 2 + 3 = 13 \pmod 5 = 3$

Example 12.5

Evaluate: $(7+29) \bmod 5$

Solution

$(7+29) = 36 \bmod 5 = 1$

Example 12.6

Which of the following satisfies: $3x+1=4(\text{mod}\,9)$?

Solution

$$\begin{aligned}
&3x+1=4(\text{mod}\,9)\\
&When\ x=4\\
&3(4)+1=12+1=13\ (\text{mod}\ 9)=4\\
&\therefore x=4
\end{aligned}$$

Example 12.7

If $25\ (\text{mod}\ 7)=x,\ find\ x$

Solution

$$\begin{aligned}
&25\ (\text{mod}\ 7)=4\\
&\qquad\therefore x=4
\end{aligned}$$

Example 12.8

(a) Draw the addition table for the set $\{2,3,5,7\}$ in arithmetic modulo 9

(b) From your table find: $(i)\ (2\oplus 3)\oplus 2 \qquad (ii)\ n\ \ if\ \ 3\oplus n=1$

Solution

(a) Addition $\oplus$ table modulo 9

$\oplus$	2	3	5	7
2	4	5	7	0
3	5	6	8	1
5	7	8	1	3
7	0	1	3	5

(b)

$(i)\ (2\oplus3)\oplus2=5\oplus2=7\,\text{mod}9=7$

$(ii)\ When\ n=7\Rightarrow3\oplus7=10\,\text{mod}\,9=1\therefore n=7$

12.3 **Modulo Multiplication**

The product of numbers in a given modulo is obtained by first multiplying the numbers and converting to the given modulo.

Example 12.9

(i) Copy and complete the following table for multiplication modulo 7 on the set $\{1,2,3,6\}$.

$\otimes$	1	2	3	6
1	1	2	3	6
2	2	4	6	-
3	3	-	2	4
6	6	5	-	-

(ii) Use your table to find the truth set of $n\otimes n\otimes6=5$

Solution

(i)

$\otimes$	1	2	3	6
1	1	2	3	6
2	2	4	6	$\underline{5}$
3	3	$\underline{6}$	2	4
6	6	5	$\underline{4}$	$\underline{1}$

(ii) $n \otimes n \otimes 6 = 5$

From table, when $n = 3$

$$3 \otimes 3 \otimes 6 = 2 \otimes 6 = 5$$

Hence, $n = 3$

Example 12.10

(i) Copy and complete the following table for multiplication modulo 8 on the set

$\{2,3,4,5,6,7\}\}$

$\otimes$	2	3	4	5	6	7
2	4	6	0	2	4	6
3	6	1	4	-	2	5
4	0	4	-	4	0	4
5	-	7	4	1	-	3
6	4	2	0	6	4	-
7	6	-	4	-	2	1

(ii) From the table, find

(α) $(3 \otimes 5) \otimes 6$ (β) $truth\ set\ of\ n \otimes n = 1$

Solution

(i)

$\otimes$	2	3	4	5	6	7
2	4	6	0	2	4	6
3	6	1	4	$\underline{7}$	2	5
4	0	4	$\underline{0}$	4	0	4
5	$\underline{2}$	7	4	1	$\underline{6}$	3
6	4	2	0	6	4	$\underline{2}$
7	6	$\underline{5}$	4	$\underline{3}$	2	1

(ii) (α)

$(3 \otimes 5) \otimes 6 = 7 \otimes 6 = 2$

(β)

$n \otimes n = 1$

$From\ \ table\ \ when\ \ n = 3$

$\Rightarrow 3 \otimes 3 = 1$

$When\ \ n = 5$

$\Rightarrow 5 \otimes 5 = 1$

$When\ \ n = 7$

$\Rightarrow 7 \otimes 7 = 1$

Hence, truth set is $\{n : n = 3,5,7\}$

Example 12.11

(a) Draw a table for multiplication $(\otimes)$ mod 7 on the set:
$P = \{2,3,4,5,6\}$

(b) Use your table to find on the set P, the truth set of $n \otimes (n \otimes 6) = 3$

Solution

(a)

$\otimes$	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

(b) Given that: $n \otimes (n \otimes 6) = 3$

When $n = 2$

$\Rightarrow 2 \otimes (2 \otimes 6) = 2 \otimes 5 = 3$ *and*

when $n = 5$

$\Rightarrow 5 \otimes (5 \otimes 6) = 5 \otimes 2 = 3$

Hence, truth set is $\{n : n = 2, 5\}$

Example 12.12

(a) Draw a table for multiplication $(\otimes)$ in modulo 8 on the set $\{3, 5, 7\}$

(b) Using your table, (i) evaluate: $(7 \otimes 3) \otimes 5$ *and* $7 \otimes (3 \otimes 5)$

(ii) find the truth set of $n \otimes n = 1$

Solution

(a)

$\otimes$	3	5	7
3	1	7	5
5	7	1	3
7	5	3	1

(b)

(i) $(7 \otimes 3) \otimes 5 = 5 \otimes 5 = 1$ *and* $7 \otimes (3 \otimes 5) = 7 \otimes 7 = 1$

(ii) $n \otimes n = 1$

When $n = 3$

$\Rightarrow 3 \otimes 3 = 1$

When $n = 5$

$\Rightarrow 5 \otimes 5 = 1$

$When\ n = 7$

$\Rightarrow 7 \otimes 7 = 1$

Hence, the truth set is $\{n : n = 3,5,7\}$

Example 12.13

(a) Draw a table of multiplication $(\otimes)$ in modulo 8 on the set $\{2,3,5,7\}$

(b) Use your table to find the solution set of: (i) $3 \otimes n = 5$
 (ii) $n \otimes n = 1$

Solution

(a)

$\otimes$	2	3	5	7
2	4	6	2	6
3	6	1	7	5
5	2	7	1	3
7	6	5	3	1

(b)

(i) $3 \otimes n = 5$

When $n = 7$

$\Rightarrow 3 \otimes 7 = 5$

Therefore, the truth set is $\{n : n = 7\}$

(ii) $n \otimes n = 1$

When $n = 3$

$\Rightarrow 3 \otimes 3 = 1$

When $n = 5$

$\Rightarrow 5 \otimes 5 = 1$

When $n = 7$

$\Rightarrow 7 \otimes 7 = 1$

Hence, the truth set is $\{3,5,7\}$

Example 12.14

(a) Copy and complete the tables for addition $(\oplus)$ and multiplication $(\otimes)$
modulo 5.

Addition modulo 5

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	-
2	2	-	4	-	1
3	3	4	0	1	-
4	4	0	1	2	-

Multiplication modulo 5

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	-	-
3	0	3	1	-	2
4	0	4	-	2	-

(a) From the tables find: $(i)\ (2\otimes 4)\oplus 4 \quad (ii)\ (4\oplus 4)\otimes 2$

Solution
(a)
Addition modulo 5

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	$\underline{0}$
2	2	$\underline{3}$	4	$\underline{0}$	1
3	3	4	0	1	$\underline{2}$
4	4	0	1	2	$\underline{3}$

Multiplication modulo 5

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	$\underline{1}$	$\underline{3}$
3	0	3	1	$\underline{4}$	2
4	0	4	$\underline{3}$	2	$\underline{1}$

(b)

(i) $(2 \otimes 4) \oplus 4 = 3 \oplus 4 = 2$ (ii) $(4 \oplus 4) \otimes 2 = 3 \otimes 2 = 1$

Example 12.15

If multiplication (*) is defined on {1, 2, 3, 4}, a set of numbers in modulo
five, find the value of n for which 2*(3*n) = 2

Solution

Multiplication table for modulo five

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$For\ \ 2*(3*n)=2$

$When\ \ n=1$

$\Rightarrow 2*(3*1)=2*3=1$

$When\ \ n=2$

$\Rightarrow 2*(3*2)=2*1=2$

$When\ \ n=3$

$\Rightarrow 2*(3*3)=2*4=3$

$When\ \ n=4$

$\Rightarrow 2*(3*4)=2*2=4$

Hence,

$n=2$

Example 12.16

(a) Copy and complete the multiplication table modulo 5 on the set $\{1, 2, 3, 4\}$.

*	1	2	3	4
1	1		3	
2		4	1	
3	3			2
4		3		1

(b) From the table,

(i) Solve the expression $2n*4=3$

(ii) Find the value of n for which $2*(3*n)=2$

Solution

(a)

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(b)

(i) When n = 1

$2(1) * 4 = 2 * 4 = 3 \Rightarrow n = 1$

(ii) When n = 2

$2 * (3 * 2) = 2 * 1 = 2 \Rightarrow n = 2$

Example 12.17

(a) Draw the table for:

(i) addition $\oplus$ modulo 7

(ii) multiplication $\otimes$ modulo 7

on the set {0, 1, 2, 3, 4}

(b) from your tables, evaluate:

(i) $m \otimes m = 2$

(ii) $m \oplus (m \otimes 4) = 5$

(iii) $m \otimes (m \oplus 3) = 0$

Solution

(a)

(i)

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	0
4	4	5	6	0	1

(ii)

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	1
3	0	3	6	2	5
4	0	4	1	5	2

(b)

(i) $m \otimes m = 2$

$When \;\; m = 3$

$\Rightarrow 3 \otimes 3 = 2$

$When \;\; m = 4$

$\Rightarrow 4 \otimes 4 = 2$

$Hence, \;\; m = 2, 4$

(ii) $m \oplus (m \otimes 4) = 5$

$When \;\; m = 1$

$\Rightarrow 1 \oplus (1 \otimes 4) = 1 \oplus 4 = 5$

$When \;\; m = 2$

$\Rightarrow 2 \oplus (2 \otimes 4) = 2 \oplus 1 = 3$

$When \;\; m = 4$

$\Rightarrow 4 \oplus (4 \otimes 4) = 4 \oplus 2 = 6$

$Hence, \;\; m = 1$

(iii) $m \otimes (m \oplus 3) = 0$

$When \;\; m = 0$

$\Rightarrow 0 \otimes (0 \oplus 3) = 0 \otimes 3 = 0$

$When \;\; m = 4$

$\Rightarrow 4 \otimes (4 \oplus 3) = 4 \otimes 0 = 0$

$Hence, \;\; m = 0, \; 4$

EXERCISE

QUE. A

(a) Copy and complete the multiplication $(\otimes)$ table and addition $(\oplus)$ table for modulo 7 below.

multiplication $(\otimes)$ modulo 7

$\otimes$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	-	5	1	4
4	4	-	5	-	6	3
5	5	3	1	6	-	2
6	-	5	-	3	2	1

Addition $(\oplus)$ modulo 7

$\oplus$	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	-	0	1
3	4	-	6	0	1	-
4	5	6	0	1	2	3
5	6	0	-	2	3	4
6	0	1	2	3	4	-

(b) From the tables, (i) find n if $n \otimes (n \oplus n) = 1$

(ii) evaluate $(4 \oplus 5) \otimes 3$

QUE. B

(i) Draw the addition $(\oplus)$ and multiplication $(\otimes)$ tables for the set: $\{2,3,5,7\}$ in arithmetic modulo 9

(ii) From your tables find; (α) the value of $(5\otimes 5)\oplus(3\otimes 7)$

(β) the truth set of $n\otimes 3=6$

QUE. C

The operation $*$ is defined on the set R, of real numbers by

$m*n=(m+n+mn)\text{ mod }12$

(a) Draw a table for the operation $*$ on the set $\{1,2,3,4,5\}$. From your table (b) show that the operation $*$ is commutative on the set $\{1,2,3,4,5\}$.

QUE. D

(a) Draw the addition $(\oplus)$ and multiplication $(\otimes)$ tables on the set $T=\{2,5,7,11\}$ in arithmetic modulo 12

(b) From your tables (i) evaluate $(5\otimes 7)\oplus(7\otimes 11)$

(ii) find n if $n\otimes(n\oplus 2)=11$

QUE. E

Simplify the following:

1. 27(mod 7) 2. 20(mod 8) 3. 47(mod 3)
4. 15(mod 4) 5. 51(mod 9) 6. 66(mod 4)
7. 72(mod 7) 8. 23(mod 3) 9. 29(mod 8)
10. 32(mod 12) 11. 57(mod 7) 12. 68(mod 7)
13. 94(mod 8) 14. 107(mod 9) 15. 75(mod 4)
16. 98(mod 5) 17. 87(mod 11) 18. 111(mod 13)
19. 221(mod 12) 20. 391(mod 15)

QUE. F

Find the sum of the following in the moduli given.

1. 29+9(mod 6) 2. 31+14(mod 7) 3. 47+37(mod 9)
4. 57+39(mod 9) 5. 65+49(mod 7) 6. 49+66(mod 11)
7. 120+77(mod 13) 8. 17+33+46(mod 15)
9. -8+11(mod 13) 10. 14+67+35(mod 7) 11. 24+26(mod 5)
12. 33+16(mod 4) 13. 71+42(mod 9) 14. 123+621(mod 7)
15. 51+64(mod 9) 16. 14+17(mod 3) 17. 141+69+9(mod 13)
18.168+69(mod 11) 19. 321+122(mod 12) 20. 97+133(mod 9)

QUE. G

Find the following products in the given moduli.

1. 5x7(mod 4) 2. 6x13(mod 5) 3. 8x9(mod 6)
4. 16x9(mod 11) 5. 23x13(mod 9) 6. 18x11(mod 7)
7. 64x17(mod 8) 8. 29x39(mod 12) 9. 79x19(mod 13)
10. 55x29(mod 15) 11. 13x11(mod 7) 12. 27x19(mod 8)
13. 64x13(mod 11) 14. 37x32(mod 12) 15. 54x72(mod 12)
16. 65x37(mod 13) 17. 89x37(mod 15) 18. 122x46(mod 14)
19. 68x64(mod 16) 20. 97x111(mod 9)

QUE. H

Find the solution sets for the following equations.

1. $2x \equiv 2(\text{mod } 4)$ 2. $3x \equiv 1(\text{mod } 4)$ 3. $2x+1 \equiv 3(\text{mod } 4)$
4. $x^2 \equiv 1(\text{mod } 4)$ 5. $6+y \equiv 2(\text{mod } 8)$ 6. $4+m \equiv 3(\text{mod } 12)$
7. $x+5 \equiv 3(\text{mod } 7)$ 8. $8+x \equiv 3(\text{mod } 12)$ 9. $4+n \equiv 2(\text{mod } 6)$
10. $3x+4 \equiv 0(\text{mod } 5)$ 11. $x^2 \equiv 4(\text{mod } 5)$ 12. $x^2 \equiv 3(\text{mod } 5)$
13. $x^2+1 \equiv 0(\text{mod } 5)$

CHAPTER 13
APPROXIMATIONS
(SIGNIFICANT FIGURES, STANDARD FORMS AND DECIMAL PLACES)

Approximating a number means writing the number to the ***exact value*** or ***very close*** to the exact value. For instance, in a market, if the total number of market women counted is 34987, the ***approximation*** of this number is 35000.
Here, we shall be looking at three aspects, namely: ***Significant Figures, Standard Forms*** and ***Decimal places.***

13.1 **Significant Figures**

It is necessary to know how to determine the ***significant figures*** of any given numeral (number). We determine significant figures from the first ***non-zero*** digit from the left to the last non-zero numeral on the right, including any zeros which come 'inside' the number..
We can see that the numbers 24.0700, 0.240700 and 0.000240700 have only four significant figures. The two zeros at the end of each number can be omitted without affecting the value of the number. The three zeros immediately after the decimal point in 0.000240700 cannot be omitted but they are still not significant.
The fact that they are there tells us that the first significant digit is in the fourth decimal place.

NB

Zeros after the decimal point are ***not*** significant but zeros that appear in between
non-zero digits ***are*** significant.

Example 13.1

For each of the following real numbers, state whether the zeros are

(a) Significant
(b) Not significant and can be omitted

(c) Not significant and cannot be omitted

1. 7032 2. 9001 3. 2730 4. 0917 5. 0.2170 6. 0.8021
7. 0.3702 8. 0.00589 9. 0.12300 10. 9.015

Solution

1. (a) 2. (a) 3. (c) 4. (b) 5. (b) 6. (a) 7. (a)
8. (c) 9. (b) 10. (a)

13.2 **Correcting Or Rounding Numbers To A Given Number Of Significant Figures**

Steps

1. Locate the digit that corresponds to the required number of significant figures you are correcting to
2. Identify the digit next to the digit located in step 1. If this digit is ***less*** than 5, leave the required significant figure as it is but if it is ***greater*** than or equal to 5, add 1 to the last significant digit.
3. Replace any remaining digits with zeros to maintain the value of the given number if necessary.

NB

When the number you are correcting is a ***decimal***, you ignore or discard the rest of the digits after the required number of significant figures but when the number is ***whole*** or ***non-decimal,*** you covert all other digits after the required number of significant figures to zeros to retain the place value of the number.

Example 13.2

Express the following correct to three significant figures:

(1) 75882
(2) 1.02616

(3) 39763
(4) 0.0030071
(5) 0.024561

Solution

1. $75882 = 75900$ to three significant figures
NB: digits after third significant figure are made zeros since number is whole
2. $1.02616 = 1.03$ to three significant figures
NB: the zero between $1\ and\ 2$ is significant since is found in between two significant figures (non-zeros)
3. $39763 = 39800$ to three significant figures
NB: 1 is added to 7 to make it 8 since the next digit after it is more than 5
4. $0.0030071 = 0.003$ to three significant figures
5. $0.024561 = 0.025$ to three significant figures
NB: 1 is added to 4 to make it 5 since the next digit after it is ≥ 5

Example 13.3

Express the 196650 to the **nearest** thousand.

Solution

196650 = 197000

NB

The nearest thousand is the same as to three significant figures.

13.3 **Standard Forms**

Expressing a number in standard form means to express it as a product of ten (10).

NB

When the number given is a ***whole number***, we move from the last digit on the ***RHS*** to the ***LHS*** putting a decimal point after the first digit. We then multiply result by 10 raised to the ***positive power*** of the number of times we moved from ***RHS*** to the decimal point located. Thus, $256000 \ in \ standard \ form \ is \ 2.56\times10^{5}$

Again, when the number given is already a ***decimal numeral*** with more than one non-zero to the ***LHS*** of the decimal point, we move the decimal point to the first digit. We then multiply result by 10 raised to the ***positive power*** of the number of times we moved from the original decimal point. Thus, $46.52 \ in \ standard \ form \ is \ 4.652\times10^{1}$

Sometimes too, decimal numeral given will have several zeros from the first digit. Thus, we move and place the decimal point after the first non-zero digit and multiply result by 10 raised to the ***negative power*** of the number of times moved from the original decimal point. Thus, $0.0098 \ in \ standard \ form \ is \ 9.8\times10^{-3}$

Example 13.4

Express the following in standard form:

1. 98000
2. 77×93
3. 0.0098
4. $(3.6\times10^{-2})(9.5\times10^{4})$
5. 0.000316×10^{-7}

Solution

1. $98000=9.8\times10^{4}$

NB

we moved four steps from RHS to LHS

2. $77\times93=7161=7.161\times10^{3}$

NB

we moved three steps from RHS to LHS

3. $0.0098 = 9.8\times10^{-3}$

NB

we moved three steps from LHS to RHS

4. $(3.6\times10^{-2})(9.5\times10^{4}) = (3.6\times9.5\times10^{-2}\times10^{4}) = 34.2\times10^{2} = 3.42\times10^{3}$

NB

We still moved one step to the left of the decimal point since there is more than one digit to the left of the decimal point

5. $0.000316\times10^{-7} = 3.16\times10^{-4}\times10^{-7} = 3.16\times10^{-11}$

NB

we moved four steps from LHS to RHS

Example 13.5

Express the sum 200.2 and 4.27x10^3 in standard form.

Solution

$$200.2 + 4.27\times10^{3} = 0.2002\times10^{3} + 4.27\times10^{3} = 10^{3}(0.2002 + 4.27)$$
$$= 10^{3}(4.4702) = 4.4702\times10^{3}$$

Example 13.6

Multiply 3.5x10^{-6} by 4.2x10^4 and give your answer in standard form.

Solution

$$3.5\times10^{-6}\times4.2\times10^{4}=3.5\times4.2\times10^{-6+4}=14.7\times10^{-2}=1.47\times10^{-1}$$

Example 13.7

Without using tables or calculators, evaluate the following leaving your answers in standard form

(1) $\dfrac{(0.00042\times10^{-8})(15000)}{(5000\times10^{7})(0.0021\times10^{14})}$

(2) $\dfrac{0.0048\times0.81}{0.0027\times0.004}$

Solution

$$(i)\ \frac{(0.00042\times10^{-8})(15000)}{(5000\times10^{7})(0.0021\times10^{14})}=\frac{(42\times10^{-5}\times10^{-8})(15\times10^{3})}{(5\times10^{3}\times10^{7})(21\times10^{-4}\times10^{14})}$$

$$=\frac{(42\times10^{-13})(15\times10^{3})}{(5\times10^{10})(21\times10^{10})}$$

$$=\frac{(42\times15)(10^{-13}\times10^{3})}{(5\times21)(10^{10}\times10^{10})}=\frac{630\times10^{-10}}{105\times10^{20}}=\frac{6\times10^{-10}}{10^{20}}$$

$$=6\times10^{-10-20}=6\times10^{-30}$$

$$(ii)\ \frac{0.0048\times0.81}{0.0027\times0.004}=\frac{0.0048\times0.81\times10000000}{0.0027\times0.004\times10000000}$$

$$=\frac{48\times81\times10}{27\times4}=36\times10=360$$

NB: Here, we compare the total number of decimal places for the numerator and denominator, then multiply both top and down by the greater one (attaching that number as zeros to 1) to get rid of all decimal points. Thus, here the denominator has 7 decimal places, so we multiplied both top and down by 10000000.

Alternatively, we could just change each decimal into standard form and simplify as in (1).

13.4 **Decimal Places (d.p)**

Decimal places in a number is obtained by counting from the first digit after the decimal point to the ***RHS***. The first digit after the decimal point is called the ***first decimal place***, the second is the ***second decimal place*** and so on. Thus, in 7.3456, 3 is the first decimal place, 4 is the second decimal place and so on.

13.4.1 **Correcting Numbers To A Given Number Of Decimal Places**

Steps

1. Locate the digit that corresponds to the required number of decimal places you are correcting to
2. Identify the digit next to the digit located in step 1. If this digit is ***less*** than 5, leave the required decimal place as it is but if it is ***greater*** than or equal to 5, add 1 to the last decimal place required

Example 13.8

Correct the following to the indicated number of decimal places

(1) 32.2546 *to* 1 $d.p$ (2) 56.8712 *to* 2 $d.p$ (3) 87.1237 *to* 3 $d.p$

(4) 3.9997 *to* 3 $d.p$ (5) 299.996 *to* 2 $d.p$

Solution

1. $32.2546 = 32.3$ *to* $1dp$ 2. $56.8712 = 53.87$ *to* 2 dp

3. $87.1237 = 87.124$ *to* 3 dp 4. $3.9997 = 4.000$ *to* 3 dp

5. $299.996 = 300.00$ *to* 2 dp

EXERCISE

QUE. A

Correct the following to three significant figures

(1) 5045.0049
(2) 7.0959
(3) $(0.13)^3$

QUE. B

Simplify the following leaving your answer in standard form

(1) 0.00096
(2) 73×89
(3) $\sqrt{\dfrac{0.0048 \times 0.81 \times 10^{-7}}{0.27 \times 0.04 \times 10^{6}}}$
(4) $\dfrac{0.125 \times 0.00576}{0.0015 \times 0.32}$
(5) $\sqrt{\dfrac{a^2 + b^2}{c}}$ $if\ a = 4.0 \times 10^{-2}, b = 3.0 \times 10^{-2}\ and\ c = 100$

(6) $\dfrac{7.25 \times (0.16)^2}{0.004}$
(7) $\dfrac{0.0245 \times 1.2}{0.08 \times 1.75}$

QUE. C

Correct the following to the indicated decimal places
(1) 699.999 $to\ 2\ d.p$ (2) 0.45998 $to\ 3\ d.p$ (3) 1.05849 $to\ 3\ d.p$
(4) 15.8489 $to\ 3\ d.p$ (5) 1897.07 $to\ 2\ d.p$

QUE. D

(1) Use tables to evaluate $\frac{(17.3)^2 \times 4.97}{\sqrt[3]{7850000}}$

(2) Express the sum of 200.2 and 4.27×10^3 in standard form

CHAPTER 14

QUADRATIC EQUATIONS

A ***quadratic equation*** in ***one*** variable is a ***second-degree*** polynomial equation of the form: $ax^2 + bx + c = 0$, where x is a variable and $a, b \text{ and } c$ are real numbers with $a \neq 0$. A ***solution*** of an equation is called a ***root*** of the equation. A real number solution of an equation is called a ***real root*** and an imaginary number solution is called an ***imaginary root.***

14.1 **Methods Of Solving Quadratic Equations**

1. Solution by factoring
2. Solution by square root
3. Solution by completing the square
4. Solution by quadratic formula

14.1.1 **Solution By Factoring**

This method which rests on the ***zero property*** of numbers is used when the quadratic equation can be written as the product of two ***first-degree*** factors.

Steps

1. Rewrite the equation with zero on the RHS
2. Multiply the coefficient of x^2 by the constant (the third term)
3. Look for two factors of the result of step 2 above such that their sum is the coefficient of x
4. Multiply each factor by x and add the products by just placing the 'plus' sign in between them
5. Replace the term in x (the second term) by the result of step 4
6. Factor LHS and use the zero factor property to set each factor equal to zero
7. Solve these simpler equations formed in step 6

Example 14.1

Solve the following by the factoring method

1. $x^2 + 3x - 10 = 0$
2. $6x^2 - 5x - 6 = 0$
3. $5x^2 = (x+2)(x+3)$
4. $4x - x^2 - 3 = 1 - x$
5. $3x + \frac{8}{x} = 10,\ x \neq 0$

Solution

1. $x^2 + 3x - 10 = 0$

Step 1

$x^2 + 3x - 10 = 0$

Step 2

$1 \times -10 = -10$

Step 3

$5 \ and \ -2$

Step 4

$5x + (-2x) = 5x - 2x$

Step 5

$x^2 + 5x - 2x - 10 = 0$

Step 6

$x(x+5) - 2(x+5) = 0$

ie. Always repeatbracket in second term as in first term

$\Rightarrow (x+5)(x-2) = 0$

Step 7

$For \ x + 5 = 0$

$\Rightarrow x = -5$

$For \ x - 2 = 0$

$\Rightarrow x = 2$

2. $6x^2 - 5x - 6 = 0$

$\Rightarrow 6x^2 + 4x - 9x - 6 = 0$

$2x(3x+2) - 3(3x+2) = 0$

$(3x+2)(2x-3)=0$

Either $3x+2=0$ *Or* $2x-3=0$

$\Rightarrow x=-\dfrac{2}{3}$ $\Rightarrow x=\dfrac{3}{2}$ *or* $1\tfrac{1}{2}$

3. $5x^2=(x+2)(x+3)$

Expand RHS

$\Rightarrow 5x^2=x^2+3x+2x+6$

$5x^2=x^2+5x+6$

$\Rightarrow 5x^2-x^2-5x-6=0$

$4x^2-5x-6=0$

$4x^2-8x+3x-6=0$

$4x(x-2)+3(x-2)=0$

$(x-2)(4x+3)=0$

$x-2=0$ *Or* $4x+3=0$

$\Rightarrow x=2$ $\Rightarrow x=-\dfrac{3}{4}$

Hence, truth set is $\{x:x=-\dfrac{3}{4},2\}$

4. 4. $4x - x^2 - 3 = 1 - x$

$\Rightarrow 4x - x^2 - 3 - 1 + x = 0$

$5x - x^2 - 4 = 0 \quad$ *Send all LHS terms to RHS and vice versa*

$\Rightarrow x^2 - 5x + 4 = 0$

$x^2 - x - 4x + 4 = 0$

$x(x-1) - 4(x-1) = 0$

$(x-1)(x-4) = 0$

Either

$x - 1 = 0 \Rightarrow x = 1$

Or

$x - 4 = 0 \Rightarrow x = 4$

5. $3x + \frac{8}{x} = 10$

$\Rightarrow 3x^2 + 8 = 10x$

$3x^2 - 10x + 8 = 0$

$3x^2 - 6x - 4x + 8 = 0$

$3x(x-2) - 2(x-2) = 0$

$(x-2)(3x-2) = 0$

Either

$x - 2 = 0 \Rightarrow x = 2$

Or

$3x - 2 = 0 \Rightarrow x = \frac{2}{3}$

Example 14.2

Find the solution set of the equation: $x^2 + 5x - 14 = 0$

Solution

$x^2 + 5x - 14 = 0$

$\Rightarrow x^2 + 7x - 2x - 14 = 0$

$x(x+7) - 2(x+7) = 0$

$\Rightarrow (x+7)(x-2) = 0$

Either

$x + 7 = 0 \Rightarrow x = -7$

Or

$x - 2 = 0 \Rightarrow x = 2$

$\therefore$ *Solution set is* $\{-7, 2\}$

Example 14.3

Factorize the following:

$(i)\ 4a^2 - 60a + 225$

$(ii)\ 25a^2 - 4$

$(iii)\ 6pq - 3rs + 3qs - 6pr$

Solution

$(i)\ 4a^2 - 60a + 225$

Equate to zero and factorize

$4a^2 - 60a + 225 = 0$

$\Rightarrow 4a^2 - 30a - 30a + 225 = 0$

$2a(2a-15) - 15(2a-15) = 0$

$\Rightarrow (2a-15)(2a-15)(2a-15) = 0$

$\Rightarrow (2a-15)^3 = 0$

Hence,

$4a^2 - 60a + 225 = (2a-15)^3$

$(ii)\ 25a^2 - 4 = 5^2a^2 - 2^2 = (5a-2)(5a+2)$

ie. by differenceof two squares

$(iii)\ 6pq - 3rs + 3qs - 6pr\ (Group\ liked\ terms)$

$= (6pq + 3qs) - (3rs + 6pr) = 3q(2p+s) - 3r(s+2p) = (3q-3r)(2p+s)$

$= 3(q-r)(2p+s)$

14.1.2 **Solution By Square Root**

This method is used to solve quadratic equations that do not have the first-degree term. That is, equations of the special form: $a^2 + c = 0 \quad a \neq 0$. It makes direct use of the square root property which states that if $A^2 = C, then\ A = \pm\sqrt{C}$. Here, the ***difference of two squares*** can also be used when the terms can be expressed as difference of two squares..

Example 14. 4

Solve the following by the square root property

(1) $2x^2 - 3 = 0$ (2) $3x^2 - 27 = 0$ (3) $(x + \frac{1}{2})^2 = \frac{5}{4}$ (4)$(a - 1)^2 = 9$

Solution

(1) $2x^2 - 3 = 0$

$2x^2 = 3$

$x^2 = \frac{3}{2}$

$x = \pm\sqrt{\frac{3}{2}}$

$\therefore$ solution set is $\left\{-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right\}$

2. $3x^2 - 27 = 0$ *Add 27 to both sides*

$\Rightarrow 3x^2 - 27 + 27 = 0 + 27$

$3x^2 = 27$ *Divide through by 3*

$$\frac{3x^2}{3} = \frac{27}{3}$$

$\Rightarrow x^2 = 9$ *Take square root*

$\Rightarrow x = \pm\sqrt{9} = \pm 3$

$\therefore$ *solution set is* $\{-3, 3\}$

3. $\left(x + \frac{1}{2}\right)^2 = \frac{5}{4}$ *Take square root*

$$\Rightarrow \left(x + \frac{1}{2}\right) = \pm\sqrt{\frac{5}{4}}$$

$$\Rightarrow x = \pm\sqrt{\frac{5}{4}} - \frac{1}{2}$$

4. $(a-1)^2 = 9$

$a - 1 = \pm\sqrt{9}$

$a - 1 = \pm 3$

$a = 1 \pm 3$

$\Rightarrow a = 1 - 3 = -2$

Or

$a = 1 + 3 = 4$

$\therefore$ *solution set is* $\{-2, 4\}$

14.1.3 **Solution By Completing The Square**

This method can be used to solve ***all*** quadratic equations. The method is based on the process of transforming the standard quadratic equation $ax^2 + bx + c = 0$ into the form $(x + A)^2 = B$ where A *and* B are constants. We then use the square root property to solve the second equation.

Steps

1. Ensure that the coefficient of x^2 is 1. If the coefficient of x^2 is not 1 then divide every term by it
2. Isolate the x^2 *and* x terms on the left-hand side
3. Pick the coefficient of x and divide it by 2
4. Square the result of step 3
5. Add the result of step 4 to both the LHS and RHS of the equation in step 2
6. Factor the LHS as the square of a binomial (two term expression) where the first term is the variable in the equation and the second term is the result of step 3
7. Apply the square root property and solve for x

Example 14.5

Solve the following by method of completing the square:

1. $x^2+6x+5=0$ 2. $2x^2+3x-2=0$

3. $x^2-3x-6=0$ 4. $2x^2+x-6=0$

Solution

(1) $x^2+6x+5=0$

Step 2

$x^2+6x=-5$

Step 3

$Coefficient\ is\ 6,\ hence\ \frac{1}{2}\times 6=3$

Step 4

$3^2=9$

Step 5

$\Rightarrow x^2+6x+9=-5+9$

$x^2+6x+9=4$

Step 6

$(x+3)^2=4$ Here we had the second term on LHS by taking half the coefficient of x

Step 7

$x+3=\pm\sqrt{4}=\pm 2$

$Either\ \ x+3=2 \qquad Or\ \ x+3=-2$

$x=2-3=-1 \qquad\qquad x=-2-3=-5$

$\therefore\ solution\ \ set\ \ is\ \ \{-5,\ -1\}$

(2) $2x^2+3x-2=0$

Step 1

We first make the coefficient of x^2 become 1 by dividing through by 2.

$\Rightarrow x^2+\frac{3}{2}x-1=0$

Step 2

$x^2+\frac{3}{2}x=1$

Step 3

$\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$

Step 4

$\left(\frac{3}{4}\right)^2=\frac{9}{16}$

Step 5

$x^2+\frac{3}{2}x+\frac{9}{16}=1+\frac{9}{16}$

$\Rightarrow x^2+\frac{3}{2}x+\frac{9}{16}=\frac{25}{16}$

Step 6

$\left(x+\frac{3}{4}\right)^2=\frac{25}{16}$ Here we had the second term on LHS by taking half the coefficient of x

$x+\frac{3}{4}=\pm\sqrt{\frac{25}{16}}$

$x+\frac{3}{4}=\pm\frac{5}{4}$

Either

$$x+\frac{3}{4}=\frac{5}{4} \qquad Or \qquad x+\frac{3}{4}=-\frac{5}{4}$$

$$\Rightarrow x=\frac{1}{2} \qquad\qquad \Rightarrow x=-2$$

Therefore solution set is $\left\{-2,\frac{1}{2}\right\}$

3. $x^2-3x-6=0$

Step 2

$x^2-3x=6$

Step 3

$\frac{1}{2}\times -3=-\frac{3}{2}$

Step 4

$\left(-\frac{3}{2}\right)^2=\frac{9}{4}$

Step 5

$x^2-3x+\frac{9}{4}=6+\frac{9}{4}$

$\Rightarrow x^2-3x+\frac{9}{4}=\frac{33}{4}$

Step 6

$\left(x-\frac{3}{2}\right)^2=\frac{33}{4}$

Step 7

$x-\frac{3}{2}=\pm\sqrt{\frac{33}{4}}$

$\Rightarrow x=\frac{3}{2}\pm\sqrt{\frac{33}{4}}=\frac{3\pm\sqrt{33}}{2}$

$\therefore solution\ \ set\ \ is\ \left\{\frac{3-\sqrt{33}}{2},\frac{3-\sqrt{33}}{2}\right\}$

4. $2x^2+x-6=0$

Step 1

$$\frac{2x^2}{2}+\frac{x}{2}-\frac{6}{2}=0$$

$$x^2+\frac{x}{2}-3=0$$

Step 2

$$x^2+\frac{x}{2}=3$$

Step 3

$$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$$

Step 4

$$\left(\frac{1}{4}\right)^2=\frac{1}{16}$$

Step 5

$$x^2+\frac{x}{2}+\frac{1}{16}=3+\frac{1}{16}$$

$$x^2+\frac{x}{2}+\frac{1}{16}=\frac{49}{16}$$

Step 6

$$\left(x+\frac{1}{4}\right)^2=\frac{49}{16}$$

Step 7

$$x+\frac{1}{4}=\pm\sqrt{\frac{49}{16}}=\pm\frac{7}{4}$$

$$x=-\frac{1}{4}\pm\frac{7}{4}$$

$$\Rightarrow x=-\frac{1}{4}-\frac{7}{4}=\frac{-8}{4}=-2$$

Or

$$x=-\frac{1}{4}+\frac{7}{4}=\frac{6}{4}=\frac{3}{2}$$

$$\therefore \text{solution set is } \left\{-2,\ \frac{3}{2}\right\}$$

14.1.4 **Solution By Quadratic Formula**

The solution to $ax^2 + bx + c = 0,\ with\ a \neq 0$ is given by the ***quadratic formula***: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a is coefficient of x^2, b is coefficient of x and c last term (usually a constant). This formula can be used to solve all quadratic equations by simple substitution.

Example 14.6

Solve the following by the quadratic formula

1. $x^2 + 2x - 15 = 0$ 2. $2x + \dfrac{3}{2} = x^2$

3. $4x^2 = 12x - 9$ 4. $x^2 + \dfrac{3}{2} = -3x$

Solution

1. $x^2 + 2x - 15 = 0$

$a = 1, b = 2\ and\ c = -15$

Using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-15)}}{2(1)} = \frac{-2 \pm \sqrt{4 + 60}}{2} = \frac{-2 \pm \sqrt{64}}{2}$$

$$= \frac{-2 \pm 8}{2}$$

Either $x = \dfrac{-2 + 8}{2} = 3$ $\quad Or\ \ x = \dfrac{-2 - 8}{2} = -5$

Hence, solution set is $\{-5, 3\}$

2. $2x + \dfrac{3}{2} = x^2$

$\Rightarrow 4x + 3 = 2x^2$

$\Rightarrow 2x^2 - 4x - 3 = 0$

$Where\ \ a = 2,\ b = -4\ \ and\ \ c = -3$

Substitute into;

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-3)}}{2(2)}=\frac{4\pm\sqrt{16+24}}{4}=\frac{4\pm\sqrt{40}}{4}$$

$$=\frac{4\pm2\sqrt{10}}{4}=\frac{2(2\pm\sqrt{10})}{4}=\frac{2\pm\sqrt{10}}{2}$$

$$\therefore\ solution\ set\ is\ \left\{\frac{2\pm\sqrt{10}}{2}\right\}$$

3. $4x^2=12x-9$

$\Rightarrow 4x^2-12x+9=0$

Where $a=4,\ b=-12$ *and* $c=9$

Substitute into;

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}=\frac{12\pm\sqrt{144-144}}{8}=\frac{12\pm\sqrt{0}}{8}=\frac{12\pm0}{8}=\frac{12}{8}=\frac{3}{2}$$

$$\therefore\ solution\ set\ is\ \left\{\frac{3}{2}\right\}$$

4. $x^2+\frac{3}{2}=-3x$

$\Rightarrow 2x^2+3=-6x$

$\Rightarrow 2x^2+6x+3=0$

Where $a=2,\ b=6$ *and* $c=3$

$$\Rightarrow x=\frac{-6\pm\sqrt{6^2-4(2)(3)}}{2(2)}=\frac{-6\pm\sqrt{36-24}}{4}=\frac{-6\pm\sqrt{12}}{4}=\frac{-6\pm2\sqrt{3}}{4}=\frac{2(-3\pm\sqrt{3})}{4}$$

$$=\frac{-3\pm\sqrt{3}}{2}$$

$$\therefore\ solution\ set\ is\ \left\{\frac{-3\pm\sqrt{3}}{2}\right\}$$

14.2 **Setting Up And Solving Word Problems**

A ***word problem*** is a mathematical problem formed by using English sentences.

Steps

1. Read the problem and know what is to be found and what is given
2. Let one of the unknown quantities be represented by a variable say x and try to represent any other unknown quantity in terms of x
3. Form an equation relating the unknown quantities to the known quantities
4. Solve the equation and write answers to all questions asked in the problem

Example 14.7

The sum of a number and its reciprocal is $\frac{13}{6}$. Find the number.

Solution

Let x be the number

Then, the sum of the number and its reciprocal is $x+\frac{1}{x}$

Implies, $x+\frac{1}{x}=\frac{13}{6}$ since reciprocal of x is $\frac{1}{x}$

$x(6x)+\frac{1}{x}(6x)=\frac{13}{6}(6x)$ i.e. multiplying both sides by $6x$

$6x^2+6=13x$ Solve quadratically

$\Rightarrow 6x^2-13x+6=0$

$(2x-3)(3x-2)=0$

$2x-3=0 \quad or \quad 3x-2=0$

$x=\frac{3}{2} \qquad x=\frac{2}{3}$

The number is $\frac{3}{2}$ *and its reciprocal is* $\frac{2}{3}$

Example 14.8

Find two numbers such that their sum is 21 and product is 104

Solution

Let the numbers be $x \text{ and } y$
From question,
$x + y = 21 ------(1)$
$xy = 104 ------(2)$
From (1),
$x = 21 - y ------(3)$
Put (3) into (2)
$\Rightarrow (21 - y)y = 104$
$21y - y^2 = 104$
$y^2 - 21y = -104$
$y^2 - 21y + 104 = 0$
$y^2 - 13y - 8y + 104 = 0$
$y(y - 13) - 8(y - 13) = 0$
$(y - 8)(y - 13) = 0$
$\therefore y = 8 \text{ or } 13$
Put $y = 8$ into (3)
$\Rightarrow x = 21 - 8 = 13$
Again, put $y = 13$ into (3)
$x = 21 - 13 = 8$
Hence, the numbers are 8 *and* 13

Example 14.9

The sum of two consecutive odd numbers is 12. *Find the numbers*.

Solution

Let the first odd number be x
Then the second odd number will be x+2 (Since every two consecutive odd numbers differ from each other by 2)
From question,

$x+(x+2)=12$
$2x+2=12$
$2x=10$
$\Rightarrow x=5$
Therefore the first odd number is 5 and the second odd number is 7

14.2.1 **Finding Equations When Given The Roots**

In finding the equation of any quadratic equation when given its roots, we use the relation below:
$x^2-(Sum\ of\ roots)x+(Product\ of\ roots)=0$
Where x is the variable in the given quadratic equation

Example 14.10

Obtain the equation whose roots are $1\frac{2}{3}\ and\ -1\frac{2}{3}$

Solution

$Sum\ of\ roots=\frac{5}{3}+-\frac{5}{3}=0$
$Product\ of\ roots=\frac{5}{3}\times-\frac{5}{3}=-\frac{25}{9}$
Using $x^2-(Sum\ of\ roots)x+(Product\ of\ roots)=0$,
$\Rightarrow x^2-(0)x+\left(-\frac{25}{9}\right)=0$
$x^2-\frac{25}{9}=0$
$\Rightarrow 9x^2-25=0\ is\ the\ required\ equation$

Example 14.11

Given that $2\ and\ -3$ are the roots of the equation $ax^2+bx+c=0,\ find\ a,b\ and\ c$

Solution

Sum of roots is $2+(-3)=-1$
Product of roots is $2(-3)=-6$
Substitute into $x^2-(Sum\ of\ roots)x+(\text{Pr}oduct\ of\ roots)=0$
Implies, $x^2-(-1)x+(-6)=0$
$x^2+x-6=0$
Comparing with $ax^2+bx+c=0$
We get $a=1, b=1\ and\ c=-6$

EXERCISE

QUE. A

Using the methods (i) factoring (ii) Completing the squares (iii) formula, find the truth set of the following equations:

(1) $5x^2+13x-6=0$
(2) $3(2x+9)=x^2$
(3) $x^2-3x-10=0$
(4) $3-2x-x^2=0$
(5) $4y^2+5y-21=0$
(6) $4x^2-16x+15=0$
(7) $2x^2-7x-15=0$
(8) $2x^2+5x-12=0$
(9) $2x^2+7x+2=0$

QUE. B

The equation $px^2+16x+4=0$ is satisfied by $x=-\frac{2}{3}$, find
$(i)\ p$ (ii) the other values of x which makes the equation true

QUE. C

Solve the following equations by the square root property

(1) $3x^2 - 5 = 0$ (2) $2x^2 + 8 = 0$ (3) $\left(x + \frac{1}{3}\right)^2 = \frac{2}{9}$

QUE. D

The sum of two numbers is 23 and their product is 132. Find the numbers

The sum of a number and its reciprocal is $\frac{10}{3}$. Find the number.

Find two consecutive positive even integers whose product is 168 (***Hint*** : let $x =$ the first even integer , then $x + 2 =$ second even integer since the difference between two consecutive even integers is 2 hence, $x(x+2) = 168$)

QUE. E

Factorize $7x^2 + 4x + 3 + mx + 3m$

QUE. F

Complete the square in the following expressions.

1. $x^2 + 5x + 3$ 2. $x^2 - 7x + 3$ 3. $2x^2 + 6x + 4$ 4. $3x^2 + 9x + 7$

5. $4x^2 + 10x + 2$ 6. $x^2 + 3x + 4$ 7. $2x^2 + 8x + 5$ 8. $3x^2 + 4x + 9$

9. $3x^2 + 7x$ 10. $3x^2 + 4x + 9$

QUE. G

Solve the following equations:

1. $x^2 + 5x + 3 = 0$ 2. $x^2 - 6x + 13 = 0$ 3. $x^2 + 3x + 2 = 0$

4. $3x^2 + 6x - 7 = 0$ 5. $4x^2 + 8x + 3 = 0$ 6. $x^2 + 6x + 4 = 0$

7. $3x^2 + 8x + 5 = 0$ 8. $2x^2 + 5x + 4 = 0$ 9. $5x^2 + 12x + 6 = 0$

10. $3x^2 + 7x + 4 = 0$

QUE. H

Using the quadratic formula, solve the following equations leaving your answer in surd form where necessary.

1. $3x^2 + x - 2 = 0$ 2. $3x^2 + 7x + 1 = 0$ 3. $16x^2 + 8x + 1 = 0$

4. $4x^2 + 3x + 6 = 0$ 5. $1 + 4x - 32x^2 = 0$ 6. $9x^2 - 6x + 1 = 0$

7. $2x^2 - 6x + 30 = 0$ 8. $17x^2 + 8x + 6 = 0$ 9. $84x^2 + 5x - 1 = 0$

10. $8x^2 + 3x + 1 = 0$

CHAPTER 15
THE NUMERATION SYSTEM
(NUMBER BASES)

A number written in base ten (10) is called ***decimal numeral.*** Hence, the ***decimal system*** is the numeration system in base ten. Numbers can be converted from base ten to other bases and vise versa. The number of ***digits*** for every base is the base number. Thus, ***base*** 6 has six digits i.e. $0,1,2,3,4,5$ and ***base*** 4 has four digits.
i.e. $0,1,2,3$.
The ***base*** of a number is written as a ***subscript*** after the number. Thus, ***seven*** in base ***five*** is written as 7_5.

15.1 **Converting From Other Bases To Number Base Ten**

Example 15.1

Convert 1203_{five} to a base ten numeral

Solution

15.1.1 **Approach 1 (Expanded form Approach)**

Steps

1. Number the digits from the last digit on the RHS starting from zero
2. Multiply each digit by the base raised to the power its corresponding
numbering in step 1
3. Sum the terms in step 2 for the answer in base ten

$$\begin{aligned}1203_{five} &= (1\times 5^3)+(2\times 5^2)+(0\times 5^1)+(3\times 5^0)\\ &= 125+50+0+3\\ &= 178_{ten}\end{aligned}$$

15.1.2 **Approach 2 (Repeated or Continued Multiplication Approach)**

Each digit starting from the LHS is multiplied by the base and the preceding digit to it is added to the result of the product and the process repeated until the last digit which is only added to the result of multiplying the previous digit before it.

i.e. $1\times 5=5+2=7\times 5=35+0=35\times 5=175+3=178_{10}$

Example 15.2

Convert 10101_2 to decimal numeral

Solution

Approach 1

$$\begin{aligned}10101_2 &= (1\times 2^4)+(0\times 2^3)+(1\times 2^2)+(0\times 2^1)+(1\times 2^0)\\ &=16+0+4+0+1\\ &=21_{10}=21\end{aligned}$$

Approach 2

$1\times 2=2+0=2\times 2=4+1=5\times 2=10+0=10\times 2=20+1=21$

NB

A number in base ten can be written without attaching the base as in the result of example 15.2 above.

Example 15.3

Convert 10011_2 to a numeral in base 10

Solution

$$10011_2 = (1\times 2^5) + (0\times 2^4) + (0\times 2^3) + (1\times 2^2) + (1\times 2^1)$$
$$= 32 + 0 + 0 + 4 + 2 = 38$$

Example 15.4

Find x if $34_x = 54_{six}$

Solution

Since both bases are not in base ten, we covert each base to base ten and solve resulting equation for x

Thus,

For LHS,

$$34_x = (3\times x^1) + (4\times x^0)$$
$$= 3x + 4$$

For RHS,

$$54_{six} = (5\times 6^1) + (4\times 6^0)$$
$$= 30 + 4 = 34$$

Equating results for LHS to RHS gives,

$$3x + 4 = 34$$
$$\Rightarrow 3x = 30$$
$$\therefore x = 10$$

Example 15.5

If $105_{12} = 302_x$, find x

Solution

$$105_{12} = (1\times 12^2) + (0\times 12^1) + (5\times 12^0)$$
$$= 149$$
$$302_x = (3\times x^2) + (0\times x^1) + (2\times x^0)$$
$$= 3x^2 + 2$$

$\Rightarrow 149 = 3x^2 + 2$

$149 - 2 = 3x^2$

$\frac{147}{3} = \frac{3x^2}{3}$

$49 = x^2$

$\Rightarrow \pm 7 = x$

Hence, $x = 7$

NB

We chose $+7$ because there is no negative base.

Example 15.6

Solve for x if $133_x = 43_7$

Solution

$133_x = x^2 + 3x + 3$

$43_7 = 31$

$\Rightarrow x^2 + 3x + 3 = 31$

$x^2 + 3x - 28 = 0$

Solve quadratically

$\Rightarrow x^2 + 7x - 4x - 28 = 0$

$x(x + 7) - 4(x + 7) = 0$

$(x + 7)(x - 4) = 0$

$x + 7 = 0 \quad or \quad x - 4 = 0$

$\Rightarrow x = -7 \quad\quad \Rightarrow x = 4$

Hence, $x = 4$ since no negative base

Example 15.7

If $233_x = 125_{seven}$, solve for x

Solution

$233_x = 2x^2 + 3x + 3$

and

$125_{seven} = 68$

$\Rightarrow 2x^2 + 3x + 3 = 68$

$2x^2 + 3x - 65 = 0$

Solving quadratically gives,

$x = 5,\ x = -\frac{13}{2}$

Therefore, $x = 5$

Example 15.8

If 136 to base n is equal to 76, find n.

Solution

$136 \textit{ to base n is equal to } 76 \textit{ means} 136_n = 76$

$\Rightarrow 136_n = 76$

$\Rightarrow (1 \times n^2) + (3 \times n) + (6 \times n^0) = 76$

$\Rightarrow n^2 + 3n + 6 = 76$

$\Rightarrow n^2 + 3n + 6 - 76 = 0$

$\Rightarrow n^2 + 3n - 70 = 0$

Solving quadratically gives;

$(n+10)(n-7) = 0 \Rightarrow n = 7$ (*Since no negative base*)

Example 15.9

Convert the following to base ten

(1) $ET6_{12}$ (2) 110.11_2

Solution

$$(1)\ ET6_{12} = (E\times 12^2)+(T\times 12^1)+(6\times 12^0)$$
$$= (11\times 144)+(10\times 12)+(6\times 1)$$
$$= 1584 \quad + \quad 120 \quad + \quad 6$$
$$= 1710$$

NB

'E' stands for the number 11 and **'T'** stands for the number 10.

$$(2)\ 110.11_2 = (1\times 2^2)+(1\times 2^1)+(0\times 2^0)+(1\times 2^{-1})+(1\times 2^{-2})$$
$$= 4 \quad + \quad 2 \quad + \quad 0 \quad + \quad \frac{1}{2} \quad + \quad \frac{1}{4}$$
$$= 6\tfrac{3}{4} = 6.75$$

NB

All digits after the decimal point are raised to the power of negatives starting with (-1) from the immediate digit after the decimal point.

Example 15.10

If $32_5 = 101_2 + 12_x$, *find the value of* x.

Solution

$32_5 = 101_2 + 12_x$

Convert both sides to base 10

$\Rightarrow (3\times 5^1)+(2\times 5^0) = (1\times 2^2)+(0\times 2^1)+(1\times 2^0)+(1\times x^1)+(2\times x^0)$

$\Rightarrow 15+2 = 4+0+1+x+2$

$17 = 7+x \Rightarrow 17-7 = x \Rightarrow x = 10$

15.2 **Converting From Decimal Numeral To Other Bases**

In converting from ***base ten*** to ***other bases***, the decimal numeral is divided repeatedly by the given base until you get a number less than the base and the remainders noted at each stage in the remainder column. This number is then placed last at the bottom of the remainder column. The answer is taken upwards from the bottom of the remainders.

Example 15.11

Express 25_{ten} as a number in base two

Solution

2	25	Remainder
2	12	1
2	6	0
2	3	0
2	1	1
	0	1

Therefore, $25_{ten} = 11001_2$

Example 15.12

If $133_x = 73_{ten}$, find x

Solution

Changing 133_x to base ten

$\Rightarrow 133_x = x^2 + 3x + 3$

$\Rightarrow x^2 + 3x + 3 = 73$

$x^2 + 3x - 70 = 0$

Solving quadratically,

$x = -10, \ or \ \ x = 7$

Hence, $x = 7$

Example 15.13

If $123_x = 38_{ten}$, find the value of x

Solution

$$123_x = x^2 + 2x + 3$$
$$\Rightarrow x^2 + 2x + 3 = 38$$
$$x^2 + 2x - 35 = 0$$

Solving quadratically,

$x = -7 \quad or \quad x = 5$

Hence, $x = 5$

Example 15.14

Arrange the following in ascending order of magnitude:
$21_{eight}, 25_{seven}, 30_{six}$

Solution

First change all the bases to base ten.
That is;

$$21_{eight} = 17_{10}$$
$$25_{seven} = 19_{10}$$
$$30_{six} = 18_{10}$$

Arranging the answers in ascending order: $17, 18, 19$
Therefore, in ascending order we have $21_8, 30_6 \text{ and } 25_7$

NB

Ascending order means ***smallest first***
Descending order also means ***biggest first***

15.3 **Conversion Between Non - Decimal Bases**

In converting a non-decimal to another non-decimal, we first convert it into base ten before converting the result into the given non-decimal base.

Example 15.15

Express 441_5 as a number in base four

Solution

First change 441_5 to base ten

$441_5 = (4\times 5^2)+(4\times 5^1)+(1\times 5^0)=121$

4	121	Remainder
4	30	1
4	7	2
4	1	3
	0	1

Therefore, $441_5 = 1321_4$

15.4 **Addition Of Number Bases**

We can only add numbers in the ***same*** base. Addition in any other number base is exactly the same as for decimal numerals except that in base ten, we borrow ten (10) whiles in other bases, we borrow the given base we are working in. For example, in base five (5), we borrow 5 and not 10.

Example 15.16

Find the sum of the following numbers: $322_{five}, 114_{five}$ *and* 324_{five}

Solution

$$\begin{array}{r} 322_5 \\ 114_5 \\ +324_5 \\ \hline 1320_5 \\ \hline \end{array}$$

Steps

Starting from the right gives: $2+4+4=10=20_5$. We then write 0 and carry the 2 to the next column. The next column gives: $2+1+2+2=7=12_5$ here we write 2 and carry the 1 to the next column. The final column gives: $3+1+3+1=8=13_5$ we then write 13

Example 15.17

What is the sum of the following numbers: $256_{eight}, 342_{eight}$ *and* 421_{eight}

Solution

$$\begin{array}{r} 256_8 \\ 342_8 \\ +421_8 \\ \hline 1241_8 \\ \hline \end{array}$$

15.5 **Subtraction Of Number Bases**

Subtraction is done in a similar way as in addition except with the sign convention.

Example 15.18

Evaluate $523_{seven} - 65_{seven}$

Solution

$$\begin{array}{r} 523_7 \\ -65_7 \\ \hline 425_7 \\ \hline \end{array}$$

Steps

Start from the right borrowing 7 from the second column.
i.e. $7+3-5=5$
Now, borrow 7 from the third column. i.e. $7+1-6=2$
(Since second column is left with 1). The last column becomes $4-0=4$

15.6 **Multiplication Of Number Bases**

Multiplication in any other base is done in the same way as in base ten.

Example 15.19

Simplify $203_{five} \times 42_{five}$, giving your answer in base 5

Solution

$$\begin{array}{r} 203_5 \\ \times 42_5 \\ \hline 411 \\ 1322 \\ \hline 14131_5 \\ \hline \end{array}$$

15.7 **Tabular Computation Of Number Bases**

Example 15.20

(a) Copy and complete the table below for addition base eight

⊕	1	3	5	7
1	2	4	6	-
3	-	6	10	12
5	6	-	-	-
7	-	12	14	16

(b) Use your table to solve the following: $(i)\ 3 \oplus x = 12 \quad (ii) n \oplus n = 6$

Solution

⊕	1	3	5	7
1	2	4	6	**10**
3	**4**	6	10	12
5	6	**10**	**12**	**14**
7	**10**	12	14	16

From the table above;

$(i)\ 3 + x = 12$

When $x = 7$

$\Rightarrow 3 + 7 = 10 = 12_8$

Hence, $x = 7$

$(ii)\ n + n = 6$

When $n = 3$

$\Rightarrow 3 + 3 = 6$

Hence, $n = 3$

EXERCISE

QUE. A

Evaluate the following:

(1) $141_{six} + 233_{six} - 102_{six}$

(2) $(111_{two} + 101_{two})(111_{two} - 101_{two})$

(3) $2102_{three} \times 122_{three}$

(4) $7T2_{12} - 29E_{12}$

QUE. B

Find the letter in the following:

(1) $243_x = 73$
(2) $223_x = 87_{ten}$
(3) $365_{seven} + 43_x = 217_{ten}$
(4) $32_5 = 101_2 + 12_x$
(5) $211_c = 320_{four}$

QUE. C

Convert the following to base ten numeral

(1) 1010_2 (2) 143_5 (3) 746_8 (4) 1231_{four} (5) 11101_{two}

QUE. D

Convert:

(1) 334_{five} to a number in base four (2) 38_{ten} to a number base five

CHAPTER 16

SEQUENCES AND SERIES

16.1 **Sequences**

16.1.1 **Definition**

A ***Sequence*** is a list of terms (usually numbers) written in a defined order in which there is a simple rule or formulae by which the terms are determined. Each number in a sequence is called a***term.***
The leading term of any sequence is called the ***first term*** denoted by ***a.***
We call $\mathbf{U_n}$ the ***nth term*** or the ***general term*** of the sequence.

16.2 **Types Of Sequences**

A sequence may be said to be ***finite*** or ***infinite.***

16.2.1 **Finite Sequence**

A ***finite sequence*** is one in which the first and last terms are known or one in which the terms of the sequence terminates.
E.gs $\{1,2,3,4,5,6,7,8\}$ *and* $\{2,4,6,8,10\}$

16.2.2 **Infinite Sequence**

An ***Infinite sequence*** is a sequence in which the last term is not known or one in which the terms do not terminate.
Egs. $\{4,2,0,-2,---\}$ *and* $\{1,3,5,7,---\}$

16.3 **Finding A Formula For The nth Term Or General Term**

We often know the terms of a sequence and want to write a formula that will produce those terms. To write a formula for the nth term of a sequence, examine the terms and look for a pattern. Each term is a function of the term number. The first term corresponds to n = 1, the second term corresponds to n = 2 and so on.

NB

Here, when a particular sequence given is identified as either arithmetic or geometric, we use the relation for finding the nth term of that sequence to compute the formula.

Example 16.1

Write the general term for the infinite sequence:
3, 5, 7, 9, 11, …

Solution

We note that even numbers are multiples of 2 and can be represented as 2n. Because the above sequence is a set of odd numbers and each odd number is 1 more than an even number, a formula for the nth term will be:

$U_n = 2n + 1$

Example 16.2

Write the general term for the infinite sequence:

$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$

Solution

To obtain the alternating signs, we use powers of -1. Because any even power of -1 is positive and any odd power of -1 is negative, we use $(-1)^{n+1}$. The denominators are the squares of the positive integers. So the nth term of this infinite sequence is given by the formula:

$$U_n = \frac{(-1)^{n+1}}{n^2}$$

Example 16.3

Write the general term for the sequence:
$6, 4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \ldots$

Solution

We note that each term is obtained by multiplying the preceding term by $\frac{2}{3}$.

Hence, the nth term is $U_n = 6\left(\frac{2}{3}\right)^{n-1}$

16.4 **Series**

The indicated sum of the terms of a sequence is called a ***series***.

16.4.1 **Summation Notation**

To describe the sum of terms of a sequence, we use summation notation. The Greek letter $\sum$ (sigma) is used to indicate sums. For instance, the sum of the first five terms of the sequence $U_n = n^2$ is written as:

$\sum_{n=1}^{5} n^2$.

We can read this notation as "the sum of n^2 for n between 1 and 5 inclusive.

To find the sum, we let n take the values 1, 2, 3, 4, 5 in the expression n^2 .

Thus, we have the sum:

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

In this context, the n is the index of summation. Other letters may also be used.

For instance, the expressions:

$$\sum_{n=1}^{5} n^2,\ \sum_{j=1}^{5} j^2 \quad and \quad \sum_{i=1}^{5} i^2$$

all have the same value.

16.4.2 **Evaluating A Sum In Summation Notation**

Example 16.4

Find the value of the expression: $\sum_{i=1}^{3}(-1)^i(2i+1)$

Solution

$$\sum_{i=1}^{3}(-1)^i(2i+1) = (-1)^1[2(1)+1]+(-1)^2[2(2)+1]+(-1)^3[2(3)+1]$$

$$= -3+5-7 = -5$$

16.4.3 **Converting A Series To A Summation Notation**

Example 16.5

Write the series in summation notation: $2+4+6+8+10+12+14$

Solution

The general term for the sequence of positive even integes is 2n. If we let n take the values from 1 through to 7, then 2n ranges from 2 through 14. So

$$2+4+6+8+10+12+14 = \sum_{n=1}^{7} 2n$$

Example 16.6

Write the series $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots+\frac{1}{50}$ in summation notation.

Solution

For this series we take the values of n from 2 through 50. We recall that the expression $(-1)^n$ produces alternating signs. The series is therefore written as:

$$\sum_{n=2}^{50}\frac{(-1)^n}{n}$$

16.5 **Arithmetic Sequences and Series**

A sequence in which each term after the first is obtained by adding a fixed amount to the previous term is called an ***arithmetic sequence or progression***.
The fixed amount is called the ***common difference*** and is denoted by the letter ***d***.
If 'a' is the first term, then the second term is a + d and the third term will be a + 2d and so on.
E.g. The set $\{1,3,5,7,9,11\}$ is an AP with a common difference of 2.

16.5.1 **Fining A Formula For The nth Term Of An Arithmetic Sequence**

In finding a formula for the nth term, U_n of an arithmetic sequence with first term 'a' and common difference 'd', we use the relation: $U_n = a + (n-1)d$

Example 16.7

Write a formula for the nth term of the arithmetic sequence:
$5, 9, 13, 17, 21, \ldots$

Solution

NB

Each term of the sequence after the first is 4 more than the previous term. Hence, the common difference is d = 4 and the first term is 5.

Thus, substituting gives;

$U_n = a + (n-1)d = 5 + (n-1)4 = 5 + 4n - 4 = 4n + 1$

$\therefore U_n = 4n + 1$

Example 16.8

Write a formula for the nth term of the arithmetic sequence:
$4, 1, -2, -5, -8, \ldots$

Solution

$d = 1 - 4 = -2 - 1 = -5 - (-2) = -8 - (-5) = -3$

$and \quad a = 4$

$\Rightarrow U_n = a + (n-1)d = 4 + (n-1)(-3) = 4 - 3n + 3 = 7 - 3n$

$\therefore U_n = 7 - 3n$

16.5.2 **Finding The Number Of Terms Of An AP**

Again, to obtain the number of terms of an AP, we use:
$U_n = a + (n-1)d = l$ Where
$a = \textit{first term}, d = \textit{common difference} \ \ and \ \ l = \textit{last term}$

Example 16.9

Find the $12th$ term of the AP of the form: $7, 6\frac{1}{4}, 5\frac{1}{2}, \ldots$

Solution

Given $7, 6\frac{1}{4}, 5\frac{1}{2}, \ldots \qquad a = 7, n = 12, d = 6\frac{1}{4} - 7 = \frac{25}{4} - 7 = -\frac{3}{4}$

From $U_n = a + (n-1)d$

$$\Rightarrow U_{12} = 7 + (12-1)\left(-\frac{3}{4}\right) = 7 + 11\left(-\frac{3}{4}\right) = 7 - \frac{33}{4} = -\frac{5}{4}$$

Example 16.10

Find the $9th$ term of the AP: $2, 4, 6, ...$

Solution

Given $2, 4, 6, ...$ $\quad a = 2,\ n = 9,\ d = 4 - 2 = 2$

Using $U_n = a + (n-1)d$

$$\Rightarrow U_9 = 2 + (9-1)2 = 2 + 8(2) = 2 + 16 = 18$$

Example 16.11

The nth term of a sequence is $5 + \frac{2}{3^{n-2}}$ $for\ n \geq 1$.

What is the sum of the fourth and fifth terms? Leave your answer in the form $\frac{x}{y}$ $where\ x\ and\ y\ are\ \text{int}\,egers$.

Solution

$$U_n = 5 + \frac{2}{3^{n-2}}$$

$$\Rightarrow U_4 = 5 + \frac{2}{3^{4-2}} = 5 + \frac{2}{9} = \frac{47}{9}$$

$$U_5 = 5 + \frac{2}{3^{5-2}} = 5 + \frac{2}{27} = \frac{137}{27}$$

$$Sum: U_4 + U_5 = \frac{47}{9} + \frac{137}{27} = \frac{141 + 137}{27} = \frac{278}{27}$$

Example 16.12

Find the third, tenth and twenty-first terms of an AP whose first term is 6 and common difference 5

Solution

$a = 6, d = 5$

$When\ \ n = 3$

$U_3 = 6 + (3-1)5 = 6 + 10 = 16$

Therefore, the third term is 16

When $n = 10$

$U_{10} = 6 + (10-1)5 = 6 + 45 = 51$

Therefore, the tenth term is 51

When $n = 21$

$\Rightarrow U_{21} = 6 + (21-1)5 = 6 + 100 = 106$

Therefore, the twenty-first term is 106

Example 16.13

Find the twelfth term of the arithmetic sequence whose first term is 2 and whose fifth term is 14.

Solution

From question, $n = 5,\ a = 2, U_5 = 14$

$\Rightarrow U_5 = 14 = 2 + (5-1)d$

$\Rightarrow 14 = 2 + 4d$

$14 - 2 = 4d$

$\frac{12}{4} = \frac{4d}{4}$

$\Rightarrow d = 3$

$Now,\ \ for\ \ 12th\ term,$

$U_{12} = 2 + (12-1)3 = 2 + 11(3) = 2 + 33 = 35$

Example 16.114

If the first and tenth terms of an AP are $3\ and\ 30$ respectively, find the fiftieth term of the sequence.

Solution

From question, $a = 3, U_{10} = 30$

$$\begin{aligned}
&But\ U_{10} = 3 + (10 - 1)d \\
&\qquad\quad = 3 + 9d \\
&\Rightarrow 30 = 3 + 9d \\
&30 - 3 = 9d \\
&\frac{27}{9} = \frac{9d}{9} \\
&\therefore d = 3 \\
&\Rightarrow U_{50} = 3 + (50 - 1)3 = 3 + 49(3) = 3 + 147 = 150
\end{aligned}$$

Hence, the fiftieth term is 150

16.5.3 **Finding The Sum Of An AP**

Generally, the sum of an AP is given by:

$S_n = \frac{n}{2}[2a + (n-1)d]$ where a is known

Or

$S_n = \frac{n}{2}(a + l)$ Where $a\ and\ l$ are known

Example 16.15

Find the sum of the first seven terms of the AP $3, 5, 7, 9, ...$

Solution

$$a = 3, n = 7, d = 5 - 3 = 7 - 5 = 2$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_7 = \frac{7}{2}[2(3) + (7-1)2] = \frac{7}{2}[6 + 12] = \frac{7}{2}[18] = 63$$

Therefore, the sum of the first seven terms of the AP is 63

Example 16.16

Find the sum of the first twenty-six terms of an AP if the first term is -7 and common difference 3.

Solution

$a=-7,\ d=3,\ n=26$

$\Rightarrow S_{26}=\dfrac{26}{2}\left[2(-7)+(26-1)3\right]=13\left[-14+75\right]=13(61)=793$

Example 16.17

Find the sum of the positive integers from 1 to 100 inclusive.

Solution

From question, $l=100,\ a=1\ \ and\ \ n=100$

$\Rightarrow Using\ \ S_n=\dfrac{n}{2}(a+l)$

$We\ \ have\ \ S_{100}=\dfrac{100}{2}(1+100)=50(101)=5050$

Example 16.18

Find the sum of the series: $12+16+20+...+84$

Solution

From question, $a=12,\ l=84,\ d=16-12=4$
First find the number of terms, n of the sequence
by using: $l=a+(n-1)d$

$\Rightarrow 84 = 12 + (n-1)4$

$\quad = 12 + 4n - 4$

$\Rightarrow 84 = 8 + 4n$

$84 - 8 = 4n$

$\frac{76}{4} = \frac{4n}{4} \Rightarrow n = 19$

Now we find the sum of these 19 *terms*

$\Rightarrow S_n = \frac{n}{2}(a+l) = S_{19} = \frac{19}{2}(12+84) = 912$

Example 16.19

Find the sum of the first 13 terms of an AP whose $8th$ term is 18 and $11th$ term is 24

Solution

Using $U_n = a + (n-1)d$

$\Rightarrow U_8 = a + 7d$

$\Rightarrow a + 7d = 18 ----(1)$ since $8th$ *term is* 18

$U_{11} = a + 10d$

$\Rightarrow a + 10d = 24 ----(2)$ Since 11th term is 24

$(2)-(1)$

$\Rightarrow \frac{3d}{3} = \frac{6}{3}$

$\therefore d = 2$

Put $d = 2$ into eqn (1)

$\Rightarrow a + 7(2) = 18$

$a + 14 = 18$

$a = 4$

$\therefore S_{13} = \frac{13}{2}[2(4) + 12(2)] = 208$

Example 16.20

In a sequence of numbers $a_1, a_2, a_3, a_4, a_5,$ each number is twice the preceding number. If $a_5 - a_1 = 20,$ find the number a_1.

Solution

Since each number is twice the preceding number,

$$\Rightarrow a_2 = 2a_1$$
$$a_3 = 2a_2$$
$$a_4 = 2a_3$$
$$a_5 = 2a_4$$

For $a_5 - a_1 = 20$

Substituting,

Implies, $2a_4 - a_1 = 20$

$$2(2a_3) - a_1 = 20$$
$$4a_3 - a_1 = 20$$
$$4(2a_2) - a_1 = 20$$
$$8(2a_1) - a_1 = 20$$
$$16a_1 - a_1 = 20$$
$$\Rightarrow \frac{15a_1}{15} = \frac{20}{15}$$
$$a_1 = \frac{4}{3}$$

16.6 **Geometric Sequences And Series**

A sequence in which each term after the first is obtained by multiplying the preceding term by a constant is called a ***geometric sequence***.

It is also called an ***exponential sequence or geometric progression (GP).***

The constant is denoted by the letter 'r' and called the ***common ratio***.

16.6.1 **Finding A Formula For The nth Term Of A Geometric Sequence**

In finding a formula for the nth term of a GP, we use the relation: $U_n = ar^{n-1}$

Example 16.21

Find a formula for the nth term of the geometric sequence:
$3, 6, 12, 24, 48, \ldots$

Solution

From question $a = 3, r = \frac{6}{3} = 2$

$\Rightarrow U_n = 3(2)^{n-1}$

Example 16.22

Write a formula for the nth term of the geometric sequence:
$6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$

Solution

$a = 6, \ r = \frac{2}{6} = \frac{1}{3}$

$\Rightarrow U_n = 6\left(\frac{1}{3}\right)^{n-1}$

Example 16.23

Find a formula for the nth term of the geometric sequence:
$2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots$

Solution

$a = 2,\ r = -\frac{1}{2}$

$\Rightarrow U_n = 2\left(-\frac{1}{2}\right)^{n-1}$

16.6.2 **Finding The nth Term Of A GP**

Generally, we find the number of terms of a GP by using the relation: $U_n = ar^{n-1}$
Where n is the nth term and a is the first term

Example 16.24

Find the eighth term of the GP $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, ---$

Solution

Given $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, ---$ $\quad n = 8,\ a = \frac{1}{64},\ r = \frac{1}{32} \div \frac{1}{64} = \frac{1}{32} \times 64 = 2$

$U_n = ar^{n-1}$

$\Rightarrow U_8 = \frac{1}{64}(2)^{8-1} = \frac{1}{64}(2)^7 = \frac{1}{64}(128) = 2$

Therefore, the eighth term is 2

Example 16.25

Find the sixth term of the GP $324,\ 108,\ 36,\ 12, ---$

Solution

From $324,\ 108,\ 36,\ 12, ---$ $\quad n = 6,\ a = 324,\ r = \frac{108}{324} = \frac{1}{3}$

$\Rightarrow U_6 = 324\left(\frac{1}{3}\right)^{6-1} = 324\left(\frac{1}{3}\right)^5$

Example 16.26

Find the nth term of the exponential sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ---$

Solution

From $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ---$ $\quad a = \frac{1}{2},\ r = \frac{1}{4} \times 2 = \frac{1}{2}$

$\Rightarrow U_n = ar^{n-1} = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2}\left(\frac{1}{2}\right)^{n} \div \frac{1}{2} = \left(\frac{1}{2}\right)^{n}$

Example 16.27

The third and sixth terms of a GP are respectively $\frac{1}{4}$ *and* $\frac{1}{32}$. Find the first term, the common ratio and the eighth term in the sequence.

Solution

Using $U_n = ar^{n-1}$

But $U_3 = \frac{1}{4}$

$\Rightarrow U_3 = ar^2$

$\Rightarrow ar^2 = \frac{1}{4} ----(1)$

Again, $U_6 = \frac{1}{32}$

$\Rightarrow ar^5 = \frac{1}{32} ----(2)$

$(2) \div (1)$

$\frac{ar^5}{ar^2} = \frac{1}{32} \times 4$

$r^3 = \frac{1}{8}$

$\Rightarrow r = \frac{1}{2}$

Put $r = \frac{1}{2}$ into (1)

$a\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\frac{a}{4} = \frac{1}{4}$

$\Rightarrow a = 1$

Hence, $U_8 = 1\left(\frac{1}{2}\right)^{8-1} = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$

Example 16.28

Three consecutive terms of a GP have a product 343 and sum $\frac{49}{2}$, find the numbers.

Solution

Let the three terms be $\frac{a}{r}, a, ar$

Hence, product of three terms is 343, $\Rightarrow \frac{a}{r} \times a \times ar = 343$

$\Rightarrow a^3 = 343 ----(1)$

$\sqrt[3]{a} = \sqrt[3]{343}$

$\therefore a = 7$

Sum of three terms is $\frac{49}{2}, \Rightarrow \frac{a}{r} + a + ar = \frac{49}{2}$

$\Rightarrow \frac{a + ar + ar^2}{r} = \frac{49}{2}$

But $a = 7$

$\Rightarrow \frac{7 + 7r + 7r^2}{r} = \frac{49}{2}$

$49r = 14 + 14r + 14r^2$

$14r^2 + 14 - 35r = 0$

$$\Rightarrow 2r^2 - 5r + 2 = 0$$
$$2r^2 - r - 4r + 2 = 0$$
$$(2r-1)(r-2) = 0$$

Either $2r - 1 = 0 \Rightarrow r = \frac{1}{2}$

Or $r - 2 = 0 \Rightarrow r = 2$

Example 16.29

Find the first term of a geometric sequence whose fourth term is 8 and whose common ratio is $\frac{1}{2}$.

Solution

$$U_4 = 8,\ r = \frac{1}{2}$$
$$\Rightarrow U_4 = 8 = a\left(\frac{1}{2}\right)^{4-1}$$
$$\Rightarrow 8 = a\left(\frac{1}{2}\right)^3 = a\left(\frac{1}{8}\right)$$
$$\Rightarrow 8 \times 8 = a$$
$$\therefore a = 64$$

Hence the first term is 64.

16.6.3 **Finding The Sum Of n Terms Of A GP**

Generally, the sum of the first n terms of a GP is found using:

$$S_n = \frac{a(1-r^n)}{1-r} ----(1) \qquad Where\ r < 1$$

and

$$S_n = \frac{a(r^n-1)}{r-1} ----(2) \qquad where\ r > 1$$

Example 16.30

Find the sum of the first five terms of the exponential sequence $\frac{1}{2}, 1, 2, ---$

Solution

Given $\frac{1}{2}, 1, 2, ---$

$a = \frac{1}{2},\ r = 1 \div \frac{1}{2} = 2, n = 5$

Since $r > 1$, we use $S_n = \frac{a(r^n - 1)}{r - 1}$,

$$\Rightarrow S_5 = \frac{\frac{1}{2}\left(2^5 - 1\right)}{2 - 1} = 15.5$$

Example 16.31

Find the sum of the first nine terms of the GP $1, \frac{1}{2}, \frac{1}{4}, ---$

Solution

Given $1, \frac{1}{2}, \frac{1}{4}, ---$

$a = 1,\ r = \frac{1}{2}, n = 9$

Since $r < 1$, we use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$S_9 = \frac{1\left[1 - \left(\frac{1}{2}\right)^9\right]}{1 - \frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^9\right]$$

Example 16.32

If the third and sixth terms of a GP are respectively $10 \quad and \quad 80$, find the sum of the first six terms.

Solution

From $U_n = ar^{n-1}$ and since $U_3 = 10, \Rightarrow ar^2 = 10----(1)$
Also, $U_6 = 80 \Rightarrow ar^5 = 80----(2)$

$(2) \div (1)$

$$\frac{ar^5}{ar^2} = \frac{80}{10}$$

$\Rightarrow r^3 = 8 \qquad \therefore r = 2$

Put $r = 2$ into (1)

$$\Rightarrow a = \frac{10}{4} = 2.5$$

$S_6 = 2.5(2^6 - 1) = 157.5$

EXERCISE

QUE. A

The first term of a linear sequence is 19 and the common difference is -5. Find:
a. the second term   b. the third term   c. the twelfth term
d. the nth term   e. the term with a value of -81.

QUE. B

The second and third terms of a linear sequence are 53 and 46 respectively. Find the sum of the first 51 terms.

QUE. C

Prof Elvis' salary increased by ¢600 every year. If in 20years he was paid a total of ¢214,000 as his salary, find:
(i) His salary for the first year

(ii) His salary in the 20^{th} year

QUE. D

The first term of a linear sequence is 71 and the 30^{th} term is -16. Find the sum of the first 50 terms.

QUE. E

The first term of a linear sequence is 34 and the last term is 86. If the sum of the terms is 3600, find the number of terms in the sequence.

QUE. F

The first term of an exponential sequence is 30 and the constant ratio is $\frac{5}{4}$.

Find:

(i) The second term
(ii) The third term
(iii) The six term
(iv) The term which is equal to 186.2

QUE. G

Find the sum of the first 20 terms of the following exponential sequence:

(8.4, 12.6, 18.9, …)

QUE. H

Write a formula for the general term of each of the following sequences.

(a) 1, 3, 5, 7, 9, …
(b) 1, -1, 1, -1, …
(c) 0, 2, 4, 6, 8, …
(d) 3, 6, 9, 12, …

(e) 4, 7, 10, 13, …
(f) -1, 2, -4, 8, -16, …
(g) 0, 1, 4, 9, 16, …
(h) 0, 6, 12, 18, 24, …
(i) 7, 12, 17, 22, 27, …
(j) -4, -2, 0, 2, 4, …

QUE. I

Find the sum of each series:

1. $\sum_{i=1}^{4} i^2$ 2. $\sum_{j=0}^{5}(2j-1)$ 3. $\sum_{i=1}^{5} 2^{-i}$

QUE. J

Write each series in summation notation:

1. $1+2+3+4+5+6$
2. $-1+3-5+7-9+11$
3. $1+4+9+16+25+36$
4. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$

QUE. K

Find the eighth term of the sequence that has a first term of 9 and a common difference of 6

QUE. L

Find the common difference if the first term is 6 and the ninth term is 82.

QUE. M

If the common difference is -2 and the seventh term is 14, then what is the first term?

QUE. N

Find the sixth term of the sequence that has a fifth term of 13 and a first term of -3.

QUE. O

Find the sum of each given series

(a) 1+2+3+…+48
(b) 8+10+12+…+36
(c) -1+(-7)+(-13)+…+(-73)
(d) -6+(-1)+4+9+…+64
(e) 20+12+4+(-4)+…+(-92)
(f) $\sum_{i=1}^{12}(3i-7)$
(g) $\sum_{i=1}^{11}(-5i+2)$

QUE. P

Write a formula for the nth term of each geometric sequence:

(a) $\frac{1}{3}, 1, 3, 9, \ldots$
(b) 8, -4, 2, -1, …
(c) 2, -4, 8, -16, …
(d) $-\frac{1}{3}, -\frac{1}{4}, -\frac{3}{16}, \ldots$

CHAPTER 17
BINARY OPERATIONS ON SETS

The operations, addition $(+)$, subtraction $(-)$, multiplication $(\times)$ and division $(\div)$ are called ***binary operations*** since they combine two numbers to obtain another number. E.g. $3+4=7$ *or* $6\times 2=12$. However, the operation can also be different from those listed above in which case the operation will be defined.

17.1 **Properties Of Binary Operations**

17.1.1 **Commutativity**

The operation ***** defined on the set of real numbers is said to be commutative if and only if; $a*b=b*a$
Thus, $a+b=b+a$ meaning addition of real numbers is commutative.
E.g. $4+3=3+4$. Again, $a\times b=b\times a$ meaning multiplication of real numbers is commutative. E.g. $5\times 2=2\times 5$.

NB

Subtraction and division of real numbers are not commutative. Thus, $a-b\neq b-a$ and $a\div b\neq b\div a$

17.1.2 **Associativity**

The operation ***** defined on the set of real numbers is said to be associative **iff** $a*(b*c)=(a*b)*c$
Thus, $a+(b+c)=(a+b)+c$ meaning addition of real numbers is associative
E.g. $2+(3+4)=(2+3)+4$
Again, $a\times(b\times c)=(a\times b)\times c$ meaning multiplication of real numbers is associative
E.g. $6\times(7\times 8)=(6\times 7)\times 8$

NB

Subtraction and division of real numbers are not associative.

17.1.3 **Distributive Property**

The operation * defined over the set of real numbers is distributive over another operation Δ also defined on the set of real numbers **iff**. $a*(b\Delta c)=(a*b)\Delta(a*c)$

Thus, $a(b+c)=ab+ac$ meaning multiplication is distributive over addition.

E.g. $5(8+9)=(5\times 8)+(5\times 9)$

17.1.4 **Closure Property**

If an operation is performed on elements of a given set and the result belongs to the given set, then we say that the set is ***closed*** under the operation.

NB

1. The set of ***N, W, Z, Q*** and ***R*** are closed under addition.
2. Only the set of ***Z, Q*** and ***R*** are closed under subtraction
3. The sets ***N, W, Z*** and ***Q*** are not closed under division
4. In most cases the binary operation will be defined as in the examples below

Thus, Let R be the set of real numbers. A ***binary operation*** $*$ on R is a ***rule*** which combines any two numbers $a \ and \ b \ in \ R$ to produce another number.

That is $*: \ R\times R\rightarrow R$

For example $a*b=c$ defines a binary operation for any $a, b\in R$. If $a*b\in R$, then R is said to be closed under operation $*$. Binary operations are usually represented by symbols like $*, \nabla, \Delta, +, \bullet, \circ \ etc$.

Example 17.1

The operation * is defined as $a * b = a + b + ab$.
Evaluate $i)\ 3*4 \quad ii)\ 4*3$
Compare your answers in (i) and (ii). What property is that?
(iii) Show that $2*(3*4) = (2*3)*4$. What property is that?
(iv) Form a table for * on the set $Q = \{1, 2, 3, 4\}$. Is the set closed under the operation *?

Solution

Given that: $a * b = a + b + ab$

$i)\ 3*4 = 3 + 4 + (3 \times 4) = 7 + 12 = 19$

$ii)\ 4*3 = 4 + 3 + (4 \times 3) = 7 + 12 = 19$

From (i) and (ii), $3*4 = 4*3$. Hence, the operation * is commutative.

(iii) $2*(3*4) = (2*3)*4$

$2*(3 + 4 + 3 \times 4) = (2 + 3 + 2 \times 3)*4$

$2*(7 + 12) = (5 + 6)*4$

$2*19 = 11*4$

$\Rightarrow 2 + 19 + 2 \times 19 = 11 + 4 + 11 \times 4$

$21 + 38 = 15 + 44$

$59 = 59$

Since LHS = RHS, the operation * is associative

*	1	2	3	4
1	3	5	7	9
2	5	8	11	14
3	7	11	15	19
4	9	14	19	24

The set ***Q*** is ***not closed*** under the operation * since at least there is $2*2 = 8$ which does not belong to ***Q***.

Example 17.2

The operation $*$ is defined over the set R of real numbers by
$a*b=a+b+ab$
for all $a, b\in R$. Determine whether R is closed under the operation.

Solution

Let us consider $2*3=2+3+2(3)=11\in R$. Clearly, for all
$a, b\in R, a*b\in R$.
Hence R is closed under the operation $*$.

Example 17.3

The operation (•) is defined on the set {2, 4, 6} by m • n = the unit digit in the product mn.

(i) Copy and complete the table.

•	2	4	6
2	4	8	2
4		6	
6			

(ii) Use the table to solve the following equation:

$(\alpha)\ x\bullet 4=8 \qquad (\beta)\ e\bullet e=e \qquad (\gamma)\ (4\bullet f)\bullet 4=f$

Solution

(i)

•	2	4	6
2	4	8	2
4	8	6	4
6	2	4	6

(ii) From the table,

$(\alpha)\ x \bullet 4 = 8$

From table,

$2 \bullet 4 = 8 \Rightarrow x = 2$

$(\beta)\ e \bullet e = e$

From table,

$6 \bullet 6 = 6 \Rightarrow e = 6$

$(\gamma)\ (4 \bullet f) \bullet 4 = f$

From table,

$(4 \bullet 4) \bullet 4 = 6 \bullet 4 = 4 \Rightarrow f = 4$

17.2 **Identity (or Neutral Element)**

Let R be the set of real numbers and let $e \in R$ be an identity element under a binary operation $*$. Then $a * e = e * a = a$ for all $a \in R$. The identity element of any set (if it exists) under any binary operation is unique.

17.3 **Inverse Element**

Let $e \in R$ be an identity element of R under the binary operation $*$ and let $a^{-1} \in R$ be the inverse of a. Then $a * a^{-1} = a^{-1} * a = e$ for all $a \in R$.

Example 17.4

The operation $*$ is defined over the set R of real numbers by $a * b = a + b + ab$ for all $a, b \in R$. Find the inverse of a general element $a \in R$ under $*$ and state which element has no inverse. Find also the inverse of 1.

Solution

To find the inverse of any element, we must first determine the identity element $e \in R$ under $*$ such that $a * e = e * a = a$ for all $a \in R$

Thus,

$a*e = a+e+ae = a$

$\Rightarrow e(1+a) = 0$,

$e = \dfrac{0}{1+a} = 0$

This implies that $e = 0$

Since the identity element e has been determined, we then find the inverse.

That is,

$a*a^{-1} = a^{-1}*a = e = 0$

$\Rightarrow a*a^{-1} = 0$

$\Rightarrow a+a^{-1}+aa^{-1} = 0$

$\Rightarrow a^{-1}(1+a) = -a$

$\Rightarrow a^{-1} = -\dfrac{a}{1+a}$

Thus, the inverse of a is $a^{-1} = -\dfrac{a}{1+a},\ a \neq -1$

Hence, the inverse of 1 is $1^{-1} = -\dfrac{1}{1+1} = -\dfrac{1}{2}$

The element -1 has no inverse under the operation $*$

EXERCISE

QUE. A

A binary operation is defined on the set of rational numbers Q, as follows:

$a*b = a+b+2ab$

(a) Evaluate $(i)\ 3*5 \quad (ii)\ \frac{1}{2}*\frac{3}{4} \quad (iii)\ (3*5)*2$

(b) Find the value of z if $z*3 = 24$

QUE. B

Given that $a*b = a+b-ab$, *where* $a, b \in Q$, *evaluate* :

1. $2*3$ 2. $3*2$ 3. $(2*3)*5$ 4. $2*(3*5)$

QUE. C

Let $a \circ b = a^2 - 2b$, *where* $a, b \in R$. *Find*

1. $3 \circ 5$ 2. $5 \circ 3$ 3. $(3 \circ 5) \circ (5 \circ 3)$ 4. a *if* $a \circ 4 = 9$

QUE. D

Let $a * b = \dfrac{ab}{a+b}$, $a + b \neq 0$, $b \in R$

Evaluate:

1. $4 * 7$ 2. $7 * 4$ 3. $(4 * 7) * 5$ 4. $7 * 5$ 5. $(4 * 7) * (7 * 5)$

QUE. E

If $a \circ b$ denotes the greatest integer which is less than $\dfrac{ab}{a+b}$ *where* $a + b \neq 0$, *and* $a, b \in R$. *Find*

1. $4 \circ 6$ 2. $(4 \circ 6) \circ 5$ 3. $4 \circ (6 \circ 5)$ 4. $(4 \circ 6) \circ (4 \circ 5)$

QUE. F

The operation * is defined on the set Q of rational numbers by

$a * b = \dfrac{ab}{a+b}$, *where* $a + b \neq 0$.

Evaluate: 1. $(2 * 3) * 5$ 2. $2 * (3 * 5)$

Is * associative?

Is * commutative?

QUE. G

In each of the following, the operation * is defined on the set of real numbers. Determine whether or not * is (i) Commutative
(ii) Associative

1. $a * b = a + b + ab$ 2. $a * b = 4ab$ 3. $a * b = a + b - 1$

4. $a * b = a$ 5. $a * b = \dfrac{a+b}{ab}$ 6. $a * b = a^b$

QUE. H

A binary operation o is defined on the set R of real numbers by:

$$xoy = \frac{x+y}{1+xy},\ (xy \neq -1)$$

1. Is the set R closed under the operation
2. Is the operation o (i) commutative (ii) associative
3. Find the identity element for the operation o
4. Does each element of R have an inverse under the operation o?

CHAPTER 18

ALGEBRAIC EXPRESSIONS

Algebraic expressions are expressions that contain letters usually called variables. The individual terms in algebraic expressions are separated by a plus (+) or a minus
(-) sign. The number attached to a variable in each term is called its ***coefficient*** whiles the number in the expression that does not contain any variable is called the ***constant*** term (have unchangeable values). For instance, in the algebraic expression $3x+2y+7$, the coefficient of the first term $3x$ *is* 3, that of the second term $2y$ *is* 2 and the constant term is 7.

NB

Multiplication $(\times)$ and division $(\div)$ do not often occur in algebraic expressions as such. For instance, $a\times b$ is written as ab and $a\div b$ is written as $\frac{a}{b}$ in algebraic expressions.
Algebraic expressions are combined with at least one of the following operations:
$+,-,\times,\div$

18.1 Addition And Subtraction Of Algebraic Expressions

In simplifying algebraic expressions, we group liked terms close to each other in a common bracket and simplify appropriately. When adding or subtracting fractions, we first find their LCM.

Example 18.1

Simplify the following expressions:

1. $5x+3y+2x+4y$ 2. $3x^2y+5xy^2-2x^2y+xy^2$
3. $(5a+3b)-(2a-7b)$

4. $\frac{1}{3}+\frac{k}{3}+\frac{k^2}{12}$ 5. $2x^2+x+3x^2$ 6. $2p-4q+2p+4q$

Solution

1. $5x+3y+2x+4y=(5x+2x)+(3y+4y)$
$$=7x+7y$$
2. $3x^2y+5xy^2-2x^2y+xy^2=(3x^2y-2x^2y)+(5xy^2+xy^2)$
$$=x^2y+6xy^2$$

3. $(5a+3b)-(2a-7b)$
Open brackets;
$$\Rightarrow(5a+3b)-(2a-7b)=5a+3b-2a+7b$$
$$=(5a-2a)+(3b+7b)$$
$$=3a+10b$$

4. $\frac{1}{3}+\frac{k}{3}+\frac{k^2}{12}=\frac{4+4k+k^2}{12}=\frac{k^2+4k+4}{12}=\frac{(2+k)^2}{12}$

5. $2x^2+x+3x^2=5x^2+x$

6. $2p-4q+2p+4q=4p$

18.2 **Multiplication And Division Of Algebraic Expressions**

Basically, we apply the properties and laws of indices in the multiplication and division of algebraic expressions. In multiplying fractions, we multiply the numerators and denominators separately.

Example 18.2

Simplify the following expressions:

1. $5q\times7q$ 2. $-3x\times2y\times5x$ 3. $5x^3y^2\div xy$ 4. $\frac{6x^2+2xy}{5z}\times\frac{15z^2}{3x+y}$

5. $35p^5\div7p^3$ 6. $7rp\times3rp^3$

Solution

1. $5q \times 7q = 35q^2$
2. $-3x \times 2y \times 5x = (-3x \times 5x)2y = -15x^2 \times 2y = -30x^2 y$
3. $5x^3 y^2 \div xy = \dfrac{5x^3 y^2}{xy} = 5x^2 y$
4. $\dfrac{6x^2 + 2xy}{5z} \times \dfrac{15z^2}{3x + y} = \dfrac{15z^2(6x^2 + 2xy)}{5z(3x + y)} = \dfrac{3z(6x^2 + 2xy)}{3x + y} = \dfrac{3z[2x(3x + y)]}{3x + y} = 6xz$

5. $35p^5 \div 7p^3 = \dfrac{35p^5}{7p^3} = 5p^2$

6. $7rp \times 3rp^3 = 21r^2 p^4$

NB

In removing or opening brackets, we use the distributive law by multiplying the term outside the bracket by each term inside the bracket (when the outside term is only one term).
Again, in multiplying two brackets of two terms each, we pick each term in the first bracket and multiply by each term in the second bracket.

Example 18.3

Simplify the following expressions:

1. $3(2x + 3) + 3(x + 1)$ 2. $4y + 3(2y - 3) + 2$
3. $(2x - 3)(x + 7)$
4. $(2a + b)(a - b) + (2a - b)(a + b)$
5. $2x[(x + 3y) + 4(2x - y)]$

Solution

1. $3(2x + 3) + 3(x + 1) = 6x + 9 + 3x + 3 = (6x + 3x) + (9 + 3) = 9x + 12$
2. $4y + 3(2y - 3) + 2 = 4y + 6y - 9 + 2 = 10y - 7$

3. $(2x-3)(x+7) = 2x(x+7) - 3(x+7)$
$= 2x^2 + 14x - 3x - 21$
$= 2x^2 + 11x - 21$

4. $(2a+b)(a-b)+(2a-b)(a+b) = 2a^2 - 2ab + ab - b^2 + 2a^2 + 2ab - ab - b^2 = 4a^2 - 2b^2$

5. $2x[(x+3y)+4(2x-y)] = 2x[x+3y+8x-4y] = 2x^2 + 6xy + 16x - 8xy = 2x^2 + 16x - 2xy$

18.3 **Factorization**

Factorization simply means forming brackets from existing expressions by removing the highest common factor between the terms and putting it outside the bracket.

Example 18.4

Factorize the following completely:

1. $2xy - 6mn - 3my + 4nx$
2. $x^2 + ax - x - a$
3. $(h^2 - k^2) - p(h+k)$
4. $36p^2 - 49q^2$
5. $x^2 + 5x + 6$ 6. $2p^2 - pq - 3q^2$ 7. $12.5 \times 3.2 - 5.47 \times 12.5$

Solution

1. $2xy - 6mn - 3my + 4nx = (2xy + 4nx) + (-6mn - 3my)$
$= 2x(y + 2n) - 3m(2n + y)$
$= (2x - 3m)(2n + y)$

2. $x^2 + ax - x - a = (x^2 - x) + (ax - a)$
$= x(x-1) + a(x-1)$
$= (x+a)(x-1)$

3. $(h^2 - k^2) - p(h+k) = (h-k)(h+k) - p(h+k)$
$= (h+k)[(h-k) - p]$
$= (h+k)(h-k-p)$

NB

Difference of two squares is applied where the expression can be expressed as difference of two perfect squares as in the following examples.

4. $36p^2 - 49q^2 = 6^2p^2 - 7^2q^2$

$= (6p - 7q)(6p + 7q)$

5. $x^2 + 5x + 6 = x^2 + 2x + 3x + 6$

$= (x^2 + 2x) + (3x + 6)$

$= x(x + 2) + 3(x + 2)$

$= (x + 2)(x + 3)$

6. $2p^2 - pq - 3q^2 = 2p^2 + 2pq - 3pq - 3q^2$

$= 2p(p + q) - 3q(p + q)$

$= (2p - 3q)(p + q)$

7. $12.5 \times 3.2 - 5.47 \times 12.5 = 12.5(3.2 - 5.47)$

$= 12.5 \times -2.27 = -28.375$

EXERCISE

QUE. A

Simplify the following:

1. $\dfrac{3x - 9x^2}{3x}$, *where* $x \neq 0$
2. $\dfrac{a^2 - 1}{a^2 + 2a + 1}$
3. $\dfrac{x + y}{x^{-1} + y^{-1}}$
4. $\dfrac{4}{x^2 - 4} - \dfrac{1}{x^2 - 3x + 2}$

QUE. B

Factorize the following completely:

1. $4m^2 - 49x^2p^2$ 2. $9x^2y - 25yz^2$ 3. $(x + 6)^2 - 36x^2$

QUE. C

Find the following

1. Q *if* $3Q + 13^2 = 16^2$
2. R *if* $5R^2 + (22.5)^2 = (27.45)^2$
3. y *if* $13y = 187^2 - 174^2$
4. x *if* $(3.46)^2 - (1.54)^2 = 10x$

QUE. D

Without using tables or calculators, evaluate the following:

1. $53.8^2 - 46.2^2$
2. $61.5^2 - 38.5^2$
3. $(2\frac{1}{2})^2 - (1\frac{1}{2})^2$

QUE. E

If $x + \frac{6}{x} = 5$, find the value of $x^2 + \frac{36}{x^2}$

Hint: Use $\left(x + \frac{6}{x}\right)^2 = 5^2$

QUE. F

Factorise the following expressions:

1. $ay + ax$ 2. $ax - bx$ 3. $27p^2 - 9pq$
4. $5mnp - 3mnp$ 5. $2rx + 2ry + 3cx + 3cy$ 6. $ax - ay + bx - by$
7. $2au + 3av + 2nu + 3bv$ 8. $5ay - by + 15a - 3b$
9. $2ax - \frac{4}{3}ay + 15bx - 10by$ 10. $6am + 15an + 2bm + 5bn$

QUE. G

Simplify the following expressions

1. $\frac{a}{2}+\frac{3a}{5}+\frac{2a}{3}$ 2. $\frac{x+y}{3}+\frac{x-2y}{5}-\frac{x-y}{2}$

3. $\frac{3a}{a^2-25}+\frac{2}{a+5}-\frac{5}{5-a}, a \neq \pm 5$ 4. $\frac{3}{x}-\frac{2}{x-y}+\frac{5x}{x(x-y)}, x \neq 0,\ x \neq y$

5. $\frac{7y}{y+3}-\frac{y}{y-3}+\frac{5y}{3-y},\ y \neq \pm 3$ 6. $\frac{4a^2}{a^2-b^2}-\frac{2a}{a+b}-\frac{2b}{a-b},\ a \neq \pm b$

CHAPTER 19

FUNCTIONS AND MAPPINGS

19.1 **Definition**

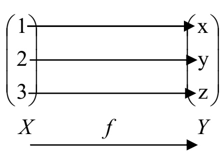
A ***function*** is a relation (a rule that associates elements of one set with elements of another set) that assigns to each element of one set (usually called the domain) to exactly one element of another set (usually called the range).
Functions are denoted by ***lower case letters*** whiles the sets involved are denoted by ***upper case letters.*** A function f connecting elements from set X to elements in set Y is denoted by: $f: X \rightarrow Y$.

NB

1. The ***domain*** of a function is simply the set of elements of the first set (say X).
2. The ***co-domain*** of a function is the set of elements of the second set (say Y).
3. The ***range*** of a function is the elements in set Y (the co-domain) which are arrowed from the first set (the domain). In short, it is the ***images*** of the given function with corresponding elements in the domain.

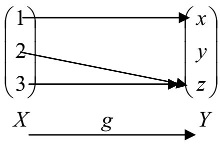
19.2 **Illustration Of Functions Using Arrow Diagrams**

An ***arrow diagram*** is simply a diagram that maps one set to another using arrows.

1. 

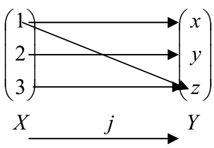
NB: Range is (x, y, z)

From the above diagram, since every element in the domain has a distinct corresponding image in the co-domain, f is defined as a ***function.***

2. 

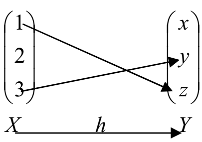
NB: Range is (x, z)

From the diagram above, since two elements in the domain can have the same image in the co-domain, g is defined as a ***function.***

3. 

NB: Range is (x, y, z)

From the diagram above, since the element 1 in the domain has two images in the
co-domain, J is ***not*** a function.

4. 

NB: Range is (y, z)

From the diagram above, since the element 2 does not have an image in the
co-domain, h is ***not*** a function.

19.3 **Functional Notations**

When the letter x in the function $f(x)$ is replaced by a scaler (number) in the notation form, we simply replace x in the given expression by the number and simplify the expression. This is called the notation of the function with respect to the given scaler.

Example 19.1

1. Given that $f : x \to x^4 - 3x^2 - 7x + 11,$ find $f(-1)$

Solution

$f(-1) = (-1)^4 - 3(-1)^2 - 7(-1) + 11 = 1 - 3 + 7 + 11 = 16$

NB

Here, -1 is an element in the domain whiles the result of finding $f(-1)$ is an image in the co-domain.

Example 19.2

If $f(x) = -2x^3 + qx + 2,$ where q is a constant and $f(-1) = 2,$ find q

Solution

$First,\ f(-1) = -2(-1)^3 + q(-1) + 2 = 2 - q + 2 = 4 - q$
But from question, $f(-1) = 2$
$\Rightarrow 4 - q = 2$
$Hence,$
$4 - 2 = q$
$\therefore q = 2$

Example 19.3

The functions f and g are defined as follows:

$f : x \to \dfrac{x-1}{2} \quad and\ g : x \to 3x + 1$

(i) Evaluate $f(-\frac{1}{2}) + 1$

(ii) Solve $f(x) = g(-2)$

Solution

$f: x \to \frac{x-1}{2} \quad and\ g: x \to 3x+1$

$(i)\ f(-\tfrac{1}{2})+1 = \frac{-\frac{1}{2}-1}{2}+1 = \frac{\frac{-1-2}{2}}{2}\ +1 \quad = -\frac{3}{2}\times\frac{1}{2}+1 = -\frac{3}{4}+1 = \frac{-3+4}{4} = \frac{1}{4}$

$(ii)\ f(x) = g(-2)$

$\Rightarrow \frac{x-1}{2} = 3(-2)+1$

$\frac{x-1}{2} = -6+1 = -5 \Rightarrow x-1 = -10 \Rightarrow x = -10+1 = -9$

$\therefore x = -9$

Example 19.4

The functions f and g are defined as:

$f: x \to 2-x^2\ \ and\ \ g: x \to \frac{1}{x-1}.$

$Evaluate\ (i)\ g(-\tfrac{1}{4}) \qquad (ii)\ \frac{f(2)}{g(3)}$

Solution

$(i)\ g(-\tfrac{1}{4}) = \frac{1}{-\frac{1}{4}-1} = -\frac{1}{\frac{5}{4}} = -1\times\tfrac{4}{5} = -\frac{4}{5}$

$(ii)\ f(2) = 2-(2)^2\ = -2$

$g(3) = \frac{1}{3-1} = \frac{1}{2} \Rightarrow \frac{f(2)}{g(3)} = \frac{-2}{\frac{1}{2}} = -2\times 2 = -4$

Example 19.5

The functions f and g are defined as:

$f: x \to x-2\ and\ \ g: x \to 2x^2-1$

$Solve: (i)\ f(x) = g(-\tfrac{1}{2}) \qquad (ii)\ f(x)+g(x) = 0$

Solution

$(i)\ g(-\tfrac{1}{2}) = 2(-\tfrac{1}{2})^2 - 1 = 2\times\tfrac{1}{4} - 1 = -\tfrac{1}{2}$

$\Rightarrow$ *from* $f(x) = g(-\tfrac{1}{2})$

$x - 2 = -\tfrac{1}{2}$

$x = 2 - \tfrac{1}{2} = 1\tfrac{1}{2} = 1.5$

$(ii)\ f(x) + g(x) = 0$

$\Rightarrow x - 2 - 2x^2 - 1 = 0$

$2x^2 + x - 3 = 0$

$\Rightarrow (2x+3)(x-1) = 0$

Either

$(2x+3) = 0 \Rightarrow x = -\tfrac{3}{2}$ *Or* $(x-1) = 0 \Rightarrow x = 1$

$\therefore x = -\tfrac{3}{2}$ *or* 1

19.4 **Sum, Differeence, Product and Quotient Functions**

Given two functions f and g, the functions $f+g,\ f-g,\ f.g$ *and* $\dfrac{f}{g}$ are defined as follows:

Sum function: $(f+g)(x) = f(x) + g(x)$

Difference function: $(f-g)(x) = f(x) - g(x)$

Product function: $(f.g)(x) = f(x).g(x)$

Quotient function: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)},$ *provided that* $g(x) \neq 0$

Example 19.6

Let $f(x) = 4x - 12$ *and* $g(x) = x - 3.$ Find the following:

$(a)\ (f+g)(x) \qquad (b)\ (f-g)(x) \qquad (c)\ (f.g)(x) \qquad (d)\ \left(\dfrac{f}{g}\right)(x)$

Solution

$(a)\ (f+g)(x) = f(x)+g(x) = 4x-12+x-3 = 5x-15$

$(b)\ (f-g)(x) = f(x)-g(x) = 4x-12-x+3 = 3x-9$

$(c)\ (f.g)(x) = (4x-12)(x-3) = 4x^2-24x+36$

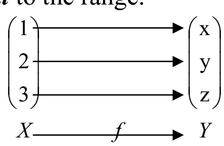
$(d)\ \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{4x-12}{x-3} = \dfrac{4(x-3)}{x-3} = 4,\ for\ x \neq 3.$

19.5 **Types Of Functions**

19.5.1 **The Onto Function**

A function is said to be an ***onto function*** if all the elements in the co-domain are arrowed from the domain. i.e. if the co-domain is ***equal*** to the range.

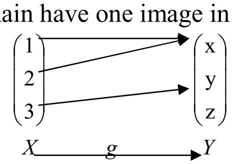
E.g.



19.5.2 **Many To One Function**

A function is said to be ***many to one*** if several elements in the domain have one image in the co-domain.

Eg.



19.5.3 **The Even Function**

A function f is said to be an ***even*** function if $f(x) = f(-x)$

Example 19.7

If $f(x) = x^2 - 1$, show that $f(x)$ is even

Solution

Take $x = 2$, $\Rightarrow f(2) = 2^2 - 1 = 4 - 1 = 3$ and $f(-2) = -2^2 - 1 = 4 - 1 = 3$
Since $f(2) = f(-2)$, $f(x)$ is even

19.5.4 **The Odd Function**

A function is said to be ***odd*** if $f(x) = -f(-x)$

Example 19.8

Show if $f(x) = x^3$ is odd.

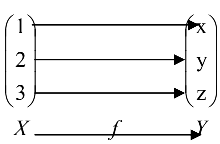
Solution

Take $x = 2$, $\Rightarrow f(2) = 2^3 = 8$ *and* $-f(-2) = -(-2^3) = -(-8) = 8$
Since $f(2) = -f(-2)$, $f(x)$ *is odd*

19.5.5 **One-To-One Function**

A function is said to be ***one to one*** if every element in the domain has ***exactly one*** image in the co-domain.

Illustration



From the diagram above, since every element of X has exactly one image in Y, $f(x)$ is one-to-one.

Example 19.9

If $f(x)=x^2-1$ is defined on the domain $\{-1,0,1\}$, show that $f(x)$ is one-to-one.

Solution

$f(-1)=-1^2-1=0, \quad f(0)=-1, \ f(1)=0$

$f(x)$ is not one-to-one since the elements -1 *and* 1 resulted to the same image.

Again, a particular function is said to be one-to-one ***mathematically*** if $f(a)=f(b) \quad and \quad a=b$

Steps

1. Replace x in the function by a. i.e. Find $f(a)$
2. Find $f(b)$ by replacing x in the function by $b.$
3. Equate the result of step 1 to the result of step 2 and make a the subject.
4. If $a=b,$ then the function is one-to-one. But if a is not equal to $b.$, it is not one-to-one. Thus, anytime a is equal to ***more than*** one answer, then, the function is not one to one.

Example 19.10

Show whether or not $f(x)=2x+1$ is one-to-one

Solution

Step 1

$f(a)=2a+1$

Step 2

$f(b)=2b+1$

Step 3

$2a+1=2b+1$

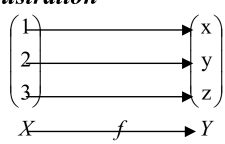
$\Rightarrow \frac{2a}{2}=\frac{2b}{2}$

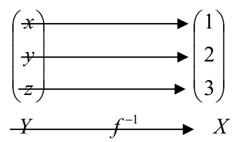
$\therefore a=b$

19.6 **Finding The Inverse Of Functions**

The inverse of a function $f(x)$ is denoted by f^{-1} or $f(x)^{-1}$

Illustration

If 

and 

Thus, if f is a function from X to Y, then f^{-1} is a function from Y to X.

Method 1

Steps

1. Let y represent the inverse function and equate it to the given function
2. Interchange the positions of x and y in step 1
3. Make y the subject of the equation in step 2 to get the required inverse function

Example 19.11

A function is defined on the set R of real numbers by $f: X \to 5x-9.$ Find the inverse of f

Solution

from $f: X \to 5x-9.,$

Step 1 let $y = 5x - 9$

Step 2 $x = 5y - 9$

Step 3 $\frac{5y}{5} = \frac{x+9}{5}$

$$\Rightarrow y = \frac{x+9}{5}$$

Therefore, the inverse of f is $\frac{x+9}{5}$

Method 2

Steps

1. Equate y to the given function
2. Make x the subject of the result of step 1
3. Interchange x and y in step 2 to get the inverse function

Illustration

From the example above, $f : X \rightarrow 5x - 9.$

Step 1

$y = 5x - 9$

Step 2

$\frac{y+9}{5} = \frac{5x}{5}$

$\Rightarrow \frac{y+9}{5} = x$

Step 3

$y = \frac{x+9}{5}$

19.7 **The Domain Of A Function**

The ***domain*** of any given function is simply the values of x for which the function is ***defined*** or has a ***solution***.

Example 19.12

.

Find the domain of the following functions:

$(i)\ \ f(x) = 5 - 3x$

$(ii)\ \ h : X \to \dfrac{3x-2}{x+4}$

$(iii)\ g : X \to \sqrt{4+x}$

$(iv)\ t : X \to \dfrac{1}{x^2 - 9}$

Solution

(i) From $f(x) = 5 - 3x$, Domain is $\{x : x \in R\}$

(ii) From $h : X \to \dfrac{3x-2}{x+4}$, first, we equate the denominator to zero and find x and secondly, the value of x found becomes the exceptions in the domain.

i.e. $x + 4 = 0 \quad \Rightarrow x = -4$

Therefore, domain is $\{x : x \in R, \textit{except } x = -4\}$

(iii) $g : X \to \sqrt{4+x}, \Rightarrow \textit{domain is } \{x : x \in R, \textit{ except } x < -4\}$

(iv) $t : X \to \dfrac{1}{x^2 - 9}, \Rightarrow \textit{domain is } \{x : x \in R, \textit{ except } x = \pm 3\}$

19.8 **The Range Of A Function**

The range of any given function is the set of values of y that makes x defined. In finding the range of any given function, first equate the function to y and make x the subject. Then, the values of y that makes the resulting expression defined is the range of the function.

Example 19.13

Find the range of the following functions:

1. $f(x) = 1 - 5x$ 2. $g(x) = 2 - x^2$ 3. $h(x) = \dfrac{x}{1+2x}$

Solution

1. From $f(x)=1-5x,$

Step 1

Let $y=1-5x$

Step 2

$\frac{5x}{5}=\frac{1-y}{5}$

$\Rightarrow x=\frac{1-y}{5}$

Therefore, the range is $\{y: y\in R\}$

2. From $g(x)=2-x^2,$

$Let\ \ y=2-x^2$

$\Rightarrow x^2=2-y$

$\therefore x=\sqrt{2-y}$

$\Rightarrow 2-y<0$

$\therefore 2<y \quad or \quad y>2$

Therefore, the range is $\{y: y\in R,\ except\ \ y>2\}$

3. $h(x)=\frac{x}{1+2x}\Rightarrow y=\frac{x}{1+2x}\Rightarrow x=y+2xy\Rightarrow x-2xy=y\Rightarrow x(1-2y)=y$

$$x=\frac{y}{1-2y}\Rightarrow Range=\{y: y\in R,\ except\ \ y=\frac{1}{2}\}$$

19.9 **Composite Functions**

The ***composite*** function usually involves more than one function together with more than one set.

Illustration

We write fg *or* fog to denote the composite function $'g'$ followed by $'f'$.

Again, gof means $'f'$ followed by $'g'$

NB:

In solving composite functions, we substitute the second function into the first function. Thus, we do this by replacing the variable in the first function by the whole of the second function. In short, we usually start the substitution from the far right of the expression.

Example 19.14

1. If $f(x) = 2x \quad and \quad g(x) = x^2 + 1,$ find $f(g(x))$

2. Two functions f and g are defined on the set R of real numbers by $f: X \to x^2 - x - 6 \quad and \quad g: X \to x - 1,\ find\ fog(3)$
3. Two functions f and g are defined by $f: X \to x - 1 \quad and \quad g: X \to 5x$. Find the values of x such that $fog(x) = 2$

Solution

1. Given that $f(x) = 2x \quad and \quad g(x) = x^2 + 1,$
   $f(g(x)) = 2(x^2 + 1) = 2x + 2$ (i.e. putting g into f)

2. $f: X \to x^2 - x - 6 \quad and \quad g: X \to x - 1,$
   First, $g(3) = 3 - 1 = 2$
   Then, $fog(3) = f(2) = 2^2 - 2 - 6 = 4 - 8 = -4$

3. From $f: X \to x - 1 \quad and \quad g: X \to 5x$,
   First, $fog(x) = 5x - 1$
   Secondly, equate result above to 2 since $fog(x) = 2$
   $\Rightarrow 5x - 1 = 2$
   $$\frac{5x}{5} = \frac{3}{5}$$
   $$\Rightarrow x = \frac{3}{5}$$

19.10 **Finding The Values Of x That Make Functions Undefined**

A rational function is said to be undefined if its denominator is zero. Thus, the function $\frac{f(x)}{g(x)}$ is said to be undefined if $g(x)=0$. i.e. in determining whether a function is undefined, we equate the denominator to zero and solve for the variable.

Example 19.15

(i) A function f is given by $f:x\rightarrow\frac{x}{2x-1}$. For what value of x is f undefined?

(ii) For what value(s) of y is $\frac{1}{y}+\frac{1}{y-1}$ undefined?

Solution

$(i)\ \ f:x\rightarrow\frac{x}{2x-1}$

For f to be undefined, $2x-1=0$

$\Rightarrow 2x=1$

$\therefore\ x=\frac{1}{2}$

Hence, the value of x for which f is undefined is $\frac{1}{2}$

(ii) the expression $\frac{1}{y}+\frac{1}{y-1}$ is undefined when $y=0\ \ and\ \ y=1$

19.11 **Finding The Rule Of A Mapping**

A mapping is described by stating the formulae or rule of its definition. A particular mapping could be in any form, say linear, exponential etc. Usually in mapping, the up values represents the x-values whiles the down values represents the y-values.

NB

In finding the rule of mappings where the difference between any two consecutive elements of the domain is constant or the same and at the same time the difference between any two consecutive elements of the co-domain is also constant, we use the relation: $y = ax + b$. Where $a = \frac{Co-domain\ constant\ difference}{Domain\ constant\ difference} = \frac{\Delta y}{\Delta x}$ and b is found by substituting any corresponding x and y values into the rule.

Illustrations

Example 19.16

The following shows a mapping $x \rightarrow y$

$$\begin{array}{cccccc} x & 0 & 3 & 6 & 9 & 12 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & -1 & 5 & 11 & 17 & 23 \end{array}$$

What linear equation connects x and y?

Solution

Since $\Delta x = 3$ and is a constant difference in domain elements and $\Delta y = 6$ also a constant difference in co-domain elements, we use the relation: $y = ax + b$. $a = \frac{\Delta y}{\Delta x} = \frac{6}{3} = 2$

For b, substitute $x = 3$, $y = 5$ (any corresponding values of x and y) into $y = ax + b$

$\Rightarrow 5 = 2(3) + b$

$\Rightarrow 5 = 6 + b$

$\Rightarrow -1 = b$

Hence, the equation that connects x and y is $y = 2x - 1$

Example 19.17

The following shows a mapping $x \rightarrow y$

$$\begin{array}{ccccccc} x & 0 & 3 & 6 & 9 & 12 & 15 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & -1 & 5 & 11 & 17 & 23 & 29 \end{array}$$

What is the rule for the mapping?

Solution

Same solution as in Example 19.7. i.e. $y = 2x - 1$.

NB

Again, In finding the rule of mappings where the ratio between any two consecutive elements of the co-domain is constant or the same, we use the relation: $y = ar^{x-b}$, . Where a is the first term of the co-domain, r is the common ratio of consecutive elements of the co-domain and b is the first term of the elements of the domain.

Example 19.18

Find the rule for the following mappings:

$$\begin{array}{cccccc} x & 0 & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 1 & 3 & 9 & 27 & 81 \end{array}$$

Solution

Using $y = ar^{x-b}$, $a = 1,\ b = 0,\ and\ r = 3$
Hence, $y = 1 \times 3^{x-0} = 3^x$

19.11.1 **Special Cases**

Example 19.19

Find the image of 3 under the mapping $y \to x^2 + 3x - 2$.

Solution

We find the image by substituting the element into the mapping given.

$$\Rightarrow y \to 3^2 + 3(3) - 2$$
$$\therefore y \to 16$$

Example 19.20

The image of x in the mapping $x \to x-6$ *is* 4, find x.

Solution

Here, we equate the rule of the mapping to the given image.
Thus, $2x-6=4 \Rightarrow x=10$

Example 19.21

Obtain the mapping table defined by the rule $y \to x+1$ on the domain $\{1,2,3\}$

Solution

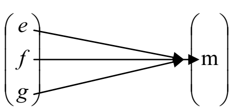
We first find the image of each element of the domain.
Thus, When $x=1,\ y=2$, when $x=2,\ y=3$ *and when* $x=3,\ y=4$

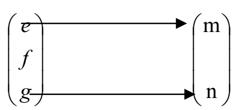
The mapping is $\begin{array}{cccc} x & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ y & 2 & 3 & 4 \end{array}$

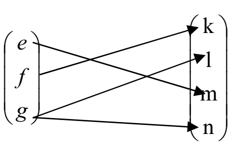
EXERCISE

QUE. A

Which of the following mappings defines a function?

(i)


(ii)


(iii)


QUE. B

Given that:

$$f(x)=\frac{1}{x-2}+\frac{3}{x-5},\ find\ f(x)\ when\ x=3$$

QUE. C

Find the values of x for which the following expressions are undefined

$(i)\ y=\dfrac{2x-5}{3x+2}$

$(ii)\ \dfrac{5x+3}{6x(x+1)}$

$(iii)\ \dfrac{x^2-9}{x^2-3x+2}$

QUE. D

The set $P=\{-2,-1,0,1,2\}$ maps onto Q by the function $f(x)=x^2-2$ where $x\in P$.

Find the elements of Q.

Draw a diagram showing the mapping between P and Q.

QUE. E

What is the image of 7 under the mapping: $x\rightarrow\dfrac{4-x}{3}$?

QUE. F

Given that $f(x)=2x-1\ and\ g(x)=x^2+1$

Find $f(1+x)$

Find the range of values of x for which $f(x)<-3$

Simplify $f(x)-g(x)$

QUE. G

Find the rule for the mapping:

$$\begin{array}{cccccc} x & 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 2 & 5 & 8 & 11 & 14 \end{array}$$

CHAPTER 20
LINEAR INEQUALITIES I
(LINEAR INEQUALITIES IN ONE VARIABLE)

20.1 **Solving Inequalities**

Inequalities are usually numerical statements involving one or more of the following signs;
$<$ and read as "less than"
$>$ and read as "greater than"
$\leq$ and read as "less than or equal to"
$\geq$ and read as "greater than or equal to"

NB

When multiplying or dividing all terms of an inequality by a negative common factor, the sign changes. Thus, $<$ changes to $>$ and $\leq$ changes to $\geq$ and vice versa.
Again, when adding or subtracting a common factor (whether negative or positive) to or from both sides of an inequality, the sign does not change.
Further, in multiplying or dividing both sides of an inequality by a positive number does not change the sign.
Generally, solving inequalities follows the same procedure as in solving linear equations.

NB

In representing inequalities on the number line, we use "O" with the arrow to represent "less than and greater than signs" and we use "●" to represent the "less than or equal to and greater than or equal to signs". On the number line, a number found on the left of another is said to be less than that on its right. It is always advisable to avoid putting too many numbers or values on the number line. Thus, we can just use 3 or 4 values on the number line. Usually, the exact value, an immediate integer less than the number and an immediate integer greater than the number are preferable.

Example 20.1

Solve the inequality $2x > 8$

Solution

$$\frac{2x}{2} > \frac{8}{2}$$
$$\Rightarrow x > 4$$

Example 20.2

Solve 4 + 3x < 10, if x is positive

Solution

4 + 3x < 10

$$3x < 10 - 4$$
$$3x < 6$$
$$\frac{3x}{3} < \frac{6}{3}$$
$$x < 2$$

Example 20.3

Find the truth set of the inequality: 2t + 5 < 4t – 5

Solution

$$2t + 5 < 4t - 5$$
$$5 + 5 < 4t - 2t$$
$$\frac{10}{2} < \frac{2t}{2}$$
$$\Rightarrow 5 < t \ \ or \ \ t > 5 \quad Hence, truth\ set\ is \{t : t > 5\}$$

Example 20.4

Which of the following number lines represents the solution of 2x + 1 > 2?

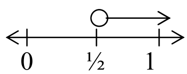
Solution

$2x + 1 > 2$

$\frac{2x}{2} > \frac{1}{2}$

$\Rightarrow x > \frac{1}{2}$

Hence;



Example 20.5

Illustrate the truth set of 3x + 2 ≥ 5x – 6 on the number line.

Solution

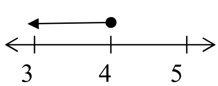
3x + 2 ≥ 5x – 6

2 + 6 ≥ 5x – 3x (grouping liked terms)

$\frac{8}{2} \geq \frac{2x}{2}$

$4 \geq x \;\; or \;\; x \leq 4$

Hence, truth set is $\{x : x \leq 4\}$



Example 20.6

Which of the following illustrations on the number lines below represents the truth set of the inequality: 3x – 9 ≥ (x – 3)?

Solution

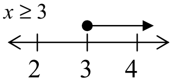
3x – 9 ≥ (x – 3)
First, open bracket

$3x - 9 \geq x - 3$

$3x - x \geq 9 - 3$

$2x \geq 6$

$\Rightarrow x \geq 3$



Example 20.7

Solve the inequality 4x + 3 ≥ 3(2x -1) and illustrate your answer on the number line.

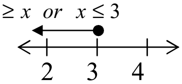
Solution

4x + 3 ≥ 3(2x – 1)
Open brackets by expanding,

$\Rightarrow 4x + 3 \geq 6x - 3$

$6 \geq 2x$

$\Rightarrow 3 \geq x \;\; or \;\; x \leq 3$



Example 20.8

Solve the inequality: 48 < ½ (1 + x)

Solution

48 < ½(1 + x)
Multiply both sides by LCM of denominator;

$\Rightarrow 2(48) < 2 \times \frac{1}{2}(1 + x)$

$96 < 1 + x$

$\Rightarrow 95 < x \;\; or \;\; x > 95$

Example 20.9

Find the truth set of the inequality: 7x + 4 < ½ (4x + 3)
Illustrate your answer on the no line.

Solution

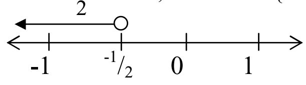
7x + 4 < ½ (4x + 3)
Multiply through by LCM of denominator of fraction

$\Rightarrow 2(7x) + 2(4) < 2 \times \frac{1}{2}(4x + 3)$

$14x + 8 < 4x + 3$

$10x < -5$

$\Rightarrow x < -\frac{1}{2}$ *Hence, truth set is* $\{x : x < -\frac{1}{2}\}$



Example 20.10

Find the values of x for which $\frac{1}{4}x - \frac{1}{3}(x - 1) \leq \frac{7}{12}$

Solution

NB

It is always convenient to multiply all terms by the LCM of the denominators before expanding brackets.

$\Rightarrow 12\times\frac{1}{4}x - 12\times\frac{1}{3}(x-1) \le 12\times\frac{7}{12}$

$3x - 4x + 4 \le 7$

$-x \le 7-4$

$-x \le 3$

$Divide\ \ through\ \ by\ -1$

$\Rightarrow x \ge -3(Since\ sign\ changes\ when\ dividing\ by\ -ve)$

Example 20.11

Find the range of values of x for which: $\frac{5-x}{3} + 2 < \frac{x-2}{2}$

Solution

$\frac{5-x}{3} + 2 < \frac{x-2}{2}$

Multiply through by LCM of denominator,

$6\times\frac{5-x}{3} + 6\times 2 < 6\times\frac{x-2}{2}$

$\Rightarrow 2(5-x) + 12 < 3(x-2)$

$\Rightarrow 10 - 2x + 12 < 3x - 6$

$22 + 6 < 3x + 2x$

$28 < 5x$

$\Rightarrow x > \frac{28}{5}$

Example 20.12

Solve $\frac{2}{5}(x+1) < 2\frac{3}{4}$

Solution

$\frac{2}{5}(x+1) < 2\frac{3}{4}$

First change the mixed fraction to improper fraction

$$\Rightarrow \frac{2}{5}(x+1) < \frac{11}{4}$$

$$20 \times \frac{2}{5}(x+1) < 20 \times \frac{11}{4}$$

$$8(x+1) < 5(11)$$

$$8x + 8 < 55$$

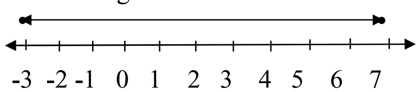
$$\Rightarrow x < \frac{47}{8}$$

Example 20.13

If P = {x: $-4 \leq x \leq 4$} and Q = {x: $2 \leq x \leq 8$}, where x is real number, which of the following is an illustration of P U Q?

Solution

P = {x: $-4 < x \leq 4$} = {-3, -2, -1, 0, 1, 2, 3, 4}
Q = {x: $-2 \leq x < 8$} = {-2, -1, 0, 1, 2, 3, 4, 5, 6, 7}
Hence, P U Q = {-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7}
NB: This will give a double number line with two limits.



Examples 20.14

If x = y and z > y, which of the following must be true?

Solution

x < z

Example 20.15

Represent $3 \leq x < 5$ on a number line

Solution

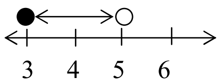
$3 \leq x < 5$

We first separate into two inequalities. i.e. from the middle to the left and from the middle to the right

Thus, $3 \leq x$ and $x < 5$

Then represent both inequalities on a single number line

i.e.



Example 20.16

Find the truth set of $4 < x - 6 < 5$ and represent your answer on the number line.

Solution

Method 1

$4 < x - 6 < 5$

First, separate the inequality into two. i.e. from the middle to the left and from the middle to the right

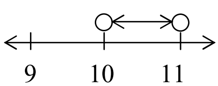
i.e. $4 < x - 6$ and $x - 6 < 5$ and solve each for x.

$$For\ 4 < x - 6$$

$$\Rightarrow 4 + 6 < x \ \Rightarrow x > 10$$

$$For\ x - 6 < 5$$

$$\Rightarrow x < 5 + 6 \ \Rightarrow x < 11\ \ Hence,\ combining\ \ gives\ 10 < x < 11$$



NB

We solve a double inequality by separating it into two and then recombining the answer into one inequality before representing on the number line.

Method 2

We could leave the inequality as it is and try to isolate the variable alone at its usual term position.

$i.e.\ for\ 4 < x - 6 < 5$

$Add\ 6\ to\ all\ terms$

$\Rightarrow 4 + 6 < x - 6 + 6 < 5 + 6$

$10 < x < 11$

Example 20.17

Solve the inequality $2 \le \frac{1}{3}(2x - 5) \le 3$

Solution

$2 \le \frac{1}{3}(2x - 5) \le 3$

$\Rightarrow 6 \le 2x - 5 \le 9$

$Add\ 5\ to\ all\ sides$

$\Rightarrow 6 + 5 \le 2x - 5 + 5 \le 9 + 5$

$\Rightarrow \frac{11}{2} \le \frac{2x}{2} \le \frac{14}{2}$

$\Rightarrow 5.5 \le x \le 7$

20.2 **Word Problems**

Example 20.18

The sum of twice a number and seven is greater than or equal to the sum of three times the same number and one. Find the range of values for the number.

Solution

Let the no be x

$\Rightarrow 2x + 7 \geq 3x + 1$

$\Rightarrow 7 - 1 \geq 3x - 2x$

$6 \geq x \quad or \quad x \leq 6$

Example 20.19

A woman went to a market to buy meat with Gh¢5. If she is to buy eggs in addition, she must have averagely between Gh¢10 and Gh¢15 on her. In what range in Ghana cedis must she have on her to enable her buy both the eggs and meat?
Illustrate your answer on the number line.

Solution

Let GH¢ x be the amount required to buy the eggs.

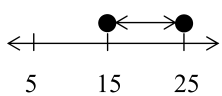
$\Rightarrow 10 \leq \frac{5+x}{2} \leq 15$

$\Rightarrow 2\times 10 \leq 2\times \frac{5+x}{2} \leq 2\times 15$

$20 \leq 5 + x \leq 30$

Subtract 5 from all terms

$\Rightarrow 15 \leq x \leq 25$



EXERCISE

QUE. A

Find the values of x which satisfy the following inequalities:
$5 - x > 1 \text{ and } 9 + x \geq 8$ Illustrate you answer on the number line.

QUE. B

For what range of values of x is: $3.2 + 1.8x > 2.6x$?

QUE. C

Find the truth set of the inequality 5x + 1 > 10 x – 9

QUE. D

Which of the following represent the solution of $2x - 3 > -1$ on the number line?

QUE. E

Illustrate the truth set of $3x + 2 \leq 5x - 6$ on the number line.

QUE. F

Solve the inequality: $\frac{1}{3}x - \frac{1}{5}(2 + x) \geq x + \frac{7}{3}$

QUE. G

Find the truth set of the inequality: $2 - \frac{1}{3}(2x - 1) > \frac{2}{3}x$

CHAPTER TWENTY 21
RATES I
(AVERAGE SPEED AND CHANGING UNITS)

Rates is an aspect that compares two or more quantities or items which are identified by the units of measurement used.
A simple example of a rate is ***speed*** which compares distance with time.

21.1 **Average Speed**

Average speed is the rate of change of ***distance*** covered to the ***time*** taken.
Mathematically, we say that average speed = $\frac{Total\ distance\ covered}{Time\ taken}$.
It is measured in meters per seconds (m/s) or kilometer per hour km/h

NB

From the above formulae,
Total distance covered = Average speed $\times$ Time taken.
Distance is measured in kilometers (km) or meters (m).
Time taken = $\frac{total\ distance\ covered}{average\ speed}$.
Time is measured in hour (h) or seconds (s).

21.2 **Standard Conversions**

$1\ Minute = 60\ seconds$
$1\ hour = 60\ Minutes$
$1\ hour = 3600\ seconds$
$1\ day = 24\ hours$
$1\ km = 1000m$
$1km = 100{,}000cm$
$1\ m = 100\ cm$
$1\ m = 1000\ mm$

$1\ cm = 10\ mm$
$1 kg = 1000 g$

NB

1. In changing minutes into seconds, we multiply the value by 60
2. In changing hours into minutes, we multiply the value by 60
3. In changing hours into seconds, we multiply the value by 3600

Example 21.1

A cyclist is riding a bicycle with wheels $1\ m$ in diameter. If the wheels make 132 revolutions each minute, find his speed in kilometers per hour. $(take\ \pi = 3{,}142)$

Solution

From question, diameter $= 1\ m$
But Circumference $= \pi d = 3.142 \times 1 = 3.142 m$
(distance round the wheel)
Therefore, $1 revolution = 3.142 m$
Time for 132 revolutions by wheel $= 1 \min = 0.01667 hrs$
Distance covered in each minute $= 132 \times 3.142 = 414.744 m$
But $414.744 m = 0.414744 km$ $(Since\ 1000 m = 1 km)$
Hence, Speed $= \dfrac{distance}{time} = \dfrac{0.414744}{0.01667} = 24.88\ kmh^{-1}$

Example 21.2

Two passenger trains, A and B, $300 km$ apart, start towards each other at the same time. They meet after $2 hours.$. If train B travels $\frac{8}{7}$ as fast as train A, what is the speed of each time?

Solution

Total distance between train A and train B $= 300km$

Let Speed of train A $= y\,kmh^{-1}$

Then Speed of train B $= \frac{8}{7}y\ km^{-1}$

From Speed $= \frac{dis\tan ce}{time}$

$\Rightarrow dis\tan ce = speed \times time$

Thus, distance covered by train A after $2\ hrs = 2\ y\,km$

Distance covered by train B after $2\ hrs = \frac{16}{7}y\,km$

$\Rightarrow 2y + \frac{16}{7}y = 300$

(Since total distance between the two trains $= 300km$)

$14y + 16y = 2100$

$30y = 2100$

$\Rightarrow y = \frac{2100}{30} = 70$

$\Rightarrow Speed\ of\ train\ A\ is\ 70\,kmh^{-1}$

Then, speed of train B is $= \frac{8}{7} \times 70 = 8 \times 10 = 80\,kmh^{-1}$

Example 21.3

A man travelling at $60\,kmh^{-1}$ used $4\,hrs$ to cover a certain distance. How long will he take to cover the same distance travelling at $80\,kmh^{-1}$.

Solution

If $60\,kmh^{-1} = 4\,hrs$

Then, $80\,kmh^{-1} = x\,hrs$

$\Rightarrow x = \frac{80 \times 4}{60} = \frac{320}{60} = 5.3\,hrs$

Hence, he used $5.3\,hrs$ to cover the same distance at $80\,kmh^{-1}$

Example 21.4

An aeroplane leaves Barcelona at $10:10pm$ and reaches Accra $4{,}415km$ away at $5:50am$ the next morning. Find correct to the nearest whole number, the average speed of the aeroplane in kmh^{-1}.

Solution

Total distance $= 4415km$
Total time $= 7hrs$ 40 $mins$.(i.e. from $10:10pm$ to $5:50am$)
But
$1hr = 60\min s$
$\Rightarrow x\,hrs = 40hrs$
$\Rightarrow x = \frac{40}{60} = 0.67 \;or\; \frac{2}{3}$
Therefore, $40\min s = \frac{2}{3}hrs$
Hence, total time $= 7\frac{2}{3}hrs \;\; or \;\; 7.67hrs$
But Speed $= \frac{distance}{time} = \frac{4415}{7.67} = 575.6kmh^{-1}$
Hence, the average speed of the aeroplane is $576kmh^{-1}$

21.3 **Changing Units**

This is the concept of changing from one unit to another.

Example 21.5

Express $13\;meters\;4\;centimeters$ in kilometers.

Solution

First, change $13meters$ to centimeters
Thus, if $100cm = 1m$
$\Rightarrow x\,cm = 13m$
$\therefore x = 13 \times 100 = 1300cm$

Therefore, $13 meters = 1300 cm$

Hence, $1300 + 4 = 1304 cm$

Now, change $1304 cm \ to \ kilometers$

$100,000 cm = 1 km$

$1304 cm = y \ km$

$\Rightarrow y = \frac{1304}{100,000} = 0.01304$

Hence, $13 \ meters \ 4 \ centimeters$ in kilometers is $0.01304 \ km$

Example 21.6

If $1cm^3$ of a metal weighs $7.3g$, express in kilograms the weight of $210 cm^3$ of the metal.

Solution

From question,

$1cm^3 = 7.3g$

Then $210cm^3 = x \ g$

$\Rightarrow x = 7.3 \times 210 = 1533$

Hence $210 cm^3 = 1533 g$

But $1000 g = 1 km$

$\Rightarrow 1533 g = y \ kg$

$y = \frac{1533}{1000} = 1.533 kg$

Therefore the weight of $210 cm^3$ metal in kilograms is $1.533 kg$

Example 21.7

Change $(i) \ 10 m/s \ to \ km/h$ $\qquad (ii) \ 100 km/h \ to \ cm/s$

Solution

$(i) \ 10 m/s \equiv \frac{10m}{1s}$

From $1000m = 1km$

$\Rightarrow 10m = \frac{10}{1000}km = 0.01km$

Again, $1h = 3600s$

$\Rightarrow 1s = \frac{1}{3600}h = 0.0002778h$

$\Rightarrow 10m/s \equiv \frac{10m}{1s} = \frac{0.01km}{0.0002778h} = 35.99712km/h$

Hence, $10m/s = 36km/h$

$(ii)\ 100km/h = \frac{100km}{1h} = \frac{10,000,000cm}{60 \times 60s} = 2778cm/s$

Example 21.8

A tap leaks at a rate of $5cm^3$ per second. How long will it take to fill a container of $30litres$ capacity? $(1litre = 1000cm^3)$

Solution

$1litre = 1000cm^3$

$\Rightarrow 30litres = 30,000cm^3$

Rate is $5cm^3$ per second

Hence, $30,000cm^3$ container can be filled in

$\frac{30,000}{5} = 6000\sec onds = \frac{6000}{3600}hrs = 1.67hrs$

Example 21.9

A car consumes $130litres$ of fuel within a distance of $250km.$ What is the rate of fuel consumption in km/l.

Solution

$km/l = \frac{km}{l} = \frac{dis\tan ce}{volume} = rate$

$\Rightarrow Rate = \frac{250}{130} = 1.9231km/l$

Example 21.10

A uniform rod of length $10m$ has mass $35kg$. What is the mass per metre?

Solution

Mass per metre $= \dfrac{mass}{length} = \dfrac{35}{10} = 3.5km/l$

Example 21.11

A bar of key soap has volume $67\,cm^3$ and a mass of $2kg$. What is its density in grams per cubic centimetre?

Solution

Density $= \dfrac{mass}{volume}$

But $1kg = 1000g \Rightarrow 2kg = 2000g$

Hence, Density $= \dfrac{2000}{67} = 29.85g/cm^3$

EXERCISE

QUE. A

Jude travelled a distance of $12\ km$ on a bicycle at an average speed of $8\ km/h$. How long did it take him to make the trip?

QUE. B

A boy cycles $7.5\ km$ to school in $27\frac{1}{2}$ minutes. Find his average speed in meters per seconds, correct to one decimal place.

QUE. C

The average petrol consumption of a car is $1\ gallon\ per\ 35\ km.$
The cost of $1\ litre$ of petrol is ¢ 747.00. Find the cost of petrol
for a journey of $280\ km$ to the nearest cedi. $(take\ 1\ gallon \equiv 4.55\ litres)$

QUE. D

A train travelling at $105 km/h$ goes through a tunnel $1575m$ long. Calculate in seconds, the time a passenger on the train spends inside the tunnel.

QUE. E

A journey made at an average speed of $40\ m/h$ took $2\frac{1}{2}\ hrs.$ How long would it take if the same journey was made at an average speed of $50\ km/h$?

QUE. F

Express $25m\ 5cm$ in centimeters.

QUE. G

A motorist travelled $40km$ at an average speed of $30km/h.$ If he made the return journey at an average speed of $50km/h.$ find the average speed of the whole journey.

QUE. H

A man covered a distance of $5km\ in\ 20\min s$ on his bicycle. Find his speed in kilometer per hour.

QUE. I

Find the average speed of a car that travels $45km\ in\ 2hrs\ 15\min s.$

QUE. J

A motorist covers a distance of $800m\ in\ 5\min s,$ find his average speed in kilometers per hour.

CHAPTER 22

VARIATIONS

This is an aspect that looks at how quantities, usually variables are connected. Variation questions are usually in two parts: The ***first part*** is the part in which values of all the variables mentioned in the question are given and the ***second part*** being the part in which all the values of the variables are given with the exception of one.

22.1 **Types Of Variations**

1. Direct Variation
2. Inverse Variation
3. Joint Variation
4. Partial Variation

22.1.1 **Direct Variation**

This type of variation is also called ***direct proportion***. Here, two variables are related in such a way that when one increases, the other one also does. Usually, the type of variation is identified by the key words mentioned in the question. For instance, when in a question we see the words directly, inversely, jointly or partly depicts direct variation, inverse variation, joint variation and partial variation respectively.

NB

The symbols $\alpha \text{ and } k$ denotes the proportionality symbol and the constant of proportionality respectively.

Steps In Solving Direct Variations

1. Write down an expression for the statement made by placing the symbol α in between the two variables
2. Replace the symbol α by the equal

3. Multiply the RHS by k
4. Substitute the values of the variables given in the first part of the question and solve for k
5. Substitute the value of k found into the equation formed in step 3 to obtain the expression connecting the given variables
6. Substitute the values of the variables given in the second part into the expression in step 5 and solve for the variable asked of in the question.

Illustration Of The First Three Steps

(i) If ‘a’ varies directly as ‘b’. We write it as:

Step 1

$a \alpha b$

Step 2

$a = b$

Step 3

$a = kb$

(ii) If ‘x’ varies as the square of ‘y’. We write it as:

Step 1

$x \alpha y^2$

Step 2

$x = y^2$

Step 3

$x = ky^2$

(iii) ‘a’ is directly proportional to the cube of ‘b’. We write it as:

Step 1

$a \alpha b^3$

Step 2

$\alpha = b^3$

Step 3

$a = kb^3$

(iv) ‘a’ is directly proportional to the square root of ‘b’. We write it as:

Step 1

$a \, \alpha \, \sqrt{b}$

Step 2

$a = \sqrt{b}$

Step 3

$a = k\sqrt{b}$

(v) 'a' varies directly as the cube root of 'b'. We write it as:

Step 1

$a \, \alpha \, \sqrt[3]{b}$

Step 2

$a = \sqrt[3]{b}$

Step 3

$a = k\sqrt[3]{b}$

Example 22.1

N varies directly as M. When $N = 8,\ M = 20$. Find the value of M when $N = 7$

Solution

Step 1

$N \, \alpha \, M$

Step 2

$N = M$

Step 3

$N = kM$

When $N = 8, M = 20$

$8 = k \times 20$

$\Rightarrow \ 8 = 20k$

$\frac{8}{20} = \frac{20k}{20}$

$\Rightarrow k = \frac{2}{5}$

Step 4

$\therefore N = \frac{2}{5}M$

Step 5

When $N = 7$

$7 = \frac{2}{5}M$

$\Rightarrow \frac{7 \times 5}{2} = \frac{2M}{2}$

$\Rightarrow M = 17.5$

Example 22.2

P varies directly as the square of Q. If $P = 3, Q = 2$.
Find Q when $P = 27$

Solution

$P \alpha Q^2$

$P = kQ^2$

When $P = 3 \ and \ Q = 2$

$3 = k \times 2^2$

$\frac{3}{4} = \frac{4k}{4}$

$\therefore k = \frac{3}{4}$

$\Rightarrow P = \frac{3}{4}Q^2$

Now, when $P = 27$

$\Rightarrow 27 = \frac{3}{4}Q^2$

$\frac{27 \times 4}{3} = \frac{3Q^2}{3}$

$36 = Q^2$

Square root both sides

$\Rightarrow Q = 6$

Example 22.3

If y varies directly as x^2, when $y = 4, x = -1$.
Find y when $x = 3$

Solution

$\Rightarrow y \alpha x^2 \Rightarrow y = kx^2$
$When\ y = 4,\ x = -1$
$\Rightarrow 4 = k \times (-1)^2 \Rightarrow 4 = k$
$\Rightarrow y = 4x^2$
$Again\ when\ x = 3$
$\Rightarrow y = 4 \times 3^2 = 4 \times 9 = 36$
$\therefore y = 36$

Example 14.4

$If\ \ x\alpha\sqrt{y}\ and\ x = 2\ when\ y = 16,\ find\ the\ value\ of\ \ y\ when\ x = 7$

Solution

$x\alpha\sqrt{y} \Rightarrow x = k\sqrt{y}$
$When\ x = 2,\ y = 16$
$\Rightarrow 2 = k \times \sqrt{16}$
$2 = k \times 4 = 4k$
$\Rightarrow k = \frac{1}{2}$
$Hence,\ x = \frac{1}{2}\sqrt{y}$
$Again,\ when\ x = 7$
$7 = \frac{1}{2} \times \sqrt{y}$
$\Rightarrow 2 \times 7 = \sqrt{y}$
$14 = \sqrt{y}\ Square\ both\ sides$
$196 = y$

Example 14.5

x varies directly as the cube root of y. When $x = 5,\ y = 125.$
Find x when $y = 8$

Solution

$x \alpha \sqrt[3]{y}$
$\Rightarrow x = k\sqrt[3]{y}$
When $x = 5$ *and* $y = 125$
$\Rightarrow 5 = k \times \sqrt[3]{125}$
$5 = k \times 5$
$\frac{5}{5} = \frac{5k}{5}$
$\Rightarrow k = 1$
Hence, $x = \sqrt[3]{y}$
When $y = 8$
$x = \sqrt[3]{8} = 2$

NB

Sometimes, a table will be given with all the values of the first variable in the first row given and some of the values of the second variable in the second row also given.
To solve such a question, we follow the steps below:
1. Write an expression for the statement by placing the symbol α in between the variables
2. Replace the symbol by the equal sign and multiplying the RHS by k
3. Locate a column in the table with values of both variables given and substitute these values into the relation in step 2 and solve for the value of k
4. Fill in each empty box by substituting the value of the variable given in the first row that corresponds to the empty box and the value of k into the relation in step 2

Example 22.5

y varies directly as the square of x. The table below shows the values of x for some selected values of y.

x	2	3	5	6
y	12		75	

(i) Find the value of the constant of proportionality and write down an expression connecting y and x

(ii) Copy and complete the table

Solution

$y \alpha x^2$

$y = kx^2$

Picking the first column, $x = 2$ *and* $y = 12$

(i)

$\Rightarrow 12 = k \times 2^2$

$12 = 4k$

$\Rightarrow k = 3$

$\Rightarrow y = 3x^2$

(ii) When $x = 3$

$y = 3(3^2) = 3 \times 9 = 27$

When $x = 6$

$y = 3(6^2) = 3 \times 36 = 108$

Hence, the completed table

x	2	3	5	6
y	12	27	75	108

22.1.2 **Inverse Variation**

Here, two variables are related in such a way that as one increases, the other one decreases.

We note that he inverse of any number or variable is obtained by dividing one (1) by the number or variable. Thus, the inverse of $2 \ is \ \frac{1}{2}$. Likewise, the inverse of x is $\frac{1}{x}$.

Steps In Solving Inverse Variations

1. Write down an expression for the statement made by placing the symbol α in between the two variables
2. Replace the symbol α by the equal sign
3. Multiply the RHS by k
4. Substitute the values of the variables given in the first part of the question and solve for k
5. Substitute the value of k found into the equation formed in step 3 to obtain the expression connecting the given variables
6. Substitute the values of the variables given in the second part into the expression in step 5 and solve for the variable asked of in the question.

Illustration Of The First Three Steps

(i) If ‘a’ varies inversely as ‘b’. We write it as:

Step 1

$a \ \alpha \frac{1}{b}$

Step 2

$a = \frac{1}{b}$

Step 3

$a = \frac{k}{b}$

(ii) If ‘x’ varies inversely as the square of ‘y’. We write it as:

Step 1

$x \ \alpha \frac{1}{y^2}$

Step 2

$x = \frac{1}{y^2}$

Step 3

$x = \frac{k}{y^2}$

(iii) 'a' is inversely proportional to the cube of 'b'. We write it as:

Step 1

$a \propto \frac{1}{b^3}$

Step 2

$\alpha = \frac{1}{b^3}$

Step 3

$a = \frac{k}{b^3}$

(iv) 'a' is inversely proportional to the square root of 'b'. We write it as:

Step 1

$a = \frac{1}{\sqrt{b}}$

Step 2

$a = \frac{1}{\sqrt{b}}$

Step 3

$a = \frac{k}{\sqrt{b}}$

(v) 'a' varies inversely as the cube root of 'b'. We write it as:

Step 1

$a \propto \frac{1}{\sqrt[3]{b}}$

Step 2

$a = \frac{1}{\sqrt[3]{b}}$

Step 3

$a = \frac{k}{\sqrt[3]{b}}$

Example 22.6

y varies inversely as x. When $y = 8, x = 3$

(i) Find a relation between y and x

(ii) Find the value of x when $y = 12$

Solution

$y \alpha \frac{1}{x}$

$y = \frac{k}{x}$

When $y = 8, x = 3$

$\Rightarrow 8 = \frac{k}{3}$

$24 = k$

(i) $y = \frac{24}{x}$

(ii) When $y = 12$

$$12 = \frac{24}{x}$$

$$\Rightarrow x = \frac{24}{12} = 2$$

Example 22.7

If y is inversely proportional to $(x+2)$ and $y = 48$ when $x = 10$, find x when $y = 30$

Solution

$$y \alpha \frac{1}{x+2}$$

$$\Rightarrow y = \frac{k}{x+2}$$

When $y = 48, x = 10$

$$\Rightarrow 48 = \frac{k}{10+2} = \frac{k}{12}$$

$$48(12) = k$$

$$\therefore k = 576$$

$$\Rightarrow y = \frac{576}{x+2}$$

When $y = 30$

$$\Rightarrow 30 = \frac{576}{x+2}$$

$$30(x+2) = 576$$

$$30x + 60 = 576$$

$$30x = 576 - 60$$

$$\frac{30x}{30} = \frac{516}{30}$$

$$\Rightarrow x = 17.2$$

Example 22.8

P varies inversely as the square of $(Q+1)$ and P is 2 when Q is 3.
(a) Write an equation connecting P and Q
(b) Find the possible values of Q when $P = 8$

Solution
(a)

$$P \alpha \frac{1}{(Q+1)^2}$$

$$P = \frac{k}{(Q+1)^2}$$

When $P = 2, Q = 3$

$$\Rightarrow 2 = \frac{k}{(3+1)^2}$$

$$2 = \frac{k}{4^2}$$

$$k = 32$$

$$P = \frac{32}{(Q+1)^2}$$

(b) When $P = 8$

$$\Rightarrow 8 = \frac{32}{(Q+1)^2}$$

$$8(Q^2 + 2Q + 1) = 32$$

Divide through by 8

$$\Rightarrow Q^2 + 2Q + 1 = 4$$

$$Q^2 + 2Q - 3 = 0$$

$$Q^2 - Q + 3Q - 3 = 0$$

$$Q(Q-1) + 3(Q-1) = 0$$

$$\Rightarrow (Q-1)(Q+3) = 0$$

Either

$$Q - 1 = 0 \quad \Rightarrow Q = 1$$

Or

$$Q + 3 = 0 \quad \Rightarrow Q = -3$$

NB

Similarly, a table will be given with all the values of the first variable in the first row given and some of the values of the second variable in the second row also given. To solve such a question, we follow the steps below:

1. Write an expression for the statement by placing the symbol α in between the variables
2. Replace the symbol by the equal sign and multiply the RHS by k
3. Locate a column in the table with values of both variables given and substitute these values into the relation in step 2 and solve for the value of k
4. Fill the empty boxes by substituting the value of k and the corresponding value given in the table into the relation in step 2

Example 22.9

P varies inversely as Q. The table below shows the value of Q for some selected values of P

Q	6	8	9	12
P	24			12

(a) Find the constant of proportionality
(b) Copy and complete the table

Solution
(a)
Step 1
$P\alpha\frac{1}{Q}$
Step 2
$\Rightarrow P=\frac{k}{Q}$
Step 3
Considering column 1, $P=24\ and\ Q=6$
$24=\frac{k}{6}$
$\Rightarrow k=144$
$\Rightarrow P=\frac{144}{Q}$
When $Q=8$
$P=\frac{144}{8}=18$
When $Q=9$
$\Rightarrow P=\frac{144}{9}=16$

(b) The completed table

Q	6	8	9	12
P	24	18	16	12

Example 22.10

If P is inversely proportional to Q and P=24 when Q=8, find P when Q=12.

Solution

$$Step\ 1$$

$$P\alpha\frac{1}{Q}$$

$$Step\ 2$$

$$\Rightarrow P=\frac{k}{Q}$$

$$Step\ 3$$

$$When\ \ P=24,\ Q=8\ \ substituting\ \ to\ \ get;$$

$$24=\frac{k}{8}\Rightarrow 24\times 8=k\Rightarrow k=192$$

$$\Rightarrow Equation\ \ becomes\ \ P=\frac{192}{Q}----(1)$$

$$Now,\ substitute\ \ Q=12\ \ \text{int}\, o\ \ (1)$$

$$\Rightarrow P=\frac{192}{12}=16$$

Example 22.11

The table below shows the corresponding values of x and y for a relation.

x	4	16	36
y	1	2	3

Using the table, find the value of k if y and x are connected by the equation: $y=\frac{\sqrt{x}}{k}$

Solution

$Picking\ correspond\imath ng\ x = 4\ and\ y = 1\ from\ table\ and\ substitute\ \text{int}o\ equation$

$1 = \frac{\sqrt{4}}{k} \Rightarrow k = 2$

$Hence,\ new\ equation\ is\ y = \frac{\sqrt{x}}{2}$

22.1.3 **Joint Variation**

Here, one variable depends on two or more variables.
In short, joint variation is the combination of either two direct variations or a combination of a direct variation and an inverse variation.

Steps In Solving Joint Variation Questions

1. Write an expression for each of the types of variations mentioned in the question by placing the proportionality symbol α appropriately.
2. Combine the two variations by multiplying their RHSs
3. Replace the α by the equal sign and multiply the RHS by 'k'
4. Substitute the values of the variables given in the first part of the question into the equation in step 3 and find the value of 'k'
5. Put back value of 'k' found into the expression in step 3 to get an expression connecting the variables mentioned in the question
6. Compute the value of the variable asked of in the question by putting back the value of 'k' and the value of the variables given in the second part of the question

Illustration Of First Three Steps

(i) If z varies directly as $x\ and\ y$. We write it as:

Step 1

$z \alpha x \quad and \quad z \alpha y$

Step 2

$z \alpha xy$

Step 3

$z = kxy$

(ii) If y varies directly as x and inversely as q. We write it as:

Step 1

$y \alpha x \ \ and \ \ y \alpha \frac{1}{q}$

Step 2

$y \alpha \frac{x}{q}$

Step 3

$y = \frac{kx}{q}$

(iii) z varies directly as x and inversely as the square of y

Step 1

$z \ \alpha \ x \ \ and \ \ z \ \alpha \ \frac{1}{y^2}$

Step 2

$z \alpha \ x\left(\frac{1}{y^2}\right) \Rightarrow z \ \alpha \ \frac{x}{y^2}$

Step 3

$z = \frac{kx}{y^2}$

(iv) P varies directly as Q and inversely as the cube root of R.
We write it as:

Step 1

$P \ \alpha \ Q \ \ and \ P \ \alpha \frac{1}{\sqrt[3]{R}}$

Step 2

$y \alpha \frac{Q}{\sqrt[3]{R}}$

Step 3

$y = \frac{Q}{\sqrt[3]{R}}$

(v) p varies directly as the square of q and inversely as r. Express p in terms of q, r and k

Step 1

$p \alpha q^2 \;\; and \;\; p \alpha \frac{1}{r}$

Step 2

$p \alpha \frac{q^2}{r}$

Step 3

$p = \frac{kq^2}{r}$

Example 22.12

P is directly proportional to R and inversely proportional to the square root of Q. If $P = 6,\ R = 10 \;\; and \;\; Q = 16$. Find P when $R = 5 \;\; and \;\; Q = 4$

Solution

Step 1

$P \alpha R \;\; and \;\; P \alpha \frac{1}{\sqrt{Q}}$

Step 2

$P \alpha \frac{R}{\sqrt{Q}}$

Step 3

$P = \frac{kR}{\sqrt{Q}}$

When $P = 6, R = 10 \ and \ Q = 16$

Step 4

$$6 = \frac{k \times 10}{\sqrt{16}}$$

$$6 = \frac{10k}{4}$$

$$24 = 10k$$

$$\therefore k = \frac{12}{5}$$

Step 5

Hence, $P = \dfrac{12R}{5\sqrt{Q}}$

Step 6

When $R = 5 \ and \ Q = 4$

$$P = \frac{12 \times 5}{5\sqrt{4}}$$

$$P = \frac{60}{5 \times 2} = \frac{60}{10} = 6$$

Example 22.13

x varies directly as the square root of t and inversely as s.
Where $x = 4, t = 9 \ and \ s = 18$

(i) Express x in terms of $s \ and \ t$

(ii) Find x when $t = 81 \ and \ s = 27$

Solution

Step 1

$x \, \alpha \sqrt{t} \quad and \quad x \alpha \dfrac{1}{s}$

Step 2

$$y \alpha \frac{\sqrt{t}}{s}$$

Step 3

$$x = \frac{k\sqrt{t}}{s}$$

Step 4

When $x = 4, t = 9$ *and* $s = 18$

$$4 = \frac{k\sqrt{9}}{18}$$

$$72 = 3k$$

$$\therefore k = 24$$

Step 5

(i) *Hence,* $x = \dfrac{24\sqrt{t}}{s}$

Step 6

(ii) When $t = 81$ *and* $s = 27$

$$x = \frac{24 \times \sqrt{81}}{27}$$

$$x = \frac{24 \times 9}{27}$$

$$x = 8$$

Example 22.14

x, y *and* z are such that x varies directly as z and inversely as the cube root of y.

If $x = 8, y = 27$ *and* $z = 4$. Find (a) an expression for x in terms of y *and* z

(b) the value of y when $x = 12,$ *and* $z = 10$

Solution

Step 1

$$x \alpha z \quad and \quad x \alpha \frac{1}{\sqrt[3]{y}}$$

Step 2

$$x \alpha \frac{z}{\sqrt[3]{y}}$$

Step 3

$$x = \frac{kz}{\sqrt[3]{y}}$$

Step 4

When x=8, y=27 and z=4

$$\Rightarrow 8 = \frac{k \times 4}{\sqrt[3]{27}} = \frac{4k}{3} \Rightarrow 8 \times 3 = 4k \Rightarrow \frac{24}{4} = \frac{4k}{4} \Rightarrow k = 6$$

Step 5

(a)

$$x = \frac{6z}{\sqrt[3]{y}}$$

Step 6

(b)

$$12 = \frac{6(10)}{\sqrt[3]{y}} = \frac{60}{\sqrt[3]{y}} \Rightarrow 12(\sqrt[3]{y}) = 60$$

$$\Rightarrow \sqrt[3]{y} = \frac{60}{12} = 5$$

Cube both sides

$$\Rightarrow y = 5^3 = 125$$

NB

Similarly, a table can be given with all the values of the first variable in the first row given and some of the values of the second variable in the second row also given. To solve such a question, we follow the steps below:

1. Write an expression for the statement by placing the symbol α in between the variables
2. Replace the symbol by the equal sign and multiply the RHS by k
3. Locate a column in the table with values of both variables given and substitute these values into the relation in step 2 and solve for the value of k
4. Fill the empty boxes by substituting the value of k and the corresponding value given in the table into the relation in step 2

Example 22.15

In the table below, $W \alpha \frac{Q}{R^2}$, where W, R and Q are positive integers.

Solve for w_2 and r_3

W	R	Q
3	4	4
w_2	1	2
8	r_3	6

Solution

Step 1 & 2

$W \alpha \frac{Q}{R^2}$

Step 3

$W = \frac{kQ}{R^2}$

Step 4

When $W = 3, Q = 4 \ and \ R = 4$

Implies, $3 = \frac{k \times 4}{4^2}$

$3 = \frac{4k}{16}$

$3 = \frac{k}{4}$

$\Rightarrow k = 12$

Step 5

Hence,

$W = \frac{12Q}{R^2}$

Step 6

Now when $W = w_2, R = 1 \ and \ Q = 2$

$w_2 = \frac{12 \times 2}{1^2} = 24$

Again, when $W = 8, R = r_3 \ and \ Q = 6$

$$8 = \frac{12 \times 6}{r_3^{\ 2}}$$

$$r_3^{\ 2} = \frac{12 \times 6}{8} = 9$$

$$\Rightarrow r_3 = 3$$

22.1.4 **Partial Variation**

Here, we shall deal with variations connecting two variables which consist of several parts or which are sums of variations.
In short, Partial variation is simply the sum of a ***constant*** (usually called the constant part) and ***one*** or ***two*** variations. Usually, partial variation questions give two set of values of all the variables mentioned of in the question. In solving such variations, we follow the steps below:

1. Represent the constant part by k_1 and the constant of proportionality by k
2. Write out expression (s) for the various types of variation (s) mentioned of in the question following their appropriate steps to get a combined equation
3. Add the constant part, k_1 to the result of step 2
4. Substitute the first and second set of values of all the variables into the result of step 3 and simplify into two simultaneous equations
5. Solve the two equations in step 4 simultaneously for values of $k_1 \ and \ k$
6. Substitute back the values of the constants $k_1 \ and \ k$ into the result of step 3 to obtain the expression connecting the variables
7. Trace the question to where you are required to find one of the variables and substitute values of the variables given, then solve for the value of the variable asked of in that part

Illustrations Of The First Three Steps

(i) If y is partly a constant and partly varies directly as x.
We write it as:

Step 1

Let $k_1 = Constant\ part\ and\ k = constant\ of\ proportionality$

Step 2

Since involves direct variation only, we have;

$y \alpha x$

$y = kx$

Step 3

$y = k_1 + kx$

(ii) If y is partly a constant and partly varies inversely as x.
We write it as:

Step 1

Let $k_1 = Constant\ part\ and\ k = constant\ of\ proportionality$

Step 2

Since involves only inverse variation, we have;

$y \alpha \frac{1}{x}$

$y = \frac{k}{x}$

Step 3

$y = k_1 + \frac{k}{x}$

(iii) If z is partly a constant and partly varies jointly as $x\ and\ y$. We write it as:

Step 1

Let k_1= constant part and k be the constant of proportionality

Step 2

$z \alpha x \quad and \quad z \alpha y$

$\Rightarrow z \alpha xy$

$\Rightarrow z = kxy$

Step 3

$z = k_1 + kxy$

Example 22.16

A variable y is partly a constant and partly varies directly as the square of x. When $x = 2, y = 6$ *and when* $x = 3, y = 10$

(i) Find the equation connecting x *and* y

(ii) What is the value of y when $x = 4$?

(iii) Find x when $y = 5$

Solution

Step 1

Let $k_1 = Constant\ part\ and\ k = constant\ of\ proportionality$

Step 2

$y \alpha x^2$

$\Rightarrow y = kx^2$

Step 3

$y = k_1 + kx^2$

Step 4 & 5

When $x = 2, y = 6$

$\Rightarrow 6 = k_1 + k \times 2^2$

$6 = k_1 + 4k$

$\Rightarrow k_1 + 4k = 6 - - - - - - - - (1)$

Again *when* $x = 3, y = 10$

$10 = k_1 + k \times 3^2$

$10 = k_1 + 9k$

$\Rightarrow k_1 + 9k = 10 - - - - - - - - (2)$

Solve (1) and (2) simultaneously

$(2) - (1)$

$\Rightarrow 5k = 4$

$\therefore k = \frac{4}{5}$

Put $k = \frac{4}{5}$ int$o$ (2)

$\Rightarrow k_1 + 9 \times \frac{4}{5} = 10$

$\Rightarrow 5k_1 + 36 = 50$

$5k_1 = 50 - 36 = 14$

$\frac{5k_1}{5} = \frac{14}{5}$

$\therefore k_1 = \frac{14}{5}$

Step 6

(i) $y = \frac{14}{5} + \frac{4}{5}x^2$

Step 7

(ii) When $x = 4$

$y = \frac{14}{5} + \frac{4}{5} \times 4^2 = \frac{14}{5} + \frac{64}{5}$

$y = \frac{78}{5}$

(iii) When $y = 5$

$\Rightarrow 5 = \frac{14}{5} + \frac{4}{5}x^2$

$25 = 14 + 4x^2$

$25 - 14 = 4x^2$

$11 = 4x^2$

$\Rightarrow x^2 = \frac{11}{4}$

$\therefore x = \sqrt{\frac{11}{4}}$

Example 22.17

The cost, C of manning a household is partly a constant and partly varies as the number, n of people in the house.
For 8 people, the cost is ¢ 70,000 and 10 people, the cost is ¢ 90,000. Find
(i) an expression for C in terms of n

(ii) the weekly cost for 12 people

Solution

Let $k_1 = Constant\ part\ and\ k = constant\ of\ proportionality$

$C \alpha n \quad \Rightarrow C = kn$

$C = k_1 + kn$

When $n = 8, C = 70000$

$\Rightarrow 70000 = k_1 + k \times 8$

$70000 = k_1 + 8k$

$\Rightarrow k_1 + 8k = 70000 - - - - - - - - (1)$

When $n = 10, C = 90000$

$\Rightarrow 90000 = k_1 + 10k$

$k_1 + 10k = 90000 - - - - - - - - (2)$

$(2) - (1)$

$2k = 20000$

$\Rightarrow k = 10000$

Put $k = 10000$ into (1)

$\Rightarrow k_1 + 8 \times 10000 = 70000$

$k_1 + 80000 = 70000$

$k_1 = 70000 - 80000 = -10000$

$(i)\ C = -10000 + 10000n$

(ii) When $n = 12$

$\Rightarrow C = -10000 + 10000(12)$

$C = -10000 + 120000 = 110000$

Example 22.18

The cost, C of producing a motor car in a certain factory is partly constant and partly varies inversely as the number n of cars produced per day.If the cost of production is 1,600 dollars when 4 cars are produced per day and 1,420 dollars when 5 cars are produced per day. Find the rate of production to bring the cost down to 1,150 dollars per car.

Solution

Let $k_1 = Constant\ part\ \ and\ \ k = constant\ of\ \ proportionality$

$C \alpha \frac{1}{n}$

$C = \frac{k}{n}$

$C = k_1 + \frac{k}{n}$

When $n = 4, C = 1600$

$\Rightarrow 1600 = k_1 + \frac{k}{4} - - - - - - - - (1)$

When $n = 5, C = 1420$

$\Rightarrow 1420 = k_1 + \frac{k}{5} - - - - - - - - (2)$

$(1) - (2)$

$180 = \frac{k}{4} - \frac{k}{5} = \frac{k}{20}$

$\Rightarrow k = 180 \times 20 = 3600$

Put $k = 3600$ into (2)

$\Rightarrow 1420 = k_1 + 720$

$\Rightarrow k_1 = 1420 - 720 = 700$

$\therefore C = 700 + \frac{3600}{n}$

When $C = 1150$

$\Rightarrow 1150 = 700 + \frac{3600}{n}$

$1150n = 700n + 3600$

$1150n - 700n = 3600$

$\frac{450n}{450} = \frac{3600}{450}$

$n = 8$

EXERCISE

QUE. A

The Resistance, R to the motion of a car is partly a constant and partly varies as the square of the speed, V. When the car is moving at $30kmh^{-1}$. The resistance is $630N\ and\ at\ 50kmh^{-1}$, the resistance is $950N$. Find

(i) an expression for R in terms of V

(ii) the resistance at $80kmh^{-1}$

QUE. B

The force of attraction, F between two bodies varies directly as the product of their masses, m and M and inversely as the square of the distance, d between them. Given that

$F = 20N\ when\ M = 25kg, m = 10kg\ and\ d = 5m,\ find$

(a) an expression for F in terms of M, m and d

(b) the distance, d when $F = 30N\ M = 7.5kg, m = 4kg$

QUE. C

The frequency of vibration, f of a stretched wire is directly proportional to the square root of the length, l and inversely proportional to the square root of the tension, T in the wire. Express T in terms of f, l and k where k is a constant.

QUE. D

Given that the cost GH ¢ C of digging a hole depends on the depth covered, d m and the number of working hours, x hrs such that $C = mx + nd,\ where\ m\ and\ n$ are constants. Using the data below, find C when x was $21hrs\ and\ d = 14m$

C / GH ¢	x / hrs	d / m
1500	9	4
1000	5	3

QUE. E

P varies inversely as the square of Q. When $Q = 4, P = 9,$ find the value of P when $Q = 9$

QUE. F

X is inversely proportional to Y. If
$X = -2 \ when \ Y = 3, \ find \ X \ when \ Y = -5$

QUE. G

Given that:
$R \alpha \frac{S^2}{\sqrt{T}} \ and \ R = 3 \ when \ T = 4 \ and \ S = 4, \ find \ R \ when \ S = 2 \ and \ T = 81$

QUE. H

$If \ y \alpha x^3 \ and \ y = 54 \ when \ x = 3, \ find \ y \ when \ x = 1.5 \ and \ obtain \ an \ expression \ for \ y \ in \ terms \ of \ x.$

QUE. I

The area A of a sector, containing a specified angle, varies as the square of its radius r. when r is 2cm, A is $3.6cm^2$.

(a) Find A when r is 7cm
(b) Find r when A is $8.1cm^2$

QUE. J

The weight of a metal ball varies directly as the cube of its diameter. A metal ball of radius 2.5cm weighs 75N. find the weight of a ball of the same metal but of radius 5cm.

QUE. K

If y varies inversely as the square of x, and y=0.25 when x=2, find (a) the constant of variation (b) y when x=4

QUE. L

The variables u, v and w are related by a joint variation. When u=2 and v=3, w=12. Find w when u=3, v=6 if

$(a)\ w\alpha uv \quad (b)\ w\alpha\frac{u}{v} \quad (c)\ w\alpha uv^2 \quad (d)\ w\alpha\frac{u}{v^2}$

QUE. M

If x varies jointly as y^2 and z^3, and x=18 when y=3 and z=6, find

(a) x in terms of y and z
(b) the value of x when y=2 and z=4

QUE. N

The variable q varies directly as r^2 and inversely as s. when r=1.5 and s=4, q=10.

(a) Find the constant of variation
(b) Find the value of q when r=3 and s=8

QUE. O

Suppose y varies directly as x andt, and inversely as the cube of s. Given that y=50 when x=8, t=5 and s=4, find

(a) The constant of variation
(b) The value of y when x=9, t=3 and s=6.

QUE. P

The variable z varies directly as the square root of x and inversely as the square of y. find z when x=36 and y=8 if z=18 when x=9 and y=5.

QUE. Q

Y varies partially as x and is partly constant. When x=0, y=1 and when x=1, y=0.

(a) Find the constants of variation in the relation between x and y.
(b) Calculate the value of y when x=2.

QUE. R

s varies partly as t and partly as the inverse of t.
when t=1, s=3 and when t=2, s=4.5

(a) Find s when t=5
(b) Find t when $s = 2\sqrt{2}$

QUE. S

If y varies directly as x and y is 7 when x is 5, find y when x is -3

QUE. T

If w varies inversely as z and w is 6 when z is 2, find w when z is -8

QUE. U

If A varies jointly as F and T and A is 6 when F is $3\sqrt{2}$ and T is 4, find A when F is $2\sqrt{2}$ $and\ T = \frac{1}{2}$.

QUE. V

If D varies directly with t and inversely with the square of s and D=12.35 when t=2.8
And s=248, find D when t=5.63 and s=6.81.

QUE. W

If y varies directly as m and y is -3 when m is $\frac{1}{4}$,
find y when m is -2.

QUE. X

If c varies directly as m and inversely as n and c is 20 when m is 10 and n is 4, find c when m is 6 and n is -3.

CHAPTER 23

TRANSFORMATION AND COORDINATES

Transformation is an area applied to plane figures. Whenever a plane figure undergoes a transformation, it may change in size, shape or position. A plane figure is transformed by transforming its vertices.

23.1 **Types Of Transformations**

1. Reflection
2. Rotation
3. Translation
4. Enlargement
5. Special mapping

NB

Reflection, Rotation and Translation are said to be rigid motions since under such transformations the size and shape does not change. But Enlargement is not a rigid motion since the size of the figure transformed changes under Enlargement.

23.1.1 **Reflection**

Reflecting a line places the image point at the opposite side of the line and we call this line the ***mirror line*** or the ***line of reflection***.

Types Of Reflection

Reflection In The x - axis or Reflection In The line y = 0

In reflecting a point in the x-axis or in the line y = 0, we negate the y-component of the point. Thus, if (x, y) is a point, then a reflection in the x-axis has the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ -y \end{pmatrix}$ *or* $(x, y) \rightarrow (x, -y)$

Example 23.1

Find the image of the point $(2,3)$ under a reflection in the x-axis.

Solution

From $(x,y) \to (x,-y)$, we negate the y-value of the given point

$\Rightarrow (2,3) \to (2,-3)$

Example 23.2

Find the image of the point $(5,-2)$ under a reflection in the line $y = 0$

Solution

From $(x,y) \to (x,-y)$, we negate the y-value of the given point

Thus, $\Rightarrow (5,-2) \to (5,2)$

Reflection In The y- axis or Reflection In The line x = 0

In reflecting a point in the y-axis or in the line x = 0, we negate the x - component of the point. Thus, if (x, y) is a point, then a reflection in the

y - axis has the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ y \end{pmatrix}$ or $(x,y) \to (-x,y)$

Example 23.3

Find the image of the point $(4,7)$ under a reflection in the y – axis.

Solution

From $(x,y) \to (-x,y)$, we negate the x-value of the given point

$\Rightarrow (4,7) \to (-4,7)$

Example 23.4

Find the image of the point $(2,0)$ under a reflection in the line $x = 0$

Solution

From $(x, y) \to (-x, y)$, we negate the x-value of the given point
$(2,0) \to (-2,0)$

Reflection In The Line x = a or x - a = 0

In reflecting a point say, (x, y) in the line x = a, we use the mapping:
$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2a - x \\ y \end{pmatrix}$ *or* $(x, y) \to (2a - x, y)$

Example 23.5

Find the image of the point $(2,0)$ under a reflection in the line $x = 3$

Solution

Using the rule $(x, y) \to (2a - x, y)$ where a=3
$\Rightarrow (2,0) \to (2(3) - 2,\ 0) = (6 - 2, 0) = (4,0)$

Example 23.6

What is the image of the point $(2, 3)$ under a reflection in the line $x + 3 = 0$?

Solution

Using the rule $(x, y) \to (2a - x, y)$ where a = -3
$\Rightarrow (2,3) \to (2(-3) - 2,\ 3) = (-6 - 2, 3) = (-8, 3)$

Reflection In The Line y = a or y - a = 0

In reflecting a point say, (x, y) in the line y = a, we use the mapping:
$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ 2a - y \end{pmatrix}$ or $(x, y) \rightarrow (x, 2a - y)$

Example 23.7

Find the image of the point $(1, 5)$ under a reflection in the line $y = 3$

Solution

Using the rule $(x, y) \rightarrow (x, 2a - y)$ where a=3
$\Rightarrow (1,5) \rightarrow (1, 2(3) - 5) = (1, 6 - 5) = (1,1)$

Reflection In The Line y = x or y - x = 0

In reflecting a point say, (x, y) in the line y = x or y – x = 0, we interchange the values of x and y. Thus, $(x, y) \rightarrow (y, x)$

Example 23.8

Find the image of the point $(1,2)$ under a reflection in the line y = x

Solution

From $(x, y) \rightarrow (y, x)$ we interchange x and y values
$(1, 2) \rightarrow (2, 1)$

Reflection In The Line y = - x or y + x = 0

In reflecting a point say, (x, y) in the line y = -x or y + x = 0, we interchange the values of x and y and negate the resulting point. Thus, $(x, y) \rightarrow (-y, -x)$

Example 23.9

Find the image of the point $(2,3)$ under a reflection in the line y = - x

Solution

Using, $(x, y) \rightarrow (-y, -x)$ we interchange x and y and after negate both values $\Rightarrow (2,3) \rightarrow (-3,-2)$

23.1.2 **Rotation**

Generally, all rotations are done in the anticlockwise direction. Even if its not indicated in a particular question, it is assumed to be in the anticlockwise direction.

Types Of Rotations

Rotation Through 90^{o} Anticlockwise About The Origin

A rotation through 90^{o} anticlockwise about the origin is the same as a rotation through 270^{o} clockwise about the origin.

In rotating a point through 90^{o} anticlockwise about the origin, we interchange x and y after which we negate the resulting x component.

Thus, the image of the point (x, y) is obtained using the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$ or $(x, y) \rightarrow (-y, x)$

Example 23.10

What is the image of the point $(7,-5)$ under a rotation through 90^{o} anticlockwise about the origin?

Solution

$(7,-5) \rightarrow (5,7)$

Rotation Through 180^{o} Anticlockwise About The Origin

In rotating a point through 180^{o} about the origin, we negate the values of x and y.

Thus, the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \end{pmatrix}$ *or* $(x, y) \rightarrow (-x, -y)$

Example 23.11

What is the image of the point $(7, -5)$ under a rotation through 180^{o} about the origin?

Solution

$(7, -5) \rightarrow (-7, 5)$

Rotation Through 270^{o} Anticlockwise About The Origin

Rotation through 270^{o} anticlockwise about the origin is the same as rotation through 90^{o} clockwise about the same origin. By this rule, we interchange the values of x and y after which we negate the resulting y component.

Thus, the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} y \\ -x \end{pmatrix}$ *or* $(x, y) \rightarrow (y, -x)$

Example 23.12

What is the image of the point $(9, -3)$ under a rotation of 270^{o} anticlockwise about the origin?

Solution

$(9, -3) \rightarrow (-3, -9)$

Rotation About A point Other Than The Origin

In rotating a point say, (x, y) about the point (a,b), we follow the steps below:

1. Subtract the centre (the second point) from the point (the first point) to be rotated. ie. $\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}$
2. Apply the required rotation rule to the result of step 1
3. Add the result of step 2 to the centre of rotation to obtain the needed image

Example 23.13

Rotate the point $(2,3)$ through 270^{o} anticlockwise about the point $(1,2)$

Solution

Step 1

$\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad or \quad (1,1)$

Step 2

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Step 3

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad or \quad (2,1)$

23.1.3 **Translation**

A point is translated by a vector called the ***translation vector.*** Generally, translation is done by adding the vector to the given point.

Thus, if the point (x, y) is translated by the vector (a,b), we have the general mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$

Mathematically, we write: **Image = point + translation vector**

Example 23.14

Find the image of the point $(2,3)$ if it is translated by the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Example 23.15

If $B^1(2,2)$ is the image of the point B under a translation by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the point B

Solution

From the relation: **Image = point + translation vector,**

Let $B = (a,b)$

$$\Rightarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore B = (1,0)$$

Example 23.16

$A^1(2,3)$ is the image of the point $A(1,2)$ under a translation vector V. Find V.

Solution

From: **Image = point + translation vector,**

$$\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}+V$$

$$\Rightarrow\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}=V$$

$$\begin{pmatrix}1\\1\end{pmatrix}=V$$

Example 23.17

P^1 is the image of $P(2,1)$ under a translation which maps $Q(3,4)$ *onto* $Q^1(7,6)$. Find the coordinates of P^1

Solution

Let translation vector be $\begin{pmatrix}a\\b\end{pmatrix}$ and using

$point=(3,4)$ *and* $image=(7,6)$

$$\begin{pmatrix}7\\6\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}7\\6\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

$$\therefore P^1=\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}6\\3\end{pmatrix}\ or\ (6,3)$$

23.1.4 **Enlargement**

Whenever a plane figure is enlarged, its size changes. Generally, in enlarging any given plane figure from the ***origin*** by a ***scale factor***, we multiply the scale factor by its vertices. Thus, if the point (x, y) is enlarged from the origin by a scale factor ***k***, the image is obtained using the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \to k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ or (kx, ky)

Example 23.18

Find the image of the point $(2,3)$ under the enlargement from the origin with scale factor -3.

Solution

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \to -3\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \end{pmatrix} \quad or \quad (-6, -9)$$

Enlargement From A Point Other Than The Origin

Whenever a given point, say (x, y) is enlarged from another point, (a,b) other than the origin, with a scale factor, we follow the steps below to find its image:

Step 1

Subtract (a,b) from the point to be enlarged (x, y). i.e. $\begin{pmatrix} x-a \\ y-b \end{pmatrix}$

Step 2

Enlarge the result of step 1 using the given scale factor.

i.e. $k\begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} k(x-a) \\ k(y-b) \end{pmatrix}$

Step 3

Add (a,b) to the result of step 2 for the required image

i.e. $\begin{pmatrix} k(x-a) \\ k(y-b) \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$

Example 23.19

Find the image of the point $(3,2)$ under an enlargement from $(1,4)$ with scale
factor 5.

Solution

Step 1
$\begin{pmatrix}3\\2\end{pmatrix}-\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}2\\-2\end{pmatrix}$
Step 2
$5\begin{pmatrix}2\\-2\end{pmatrix}=\begin{pmatrix}10\\-10\end{pmatrix}$
Step 3
$\begin{pmatrix}10\\-10\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}11\\-6\end{pmatrix}$ *or* $(11,-6)$

23.2 **Special Mappings**

These are rules used for finding the images of given points. The components of the point are substituted into the given mapping to get the image.

Example 23.20

Find the image of the point $(2,3)$ under the mapping: $\begin{pmatrix}x\\y\end{pmatrix}\rightarrow\begin{pmatrix}x+2\\2y-1\end{pmatrix}$

Solution

Using: $\begin{pmatrix}x\\y\end{pmatrix}\rightarrow\begin{pmatrix}x+2\\2y-1\end{pmatrix}$,
$\begin{pmatrix}2\\3\end{pmatrix}\rightarrow\begin{pmatrix}2+2\\2(3)-1\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$ *or* $(4,5)$

Example 23.21

Using a scale of 2 cm to 2 units on each axis, draw on a sheet of graph paper two perpendicular axes OX and OY, for the interval $-10 \le x \le 10$ *and* $-10 \le y \le 10$

(a) Draw the triangle ABC with coordinates $A(6,8), B(2,5), C(7,2)$

(b) Draw the image triangle $A^1B^1C^1$ *of* ΔABC under an enlargement with scale factor
-1 from the origin, where $A \to A^1, B \to B^1$ *and* $C \to C^1$. Label the vertices and coordinates clearly. (c) Draw the image $\Delta A^{11}B^{11}C^{11}$ *of* ΔABC under a clockwise rotation of 270^o about the origin, where $A \to A^{11}, B \to B^{11}$ *and* $C \to C^{11}$. Label the vertices and coordinates clearly. (d) (i) What transformation maps $\Delta A^1B^1C^1$ *onto* $\Delta A^{11}B^{11}C^{11}$ where $A^1 \to A^{11}, B^1 \to B^{11}$ *and* $C^1 \to C^{11}$?
(ii) What is the equation of BB^1?

Solution

(a) Given that $A(6,8), B(2,5), C(7,2)$, draw triangle ABC using the given scale and coordinates

(b) When scale factor is -1 and using $\begin{pmatrix} x \\ y \end{pmatrix} \to k\begin{pmatrix} x \\ y \end{pmatrix}$

$A(6,8) \to -1(6,8) = A^1(-6,-8),\ B(2,5) \to -1(2,5) = B^1(-2,-5),$

$C(7,2) \to -1(7,2) = C^1(-7,-2)$

Plot the points A^1, B^1 *and* C^1 on the same graph sheet indicating vertices and coordinates clearly

(c) By a clockwise rotation of 270^o about the origin, we use the mapping: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$ since clockwise rotation of 270^o is the same as anticlockwise rotation of 90^o. Thus,

$A(6,8) \to A^{11}(-8,6),\ B(2,5) \to B^{11}(-5,2),$

$C(7,2) \to C^{11}(-2,7)$

Plot the points A^{11}, B^{11} *and* C^{11} on the graph

(d) (i) Using $A^1(-6,-8) \to A^{11}(-8,6)$ By inspection, it is realized that A^1 undergone an anticlockwise rotation of 90^o about the origin to obtain A^{11}. Therefore the transformation that maps $\Delta A^1B^1C^1$ *onto* $\Delta A^{11}B^{11}C^{11}$ is an anticlockwise rotation of 90^o about the origin.

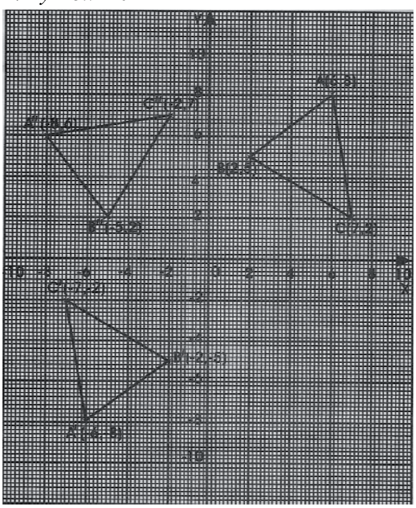
(ii) To find the equation of BB^1, first find the gradient, m of BB^1

i.e. *For* $B(2,5)$ $B^1(-2,-5)$, $m = \frac{-5-5}{-2-2} = \frac{-10}{-4} = \frac{5}{2}$

Now using the point $B(2,5)$ and the general equation:

$y - y_1 = m(x - x_1)$, the equation becomes $y - 5 = \frac{5}{2}(x-2)$

$\Rightarrow 2y - 5x = 0$



Example 23.22

(a) Using a scale of 2 cm to 2 units on both axes, draw on a graph sheet two perpendicular axes OX and OY for

$-10 \le x \le 10$ *and* $-10 \le y \le 10$

(b) Draw, labeling all the vertices clearly together with the coordinates

(i) the triangle ABC with vertices $A(4,8)$, $B(1,7)$ *and* $C(3,4)$

(ii) the image $\Delta A_1B_1C_1$ *of* ΔABC under a reflection in the line y=3

(iii) the image $\Delta A_2B_2C_2$ *of* ΔABC under an anticlockwise rotation of 90^o about the origin where $A \to A_2, B \to B_2$ *and* $C \to C_2$

(iv) the image $\Delta A_3B_3C_3$ *of* ΔABC under an enlargement with scale factor -1 from the origin where $A \to A_3, B \to B_3$ *and* $C \to C_3$ (c) What single transformation maps $\Delta A_2B_2C_2$ *onto* $\Delta A_3B_3C_3$ *where* $A_2 \to A_3, B_2 \to B_3$ *and* $C_2 \to C_3$

Solution

(a) Draw the x and y axes in the given intervals using the scale.

(b) (i) Plot the vertices $A(4,8), B(1,7)$ *and* $C(3,4)$ and join them to obtain a triangle

(ii) Using the rule for reflection in the line y = a, $(x,y) \to (x, 2a-y)$

$\Rightarrow A(4,8) \to (4, 2(3)-8) = A_1(4,-2),$

$B(1,7) \to (1, 2(3)-7) = B_1(1,-1)$

and $C(3,4) \to (3, 2(3)-4) = C_1(3,2)$

Plot the points $A_1, B_1,$ *and* C_1 and join to form a triangle

(iii) For anticlockwise rotation of 90^o**,** we use the rule:

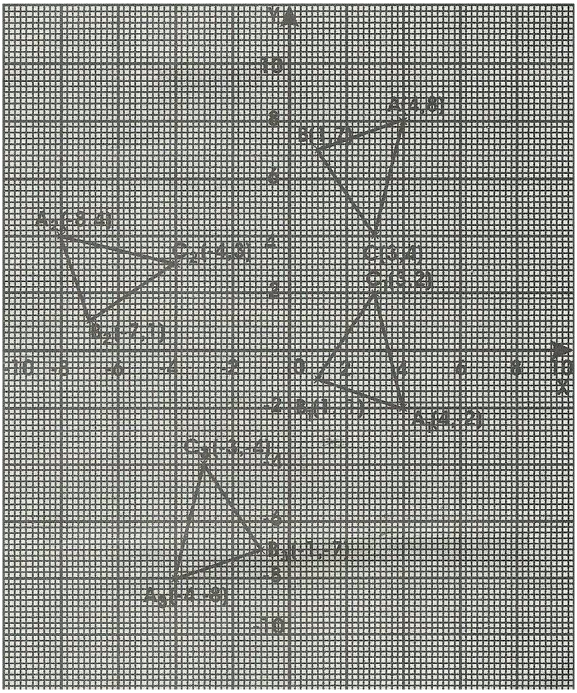
$(x,y) \to (-y,x) \Rightarrow A(4,8) \to A_2(-8,4)$

$B(1,7) \to B_2(-7,1), C(3,4) \to C_2(-4,3)$

(iv) Enlarging with scale factor -1

$\Rightarrow A(4,8) \to A_3(-4,-8)$ $B(1,7) \to B_3(-1,-7), C(3,4) \to C_3(-3,-4)$ Plot the points A_3, B_3 *and* C_3

(c) Using $A_2(-8,4) \to A_3(-4,-8)$**,** it is seen clearly that the rule mapping A_2 onto A_3 is an anticlockwise rotation of 90^o about the origin i.e. $(x,y) \to (-y,x)$



Example 23.23

(a) Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes OX and OY for the intervals $-10 \le x \le 10$ *and* $-12 \le y \le 12$

(b) Draw on the same graph sheet indicating clearly the coordinates of all vertices

(i) the square PQRS with vertices $P(2,2), Q(6,2), R(6,6), S(2,6)$

(ii) the image $P_1Q_1R_1S_1$ *of the square PQRS* under a reflection in the y – axis, where $P \to P_1, Q \to Q_1, R \to R_1$ *and* $S \to S_1$

(iii) the image $P_2Q_2R_2S_2$ *of the square PQRS* under a translation by the vector $\begin{pmatrix} 4 \\ -8 \end{pmatrix}$ *where* $P \to P_2, Q \to Q_2, R \to R_2$ *and* $S \to S_2$

(iv) the image $P_3Q_3R_3S_3$ *of the square PQRS* under a rotation of 180^o about the origin where $P \to P_3, Q \to Q_3, R \to R_3$ *and* $S \to S_3$

(c) Find the vector $\overrightarrow{P_2P_3}$

Solution

(a) Draw the X and Y axes using the given scale and intervals

(b) (i) Plot the points $P(2,2), Q(6,2), R(6,6)$ *and* $S(2,6)$ and join them to get Square PQRS

(ii) Reflection in the y – axis has the mapping:

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ y \end{pmatrix}, \Rightarrow P(2,2) \to P_1(-2,2), Q(6,2) \to Q_1(-6,2),$

$R(6,6) \to R_1(-6,6)$

Plot P_1, Q_1, R_1 *and* S_1

(iii) For the vector $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow P\begin{pmatrix} 2 \\ 2 \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -8 \end{pmatrix} = P_2(6,-6)$

$Q\begin{pmatrix} 6 \\ 2 \end{pmatrix} \to \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -8 \end{pmatrix} = Q_2(10,-6)$

$R\begin{pmatrix} 6 \\ 6 \end{pmatrix} \to \begin{pmatrix} 6 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -8 \end{pmatrix} = R_2(10,-2)$

$S\begin{pmatrix} 2 \\ 6 \end{pmatrix} \to \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -8 \end{pmatrix} = S_2(6,-2)$ Plot the points P_2, Q_2, R_2 *and* S_2 and join to get a square

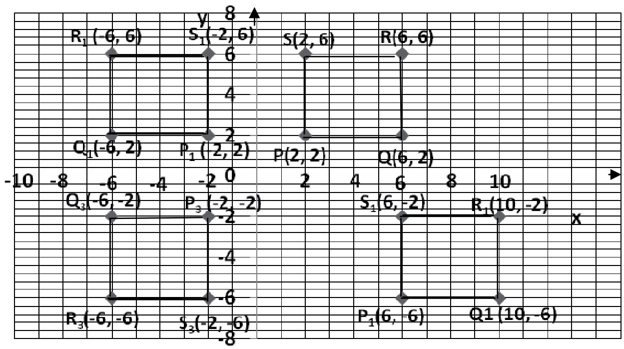
(iv) Rotation of 180^o has the mapping: $(x, y) \to (-x, -y)$

$\Rightarrow P(2,2) \to P_3(-2,-2),\ Q(6,2) \to Q_3(-6,-2)$

$R(6,6) \to R_3(-6,-6)$ *and* $S(2,6) \to S_3(-2,-6)$ Plot the points P_3, Q_3, R_3 *and* S_3

(c) $P_2 = (6,-6)$ *and* $P_3(-2,-2)$

$\overrightarrow{P_2P_3} = \overrightarrow{OP_3} - \overrightarrow{OP_2} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} 6 \\ -6 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$



Example 23.24

(a) Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axis OX and OY for the interval -10<x< and -10<y<10

(b) Draw on the same graph sheet indicating the coordinates.

(i) ΔPQR *with* $P(2,2), \overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \overrightarrow{QR} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

(ii) the $\Delta P^1Q^1R^1$ of ΔPQR under a reflection in the line $x = 0$, where $P \to P^1, Q \to Q^1, R \to R^1$

(iii) The image $\Delta P^{11}Q^{11}R^{11}$ of ΔPQR under a rotation of 180^0 about the origin where $P \to P^{11}, Q \to Q^{11}, R \to R^{11}$

(c) (i) Describe the single transformation that maps $\Delta P^1Q^1R^1$ onto $\Delta P^{11}Q^{11}R^{11}$ where $P^1 \to P^{11}, Q^1 \to Q^{11}, R^1 \to R^{11}$

(ii) find P^1R^1

Solution

(a) Draw the X and Y axes using the interval and the scale given.

(b) (i) $P = (2,2), \overrightarrow{PQ} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \overrightarrow{QR} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$\Rightarrow \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \overrightarrow{OQ} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \Rightarrow \overrightarrow{OQ} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

Hence, Q = (4, 6)

Again, $\vec{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$\Rightarrow \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \overrightarrow{OR} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \Rightarrow \overrightarrow{OR} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Hence, R = (6, 2)

Plot P, Q and R

(ii) Mapping for a reflection in the line x = 0 is given by: $(x, y) \to (-x, y)$

$\Rightarrow P(2,2) \to P^1(-2,2),\ Q(4,6) \to Q^1(-4,6)\ \ and\ \ R(6,2) \to R^1(-6,2)$

Hence, plot points P^1, Q^1, and R^1.

(iii) mapping for a rotation of 180^o about the origin is:

$(x, y) \to (-x, -y)$

$\Rightarrow P(2,2) \to P^{11}(-2,-2),\ Q(4,6) \to Q^{11}(-4,6)\ and\ \ R(6,2) \to R^{11}(-6,-2)$

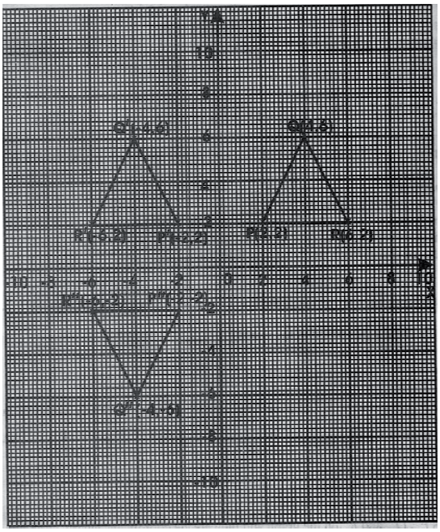
Hence, plot points P^{11}, Q^{11} and R^{11}

(c) (i)

$P^1(-2,2) \to P^{11}(-2,-2),\ Q^1(-4,6) \to Q^{11}(-4,-6)\ and\ \ R^1(-6,2) \to R^{11}(-6,-2)$

Thus, the rule is $(x, y) \to (x, -y)$ i.e. a reflection in the line y = 0

(ii) $P^1 = (-2,2)\ and\ \ R^1(-6,2) \Rightarrow P^1R^1 = \overrightarrow{OR^1} - \overrightarrow{OP^1} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$



Example 23.25

A triangle has vertices A(1, 1), B(2, 4) and C(5, 8)

(i) If the triangle is translated by the vector

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *to* $A'B'C'$, *where* $A \to A'$, $B \to B'$ *and* $C \to C'$

calculate the coordinates of A', B' *and* C'.

(ii) The triangle ABC undergoes a transformation involving rotation in anticlockwise direction through 90^{o} about the origin followed by a translation. If the final position is , $A''(2,-1)$, $B''(-1,0)$ *and* $C''(-5,3)$

determined the translation vector.

Solution

(i)

$$A\begin{pmatrix}1\\1\end{pmatrix}\to A'\left[\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\right]=\begin{pmatrix}1+1\\-1+1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}\therefore A'=(2,0)$$

$$B\begin{pmatrix}2\\4\end{pmatrix}\to B'\left[\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}\right]=\begin{pmatrix}1+2\\-1+4\end{pmatrix}=\begin{pmatrix}3\\3\end{pmatrix}\therefore B'=(3,3)$$

$$C\begin{pmatrix}-5\\3\end{pmatrix}\to C'\left[\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}5\\8\end{pmatrix}\right]=\begin{pmatrix}1+5\\-1+8\end{pmatrix}=\begin{pmatrix}6\\7\end{pmatrix}\therefore C'=(6,7)$$

(ii) For the rotation of 90^{o} anticlockwise about the origin, we use the mapping:

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-y\\x\end{pmatrix}$$

$$\Rightarrow A\begin{pmatrix}1\\1\end{pmatrix}\to\begin{pmatrix}-1\\1\end{pmatrix},\ B\begin{pmatrix}2\\4\end{pmatrix}\to\begin{pmatrix}-4\\2\end{pmatrix},\ C\begin{pmatrix}5\\8\end{pmatrix}\to\begin{pmatrix}-8\\5\end{pmatrix}$$

Now let the translation vector be $\begin{pmatrix}x\\y\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}\Rightarrow translation\ vector,\ \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\-2\end{pmatrix}$$

EXERCISE

QUE. A

(a) Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes OX and OY such that $-10\le x\le 10\ and\ -12\le y\le 12$

(b) Draw on the same graph sheet showing clearly the coordinates of all vertices

(i) the $\Delta PQR\ with\ P(4,8), \overrightarrow{QP}=\begin{pmatrix}-2\\2\end{pmatrix}\ and\ \overrightarrow{RP}=\begin{pmatrix}2\\4\end{pmatrix}$

(ii) the image $\Delta P_1Q_1R_1\ of\ \Delta PQR$ under a reflection in the line $y=-2\ where\ P\to P_1, Q\to Q_1, R\to R_1$

(iii) the image $\Delta P_2Q_2R_2\ of\ \Delta PQR$ under a translation by the vector

$\begin{pmatrix} -8 \\ 2 \end{pmatrix}$ $where\ P \to P_2, Q \to Q_2\ and\ R \to R_2$

(iv) the image $\Delta P_3Q_3R_3\ of\ \Delta PQR$ under a rotation of 180^o about the origin where $P \to P_3, Q \to Q_3, R \to R_3$ (c) Find $\overrightarrow{Q_2Q_3}$

QUE. B

(a) Using a scale of 2 cm to 2 units on each axis, draw on a sheet of graph paper two perpendicular axes OX and OY for the intervals $-10 \le x \le 10\ and\ -10 \le y \le 10$

(b) Draw on the same graph sheet indicating clearly the coordinates of all vertices

(i) the quadrilateral ABCD with coordinates $A(2,4), B(4,7), C(8,8)\ and\ D(6,3)$

(ii) The image $A_1B_1C_1D_1\ of\ ABCD$ under an anticlockwise rotation of 90^o about the origin where $A \to A_1, B \to B_1, C \to C_1\ and\ D \to D_1$

(iii) the image $A_2B_2C_2D_2\ of\ ABCD$ under a translation by the vector $\begin{pmatrix} -10 \\ -9 \end{pmatrix}$ where $A \to A_2, B \to B_2, C \to C_2\ and\ D \to D_2$

(iv) the image $A_3B_3C_3D_3\ of\ ABCD$ under a reflection in the line $y = 1$ where $A \to A_3, B \to B_3, C \to C_3\ and\ D \to D_3$

(c) Find $\overrightarrow{C_2C_3}$

QUE. C

(a) Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes OX and OY for the intervals $-10 \le x \le 10\ and\ -10 \le y \le 10$

(b) Draw on this graph sheet indicating the coordinates of all vertices

(i) $\Delta UVW\ with\ U(3,7), V(6,7)\ and\ W(3,3)$

(ii) the image $\Delta U_1V_1W_1\ of\ \Delta UVW$ under a reflection in the x – axis where $U \to U_1, V \to V_1\ and\ W \to W_1$

(iii) the image $\Delta U_2V_2W_2$ *of* $\Delta U_1V_1W_1$ under the mapping $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x+y \\ x \end{pmatrix}$

where $U_1 \rightarrow U_2, V_1 \rightarrow V_2$ *and* $W_1 \rightarrow W_2$

(c) (i) Find the equation of the line joining points V_1 *and* U

(ii) Calculate $|WW_1|$

QUE. D

(a) Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axis OX and OY for the interval -10<x< and -10<y<10

(b) Draw on the same graph sheet indicating the coordinates.

(i) ΔPQR *with* $P(2,2), \overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \overrightarrow{QR} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

(ii) the $\Delta P^1Q^1R^1$ of ΔPQR under a reflection in the line $x = 0$, where $P \rightarrow P^1, Q \rightarrow Q^1, R \rightarrow R^1$

(iii) The image $\Delta P^{11}Q^{11}R^{11}$ of ΔPQR under a rotation of 180^0 about the origin where $P \rightarrow P^{11}, Q \rightarrow Q^{11}, R \rightarrow R^{11}$

(c) (i) Describe the single transformation that maps $\Delta P^1Q^1R^1$ onto $\Delta P^{11}Q^{11}R^{11}$ where $P^1 \rightarrow P^{11}, Q^1 \rightarrow Q^{11}, R^1 \rightarrow R^{11}$

(ii) find P^1R^1

QUE. E

What transformation does $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x-3 \\ y \end{pmatrix}$ represent?

QUE. F

Find the image of the point (3, -2) under the transformation given by: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 3x \\ 3y+2 \end{pmatrix}$

QUE. G

(a) Find the image of the position vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ under the translation $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(b) If $A(2,3)$ is reflected in the x-axis, find the image A^1 *of* A

(c) In the graph above, triangle $P^1Q^1R^1$ is the image of triangle PQR after

the Transformation by the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

(d) Draw, using the scale on the graph and indicating all coordinates,

(i) Triangle $P^1Q^1R^1$

(ii) Triangle PQR before it was transformed

(iii) The image triangle $P''Q''R''$ *of* $\Delta P'Q'R'$ under a rotation about the origin through 90^o clockwise.

QUE. H

(a) Using a scale of 2cm to 2 units on both axes, draw on a sheet of graph paper, two perpendicular axes Ox and Oy for $-10 \le x \le 10$ *and* $-10 \le y \le 10$

(b) Draw, labeling all the vertices clearly together with their coordinates,

(i) triangle PQR with vertices P(2, 3), Q(8, 9) and R(7, 2)

(ii) the image $P_1Q_1R_1$ *of* ΔPQR *under a rotation through* -90^o *about* the origin, where $P \to P_1,\ Q \to Q_1$ *and* $R \to R_1$

NB:

Rotation through -90^o is the same as rotation through 270^o anticlockwise or 90^o clockwise.

(iii) The image $\Delta P_2Q_2R_2$ *of* ΔPQR *under an enlargement with scale* factor -1 from the origin, where $P \to P_2,\ Q \to Q_2$ *and* $R \to R_2$

(c) Find the bearing of R from P.

CHAPTER 24
GRAPHS OF RELATIONS
(LINEAR, QUADRATIC, SPECIAL & CUBIC RELATIONS)

24.1 **Graphs Of Linear Relations**

A ***linear relation*** is a relation in which its highest power is one (1) usually of the form $y = mx + c$. Where $m \ and \ c$ are constants.
Examples are: $y = 3x - 1 \ and \ y = -x$

NB

Linear graphs are usually straight line graphs.
To draw the graph of a linear relation with a given interval in x, we substitute the values of the interval for x and evaluate for corresponding values of y.

Likewise, in drawing a linear graph where the interval is not given, we assume any two or three values of x in which case any two consecutive numbers must be far apart.

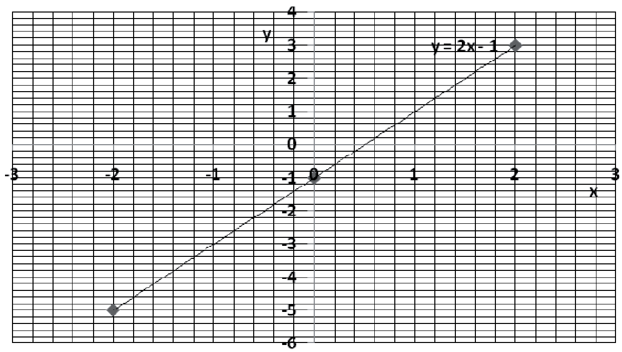
Example 24.1

Draw the graph of the relation $y = 2x - 1$

Solution

First, we choose any three values of x, say $x = -2, 0, 2$ and then find the corresponding values of the points.
i.e. when $x = -2, \ y = 2(-2) - 1 = -4 - 1 = -5$
when $x = 0, \ y = 2(0) - 1 = -1$
when $x = 2, \ y = 2(2) - 1 = 3$
Hence, we have the following table of values

x	-2	0	2
y	-5	-1	3



24.2 **Quadratic Relations**

A quadratic relation is a relation usually of the form $y = ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$. Thus, a quadratic relation can be said to be one in which the highest power of the variable is 2. Examples are: $y = 2x^2 + x - 1$ *and* $y = -x^2$

24.3 **Methods Of Drawing Graphs Of Quadratic Relations**

The two main methods are:

1. Method of substitution
2. Tabular method

24.3.1 **Method Of Substitution**

By this method, the values of x in the given range are being substituted into the given relation and corresponding values of y evaluated.

Example 24.2

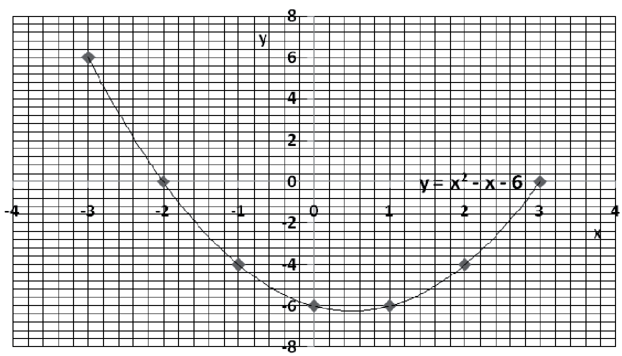
Draw the graph of the relation $y = x^2 - x - 6$ for the interval $-3 \leq x \leq 3$

Solution

When $x=-3,\ y=(-3)^2-(-3)-6=9+3-6=6$
When $x=-2,\ y=(-2)^2-(-2)-6=4+2-6=0$
When $x=-1,\ y=(-1)^2-(-1)-6=1+1-6=-4$
When $x=0,\ y=(0)^2-(0)-6=0-0-6=-6$
When $x=1,\ y=(1)^2-(1)-6=1-1-6=-6$
When $x=2,\ y=(2)^2-(2)-6=4-2-6=-4$
When $x=3,\ y=(3)^2-(3)-6=9-3-6=0$

Hence, we have the table below

x	-3	-2	-1	0	1	2	3
y	6	0	-4	-6	-6	-4	0



24.3.2 **Tabular Method**

Here, a table is drawn with x values (values of the given range) as the first row, y values (values computed) as the last row and each term of the equation is assigned a row in between the first and last rows.
Then, add values in each column excluding values in the first and last columns and write the result in each corresponding last column (i.e. the y value).

NB

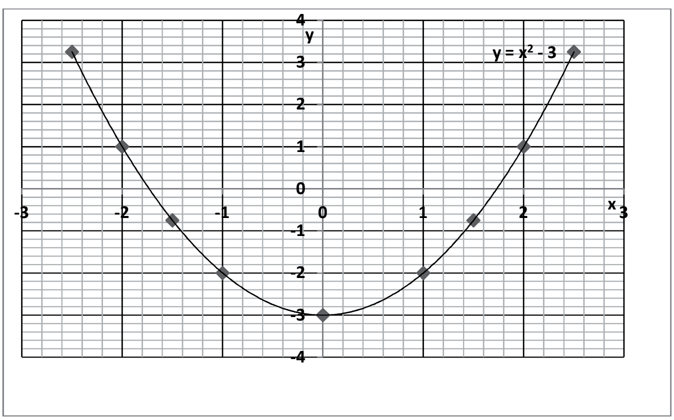
A constant term in the equation is repeatedly written in that row and all negative terms are written with the negative sign when assigning the rows to avoid confusion.

Example 24.3

Draw the graph of the relation $y = x^2 - 3x$ for the interval $-2.5 \leq x \leq 2.5$

Solution

x	-2.5	-2	-1.5	-1.0	0.0	1.0	1.5	2.0	2.5
x^2	6.25	4	2.25	1	0	1	2.25	4	6.25
$-3x$	-3	-3	-3	-3	-3	-3	-3	-3	-3
y	3.25	1	-0.75	-2	-3	-2	-0.75	1	3.25

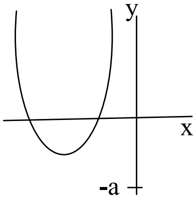


24.4 **Interpretation Of Graphs**

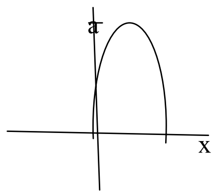
24.4.1 **Finding The Least Or Greatest Value**

In finding the least or greatest value, we trace the turning point of the curve to the
y - axis. Thus, from the graphs below, the least value is '**-a**' and the greatest value
is '**a**'.

Illustration y



Or



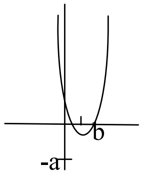
NB

All maximum graphs have greatest values and all minimum graphs have least values.

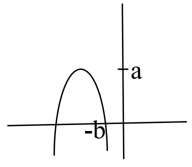
24.4.2 **Finding The Least Or Greatest Point**

The least or greatest point of a graph is the least or greatest value together with its corresponding y-value written as a point. Thus, from the graphs below, greatest point is (-b, a) and the least point is also (b, -a)

Illustration



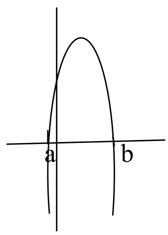
Or



24.4.3 **Finding The Equation Of Line Of Symmetry**

A ***line of symmetry*** is a vertical straight line drawn to pass through the turning point of a curve. The ***equation of a line of symmetry*** is calculated by summing the two points of intersection of the curve and the x-axis and dividing the result by 2.

Illustration



Thus, the equation of line of symmetry of the graph above is: $\frac{a+b}{2}$

24.4.4 **Finding The Values Of x For Which** $y<0 \quad or \quad y>0$

With the exception of points that lie exactly on the x-axis, all points on the part of the curve below the x-axis represents $y<0$. Likewise, the part of the curve that is above the x-axis represents $y>0$.

NB

In example 24.6, the range of values of x for which $y>0$ is $-1<x<3$.
Again, the range of values of x for which $y<0$ in example 24.6 is $x<-1 \quad or \quad x>3$, since those two parts of the curve lie below the x-axis
Similarly, in example 24.4, the range of values of x for which $y<0$ is $-3<x<2$.

Further, the range of values of x for which $y > 0$ in example 24.4 is $x < -3$ or $x > 2$ since these two parts of the graph are above the x-axis.

24.4.5 **Finding The Range Of Values Of x For Which y Decreases As x Increases**

For maximum graphs, the range of values of x for which y decreases as x increases is between the equation of line of symmetry and the last x value exactly (on RHS) on the graph. Thus, in example 24.6, the range of values of x for which y decreases as x increases is $1 < x \leq 4$ since 1 is the equation of line of symmetry and 4 is the last x value on the graph.
Similarly, for minimum graphs, the range of values of x for which y decreases as x increases is usually from the first x value on the left to the immediate x value before the equation of line of symmetry. Thus, from example 24.4, the range of values of x for which y decreases as x increases is $-5 \leq x < -0.5$

24.4.6 **Finding The Range Of Values Of x For Which y Increases As x Increases**

For maximum graphs, the range of values of x for which y increases as x increases is from the first x value on the left to a value closest to the equation of line of symmetry. In example 24.6, the range is $-2 \leq x < 1$
Likewise, for minimum graphs, the range is between the equation of line of symmetry and the last x value on the right. In example 24.4, the range is $-0.5 < x \leq 4$

NB

In both cases, the equation of line of symmetry is excluded from the range.

24.4.7 **Finding Solutions Of Equations Using Graphs**

In solving given equations from graphs, we follow the steps below:

Step 1
Rewrite the given equation to match corresponding terms as in original equation used to draw the graph.
Step 2
Compare corresponding terms of given equation with original equation. When the equation is the same on one side as the original equation, we subtract it from original equation.
Cases where one or more terms in the given equation is not the same as that in original equation, we manipulate the given equation by either adding to both sides or subtracting from both sides the difference between the term in it and the original equation.
Step 3
Subtract corresponding terms of the result of step 2 from the original equation and make y the subject.
Step 4
When the result of step 3 gives a value of y to be a number, trace a horizontal straight line through the number on the y – axis to cut the curve at two points.
But when y gives a linear equation, form a table of values and use it to draw a linear graph on the same sheet ensuring that the straight line graph cuts the curve at two points
Step 5
Now trace two vertical straight lines from the points of intersection of the horizontal line and the curve to the x – axis. The two x values found on the x – axis gives the solution of the given equation

Illustrations

(i) Use the graph of the relation $y = 2x^2 + 3x - 5$ to solve the equation $2x^2 + 3x - 5 = 0$.

Step 1
Rewriting gives: $0 = 2x^2 + 3x - 5$

Step 2 & 3
Since the corresponding terms are the same, we subtract straight away.
Thus, getting $y = 0$

Step 4
Tracing $y = 0$ becomes the x – axis. Hence, solution is the points where the curve cuts the x – axis.

(ii) Use the graph of the relation $y = x^2 + 2x - 7$ to solve the equation $x^2 + 2x - 5 = 0$

Step 1
Rewriting gives $0 = x^2 + 2x - 5$
Step 2
Since third term is different from that of original equation, we subtract 2 from it to make them the same. Thus,
$0 - 2 = x^2 + 2x - 5 - 2 \Rightarrow -2 = x^2 + 2x - 7$
Step 3
Subtract result of step 2 from original equation
Thus, we get $y + 2 = 0 \Rightarrow y = -2$
Step 4
Draw a horizontal straight line through $y = -2$ to cut the curve at two points.
Step 5
Trace these intersections to the x – axis for the solution

(iii) Use the graph of the relation $y = 2x^2 + 3x - 2$ to solve the equation $2x^2 + 3x - 2 = x + 1$
Rearranging becomes $x + 1 = 2x^2 + 3x - 2$
Since RHS is the same, we just subtract the given equation from the original one.
i.e. $y - x - 1 = 0 \Rightarrow y = x + 1$
now we draw the linear graph of the equation $y = x + 1$ to intersect the curve at two points. Trace the intersections to obtain the solution

(iv) Use the graph of the equation $y = x^2 - x + 3$ to solve the equation $x^2 + x + 5 = 0$

Rewriting becomes $0 = x^2 + x + 5$

Subtract $2x$ *and* 2 from both sides

$\Rightarrow 0 - 2x - 2 = x^2 + x - 2x + 5 - 2$

$\Rightarrow -2x - 2 = x^2 - x + 3$

Now subtracting from original gives: $y + 2x + 2 = 0 \Rightarrow y = -2x - 2$

Plotting and tracing will give the solution

Example 24.4

(a) Copy and complete the following table for the relation $y = x^2 + x - 6$ in the interval $-5 \le x \le 4$

x	-5	-4	-3	-2	-1	0	1	2	3	4
y		6		-4			-4		6	

(b) Using a scale of 2cm to 1unit on the x-axis and 2cm to 2units on the y-axis, draw the graph of the relation for the given interval.

(c) Use your graph to find

(i) The equation of the line of symmetry

(ii) The truth set of $x^2 + x = 8$

(iii) The coordinates of the minimum point

Solution

(a)

Method 1 (Method of substitution)

For $x = -5,\ y = (-5)^2 + (-5) - 6 = 25 - 5 - 6 = 14$

For $x = -3,\ y = (-3)^2 + (-3) - 6 = 9 - 3 - 6 = 0$

For $x = -1,\ y = (-1)^2 + (-1) - 6 = 1 - 1 - 6 = -6$

For $x = 0,\ y = (0)^2 + (0) - 6 = 0 + 0 - 6 = -6$

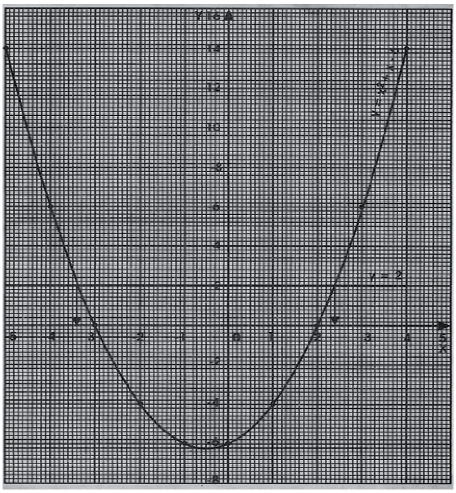
For $x = 2,\ y = (2)^2 + (2) - 6 = 4 + 2 - 6 = 0$

For $x = 4,\ y = (4)^2 + (4) - 6 = 16 + 4 - 6 = 14$

Hence, we have the following table of values:

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	14	6	0	-4	-6	-6	-4	0	6	14

(b)



(c) (i) Equation of line of symmetry $= \frac{-3+2}{2} = -\frac{1}{2} = -0.5$

i.e. dividing the sum of intersecting points of curve on x-axis by 2

(ii) For truth set of $x^2 + x = 8$

Rearranging gives $x^2 + x - 8 = 0$

Subtract from original equation

i.e. $y = x^2 + x - 6$

$-\ (0 = x^2 + x - 8)$

Giving $y = 2$

We then trace a horizontal line through $y = 2$ to cut the curve at two points.

Trace these points vertically to the x-axis for the solution truth set values

Thus, truth set is $\{x : x = -3.4,\ 2.4\}$

(iii) the coordinates of the minimum point is $(-0.5, -6.4)$
i.e. the equation of line of symmetry and its corresponding y value

Example 24.5

(a) Copy and complete the table of values for the relation $y = x^2 - 2x - 3$ for the interval $-2 \le x \le 4$

x	-2	-1.5	-1	0	1	2	2.5	3	3.5	4
y	5		0	-3				0		

(b) Using a scale of 2cm to 1unit on both axes, draw the graph of the relation $y = x^2 - 2x - 3$ for the given interval
(c) Use your graph to find
(i) The solution of $x^2 - 2x - 2 = 0$
(ii) The equation of the line of symmetry
(iii) The range of values of x for which y is negative

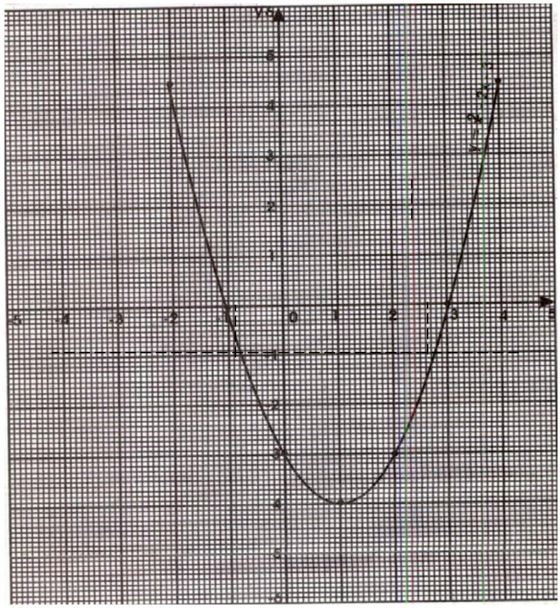
Solution

Method 2 (Tabular Method)

x	-2	-1.5	-1	0	1	2	2.5	3	3.5	4
x^2	4	2.25	1	0	1	4	6.25	9	12.25	16
$-2x$	4	3	2	0	-2	-4	-5	-6	-7	-8
-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
y	5	2.25	0	-3	-4	-3	-1.75	0	2.25	5

.

(b)



(c) (i) For solution of $x^2 - 2x - 2 = 0$

Subtracting from original equation gives

$y = -1$

Hence, tracing from the graph gives the solution $x = -0.7,\ 2.8$

(ii) equation of line of symmetry is $x = \frac{-1+3}{2} = 1$

(iii) range of values of x for which y is negative is $-1 < x < 3$

i.e. the portion of the graph that lies below the x-axis

Example 24.6

(a) Copy and complete the following table of values for the relation:

$y = (x-4)(x+2)\ \ for\ -3 \le x \le 5$

x	-3	-2	-1	0	1	2	3	4	5
y				-8					

(b) Using scales of 2cm to 1unit on the x-axis and 2cm to 2units on the
y-axis, draw the graph of $y = (x-4)(x+2)$ for $-3 \le x \le 5$

(c) Using the graph, find the:

(i) values of x for which y is decreasing

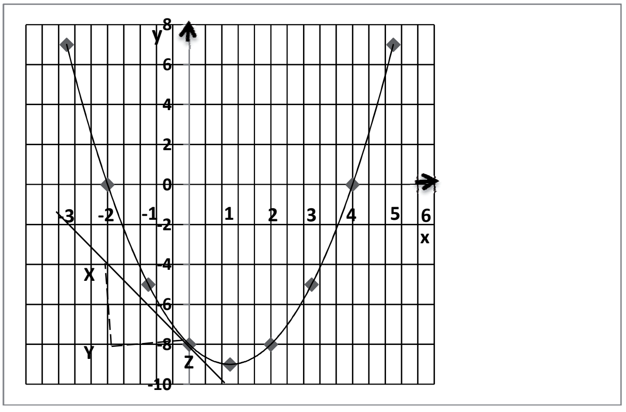
(ii) gradient of the curve at x = 0

Solution

(a)

x	-3	-2	-1	0	1	2	3	4	5
x-4	-7	-6	-5	-4	-3	-2	-1	0	1
x+2	-1	0	1	2	3	4	5	6	7
y=(x-4)(x+2)	7	0	-5	-8	-9	-8	-5	0	7

(b)



(c)

(i) From graph values of x for which y decreases is $-3 \le x < 1$

(ii) $Gradient = \dfrac{Change\ in\ y-coordinate}{Change\ in\ x-coordinates} = \dfrac{XY}{XZ} = \dfrac{-4-(-8)}{-2-0} = \dfrac{4}{-2} = -2$

Example 24.6

(a) Copy and complete the following table for the relation:
$y = 3 + 2x - x^2 \;\; for \; -2 \le x \le 4.$

x	-2	-1	0	1	1.5	2	2.5	3	3.5	4
y		0	3				1.75			-5

(b) Taking 2cm to 1 unit on both axes, draw the graph of the relation for the given interval

(c) Draw on the same graph sheet $x - y = 0$

(d) Using your graphs

(i) Solve for $3 + 2x - x^2 = x$

(ii) Find the values of x for which $3 + 2x - x^2 = 2$

Solution

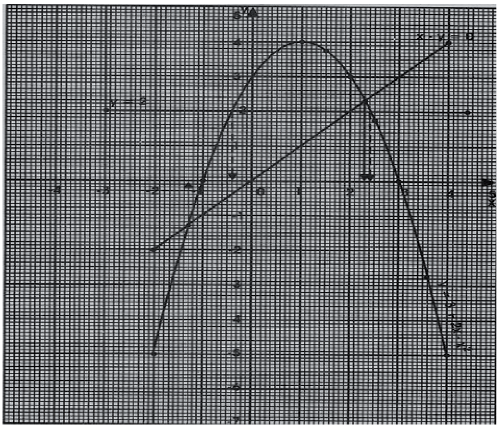
(a)

x	-2	-1	0	1	1.5	2	2.5	3	3.5	4
3	3	3	3	3	3	3	3	3	3	3
$2x$	-4	-2	0	2	3	4	5	6	7	8
$-x^2$	-4	-1	0	-1	-2.25	-4	-6.25	-9	-12.25	-16
y	-5	0	3	4	3.75	3	1.75	0	-2.25	-5

(b) Using a scale of 2cm to 1unit on both axes, draw the graph of $y = 3 + 2x - x^2$

(c) Again, for $x - y = 0 \;\; \Rightarrow y = x$ Assume any two values of x and corresponding y values and draw a linear graph

x	0	3
y	0	3



(d) (i) Solution of $3+2x-x^2=x, \Rightarrow y=x \text{ is } x=-1.3, 2.3$
i.e. the x values of the intersection of the two graphs
(ii) $3+2x-x^2=2, \Rightarrow y=2 \text{ is } x=-0.4, 2.4$
i.e. we trace with a straight line through $y=2$ horizontally to intersect the curve already drawn and then find the corresponding x values to the intersections on the x-axis.

Example 24.7

The relation for the volume $y\,cm^3$ of a tray of depth $x\,cm$ is given as: $y=x(12-x)(8-x)cm^3$
(a) Copy and complete the following tables for $y=x(12-x)(8-x)cm^3$

x	0	1	2	3	4	5	6	7	8
$x(12-x)(8-x)cm^3$	0	77		135					0

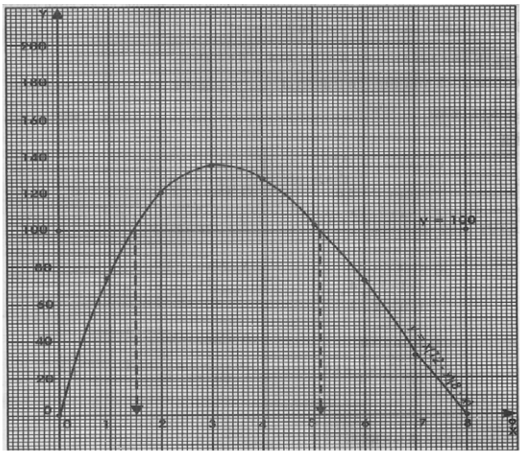
(b) Use the table of values to draw the graph of $y=x(12-x)(8-x)cm^3$ from $x=0 \text{ to } x=8$ taking 2cm to 1unit on the x-axis and 2cm to 20units on the y-axis
(c) Find the value of x if the volume of the tray is $100cm^3$
(d) What value of x gives the maximum volume of the tray?

Solution

(a)

x	0	1	2	3	4	5	6	7	8
$x(12-x)(8-x)cm^3$	0	77	120	135	128	105	72	35	0

(b)



(c) when volume is $100cm^3$, means at $y = 100cm^3$ on the graph. Thus, from the graph, the value of x when volume is $100cm^3$ is $x = 1.55cm,\ 5.1cm$

(d) the value of x that gives the maximum volume of the tray is $x = 3cm$

i.e. the same as the equation of line of symmetry

24.5 **Special Relations**

Example 24.8

(a) Copy and complete the following table of values for the relation $y = \dfrac{2x(x-1)}{x+3}$ for the interval $-1 \le x \le 6$

x	-1	0	0.5	1	2	3	4	5	6
$2x(x-1)$		0	-0.5	0	4		24	40	60
$x+3$	2		3.5	4		6	7		9
$y=\dfrac{2x(x-1)}{x+3}$			-0.14	0			3.43		6.67

(b) Using a scale of 2cm to 1unit on both axes, draw the graph of $y=\dfrac{2x(x-1)}{x+3}$ for the given interval (c) Use your graph to

(i) find the range of values of x for which y is negative

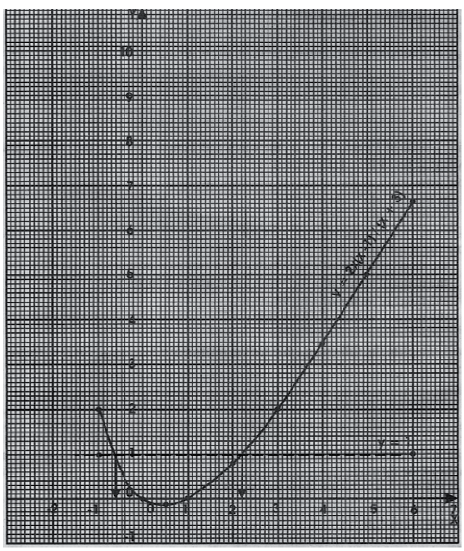
(ii) estimate correct to one decimal place the value for which $2x(x-1)=x+3$

Solution

(a)

x	-1	0	0.5	1	2	3	4	5	6
$2x(x-1)$	4	0	-0.5	0	4	12	24	40	60
$x+3$	2	3	3.5	4	5	6	7	8	9
$y=\dfrac{2x(x-1)}{x+3}$	2	0	-0.14	0	0.8	2	3.43	5	6.67

(b)



(c) (i) the range of values of x for which y is negative is $0<x<1$

(ii) For value of $2x(x-1) = x+3$,

Divide through by $x+3$

$\Rightarrow \frac{2x(x-1)}{x+3} = \frac{x+3}{x+3}$

$\Rightarrow y = 1$

Hence, by tracing, the value of x is $x = -0.5,\ 2.2$

Example 24.9

(a) Copy and complete the table of values for the relation

$y = \frac{10}{x+3} + 3x - 3$ for the interval $-2.25 \le x \le 1.5$

x	-2.25	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
y	3.58			-1.0	-0.5		1.36	2.5	

(b) Using a scale of 4cm to 1unit on the x-axis and 2cm to 1unit on y-axis, draw a graph for the relation

(c) Estimate from your graph,

(i) The solution set of the equation $(3x-3)(x+3)+10 = 0$

(ii) The values of x for which $y = 1$

Solution

(a) By method of substitution,

When $x = -2,\ y = \frac{10}{-2+3} + 3(-2) - 3 = 1$

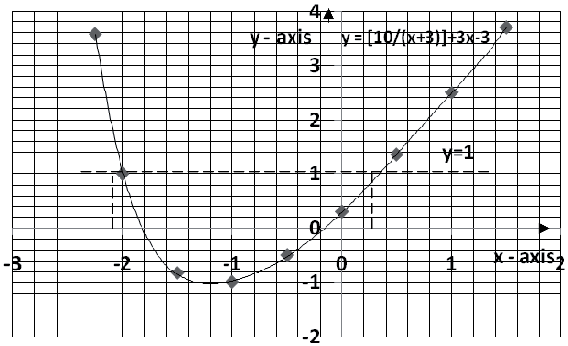
When $x = -1.5,\ y = \frac{10}{-1.5+3} + 3(-1.5) - 3 = -0.83$

When $x = 0,\ y = \frac{10}{0+3} + 3(0) - 3 = 0.3$

When $x = 1.5,\ y = \frac{10}{1.5+3} + 3(1.5) - 3 = 3.7$

x	-2.25	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
y	3.58	1.0	-0.83	-1.0	-0.5	0.3	1.36	2.5	3.7

(b)



(c) (i) for $(3x-3)(x+3)+10=0$,

Divide through by $x+3$ to conform to the original equation

Thus,

$$\frac{(3x-3)(x+3)}{x+3}+\frac{10}{x+3}=\frac{0}{x+3}$$

$$(3x-3)+\frac{10}{x+3}=0$$

Rearranging gives, $\frac{10}{x+3}+3x-3=0$

Now subtracting from original equation gives, $y=0$

Hence, solution is $x=-1.8,\ -0.15$

(ii) Values of x for which $y=1$ are $x=-2,\ 0.35$

24.6 **Cubic Relations**

A cubic relation is one in which the highest power of the variable in the relation is three (3). Thus, relations of the form:
$y=ax^3+bx^2+cx+d$ where a, b, c and d are constants and $a\neq 0$ are cubic relations.

Graphs of these relations have two turning points. When the coefficient of x^3 is positive, the graph will have a first maximum turning point on the left, followed by a minimum turning point on the right and when the coefficient of x^3 is negative, the graph will have a minimum turning point first, followed by a maximum turning point.

Example 24.10

(a) Copy and complete the table of values for the relation $y = x^3 - 12x + 3$ and for the interval $-2.5 \leq x \leq 2.5$

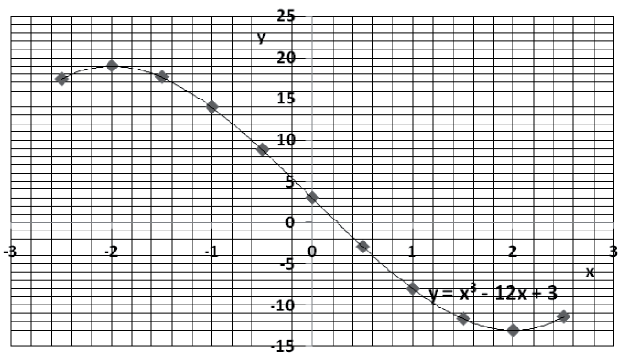
x	-2.5	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5
y	17.38			14	8.88	3		-8	-11.63		-11.38

(b) Using a scale of 2cm to 1unit on the x-axis and 2cm to 5units on the y-axis, draw a graph of the relation for the given interval
(c) Use your graph to find
(i) The greatest and least values of y
(ii) The truth set of the equation $x^3 - 12x + 3 = 0$

Solution
(a)

x	-2.5	-2	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5
y	17.38	19	17.63	14	8.88	3	-2.9	-8.0	-11.63	-13	-11.38

(b)



(c) (i) greatest value of y is 19 and its least value is -13
(ii) truth set of $x^3 - 12x + 3 = 0$ is 0.24

EXERCISE

QUE. A

(a) Copy and complete the table for the relation: $y = 7 + 4x - 3x^2$ for the interval $-3 \le x \le 4$

x	-3	-2	-1	0	0.5	1	1.5	2	2.5	3	3.5	4
y					8.25			3		-8	-15.5	

(b) Taking 2cm as a limit on x-axis and 2cm as 5units on y-axis, draw the graph of $y = 7 + 4x - 3x^2$ for the given interval. **Note:** will have a max. turning point

(c) Draw on the same graph sheet the graph of the relation $y + 2x + 2 = 0$

Note: this will be a linear graph with equation $y = -2x - 2$

(d) Using your graphs,

(i) Solve the equation $9 + 6x - 3x^2 = 0$

(ii) Find the values of x for which $y = 17 + 4x - 3x^2 = 0$

QUE. B

(a) Copy and complete the table of values for the relation $y = 3 - 2x - x^2$ for the interval $-5 \le x \le 3$

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12		0			3			

(b) Using a scale of 2cm to 1unit on x-axis and 2cm to 2units on y-axis, draw on the same graph sheet the graphs of:

(i) $y = 3 - 2x - x^2$ **Note:** will have a maximum turning point

(ii) $y + 2x + 4 = 0$ **Note:** will be a linear graph with equation $y = -2x - 4$

(c) From your graphs find

(i) The equation of the line of symmetry for the curve

(ii) The solution set of the equation $x^2 - 7 = 0$

(iii) The values of x for which the relation $y = 3 - 2x - x^2$ is greater than zero

QUE. C

(a) Copy and complete the table of values for the function $y = (x+1)(3-x)$ for the interval $-2 \le x \le 4$

x	-2.0	-1.0	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	-5.00	3.00						1.75		-2.25	-5.0

(b) Draw the graph of $y = (x+1)(3-x)$ using a scale of 2cm to 1unit on both axes for the interval $-2 \le x \le 4$ **Note:** will be a maximum graph
(c) Using your graph, find the greatest value of $(x+1)(3-x)$ and the value of x at which it occurs.

QUE. D

(a) Copy and complete the following table for the relation: $y = x^2 - 5x - 2$ in the interval $-1 \le x \le 6$

x	-1	0	1	2	3	4	5	6
y	4			-8		-6		

(b) Using a scale of 2cm to 1unit on the x-axis and 2cm to 2units on the y-axis, draw the graph of the relation **Note:** graph will have a min. turning point
(c) Use your graph to find the minimum value of y

QUE. E

(a) Copy and complete the table of values for the relation $y = \frac{1}{2}(x-3)(x+1)$ for the interval $-3 \le x \le 5$

x	-3	-2	-1	0	1	2	3	4	5
y	6			-2	-2		0		

(b) Draw on the same graph sheet and using the same area the graph of the relations $y = \frac{1}{2}(x-3)(x+1)$ and $y = -\frac{1}{2}x$ for the interval given

(c) Use your graphs to solve

$(x-3)(x+1) = 1$ and $(x-3)(x+1) = -x$

QUE. F

(a) Copy and complete the following table of values for the relation:

$y = 4 + 5x - 2x^2 \quad for \quad -3 \le x \le 5$

x	-3	-2	-1	0	1	2	3	4	5
y		-14		4	7				-21

(b) Using 2cm to 1unit on the x-axis and 2cm to 5units on the y-axis, draw the graph of $y = 4 + 5x - 2x^2$ for $-3 \le x \le 5$

(c) From your graph, find the

(i) Value of x for which y is maximum

(ii) Gradient at $x = 0$

(iii) Values of x for which $1 + 5x - 2x^2 = 0$

QUE. G

(a) Copy and complete the following table of values for the relation

$y = 10 + 6x - 3x^2 \quad for \quad -3 \le x \le 5$

x	-3	-2	-1	0	1	2	3	4	5
y				10	13		1	-14	

(b) Using a scale of 2cm to 1unit on the x-axis and 2cm to 5units on the y-axis, draw the graph of the relation for the given interval

(c) From your graph, solve the equations:

(i) $10 + 6x - 3x^2 = 0$

(ii) $5 + 2x - x^2 = 0$

(d) Find the equation of the axis of symmetry

QUE. H

(a) Copy and complete the table for the relation: $y = 7 + 3x - x^2$ for the interval $-3 \le x \le 6$

x	-3	-2	-1	0	1	2	3	4	5	6
y	-11			7				3		-11

(b) Using a scale of 2cm to 1unit on x-axis and 2cm to 2units on y-axis, draw the graph of the relation and interval given

(c) Use your graph to find

(i) The solution of $7 + 3x - x^2 = 0$ (ii) The solution of $5 + 3x - x^2 = 0$

(iii) The maximum value of y

QUE. I

(a) Copy and completer the table of values for the relation $y = 2x^2 + 5x - 3$

x	-5	-4	-3	-2	-1.5	-1	-0.5	0	1	2	2.5
y	22		0			-6		-3			22

(b) Using a scale of 2cm to 1 unit on the x-axis and 2cm to 5 units on the y-axis, draw the graph of $y = 2x^2 + 5x - 3$, *for the interval* $-5 \le x \le 2.5$.

(c) Use your graph in (b) to solve the equation: $2x^2 + 6x - 5 = 0$

QUE. J

(a) Copy and complete the table below for the relation $y = 7 - 5x - 2x^2$
for the interval: $-4 \le x \le 2$.

x	-4	$-3\frac{1}{2}$	-3	-2	-1	0	1	$1\frac{1}{2}$	2
y	-5		4			7			

(b) Using a scale of 2cm to 1unit on the x-axis and 2cm to 2units on the y-axis, draw the graph of the relation in (a)

(c) Using the graph, find the

(i) equation of the axis of symmetry of the curve

(ii) maxumum value of y

(iii) roots of the equation: $5-5x-2x^2=0$

(iv) range of values of x for which $7-5x-2x^2>0$

CHAPTER 25

PROBABILITY

25.1 **Introduction**

Probability is simply a study of the likelihood of events occurring. In short, probability is taking a portion of a whole quantity and expressing it as a ratio of the whole quantity. Thus, we divide the portion taken by the total quantity.

NB

We use $P(E)$ to denote the probability of an event occurring and the probability of the event not occurring by $P(\bar{E})$. Thus, mathematically, $P(E)+P(\bar{E})=1$. It is advisable to write down the event first as a set.

25.2 **Basic Definitions**

25.2.1 **Experiment**

An ***experiment*** is a performance whose outcome cannot be determined with certainty. An example is the throwing of a die or the tossing of a coin.

25.2.2 **Trial**

A ***trial*** is the individual performance of an experiment. An example is a single throw of a die.

25.2.3 **Random Sampling**

Random sampling is simply a selection without biasness.

25.2.4 **Sample Space**

Sample Space is the set of all the possible outcomes of a trial of an experiment. For instance, in tossing a coin, the sample space is $\{HH, HT, TH, TT\}$ denoting tail by T and head by H whiles when a die is thrown, the sample space is the set $\{1, 2, 3, 4, 5, 6\}$.

NB

The elements in a sample space are called the **sample points.**

25.2.5 **Event**

An ***event*** is obtained by putting a number of elements of a sample space together. For instance, when a coin is tossed, the event of tossing a head and a tail is $\{HT, TH\}$

NB

In probability, we use **'and'** or **'intersection'** to denote **'multiplication'**
And **'or'** and **'union'** to denote **'addition'**

25.3 **Forms of Probability**

25.3.1 **Case 1**

In our everyday to day activities, we use probability especially in dealing with various items (countable).

Example 25.1

A number is chosen from the set $\{2, 3, 4, 5, 6\}$. What is the probability that it is an even number?

Solution

First, the set for the experiment is
$\{2,3,4,5,6\}$ *which has a total of* 5 *elements*

The set for the event of even numbers is $\{2,4,\ 6\}$ *with* 3 *elements*

Hence, $P(Even)=\dfrac{number\ of\ even\ numbers}{total\ number\ of\ elements}=\dfrac{3}{5}$

Example 25.2

Mansa chooses a number at random from 1 to 20 inclusive. What is the probability that the number is divisible by 3?

Solution

Sample space is $\{1,2,3,4,5,6,---20\}$
Hence, total number of elements is 20
Set of numbers divisible by 3 is $\{3,6,9,12,15,18\}$
Number of elements divisible by 3 is 6
Hence, $P(divisible\ by\ 3)=\dfrac{6}{20}=\dfrac{3}{10}$

Example 25.3

A number is selected at random from the set of integers 1 to 20 inclusive. What is the probability that the number selected is equal to or greater than 18?

Solution

Sample space is $\{1,2,3,4,5,6,---20\}$
Total number of elements is 20
Set of elements equal or greater than 18 is $\{18,19,20\}$
Number of elements equal or greater than 18 is 3
Hence, $P(equal\ or\ greater\ than\ 18)=\dfrac{3}{20}$

Example 25.4

A bag contains 15 red balls and 6 blue balls. If one ball is picked at random from the bag, what is the probability that it is blue?

Solution

Number of red balls is 15
Number of blue balls is 6
Total number of balls is $15+6=21$

$$P(blue\ ball)=\frac{number\ of\ blue\ balls}{total\ number\ of\ balls}=\frac{6}{21}=\frac{2}{7}$$

Example 25.5

A bag contains 36 marbles which are identical. 10 are red, 12 are green and the remaining yellow. A marble is selected at random from the bag. What is the probability that the selected marble is yellow?

Solution

Total number of marbles is 36
Number of green marbles is 12
Number of red marbles is 10
Number of yellow marbles is $36-(12+10)=36-22=14$

$$P(yellow\ marble)=\frac{14}{36}=\frac{7}{18}$$

Example 25.6

There are 40 identical marbles in a box. 18 of them are blue, 14 red and 8 green. What is the probability that a marble selected at random is either blue or green?

Solution

Total number of marbles is 40
Number of blue marbles is 18
Number of red marbles is 14
Number of green marbles is 8
Number of marbles that are either blue or green is $18+8=26$

$$P(either\ blue\ or\ green\ marble)=\frac{26}{40}=\frac{13}{20}$$

Alternatively,

$$P(blue\ marbles)=\frac{18}{40}=\frac{9}{20}$$

$$P(green\ marbles)=\frac{8}{40}=\frac{1}{5}$$

Hence, $P(either\ blue\ or\ green\ marble)=P(blue)+P(green)=\frac{9}{20}+\frac{1}{5}=\frac{13}{20}$

Example 25.7

A box contains 40 pens of equal sizes. 10 of them are green and 18 are red. If a pen is chosen at random from the box, what is the probability that it is neither green nor red?

Solution

Total number of pens is 40
Number of green pens is 10
Number of red pens is 18
Number of pens that are neither green nor red is
$40-(10+18)=40-28=12$

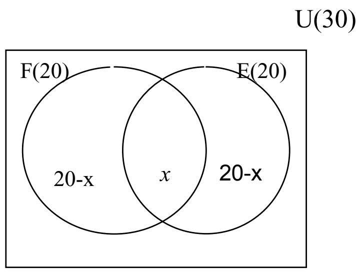
$$Hence,\ P(neither\ green\ nor\ red)=\frac{12}{40}=\frac{3}{10}$$

Example 25.8

In a class of 30 pupils, 20 speak French and 20 also speak English. Find the probability that a pupil selected at random from the class speaks both French and English.

Solution

Total number of pupils is 30
Number who speak French is 20
Number who speak English is 20
Let number who speak both languages be x



$$\Rightarrow 20 - x + x + 20 - x = 30$$
$$40 - x = 30$$
$$40 - 30 = x$$
$$x = 10$$
$$Hence,\ P(both\ languages) = \frac{10}{30} = \frac{1}{3}$$

Example 25.9

Box P contains 2 red and 4 green marbles. Another box labeled Q contains 5 yellow and 7 green marbles. A marble is selected at random from each box. Find the probability that (i) one is yellow and the other is red (ii) they are of the same colour.

Solution

For box P

Number of red marbles is 2
Number of green marbles is 4
Total number of marbles is 6

For box Q

Number of yellow marbles is 5
Number of green marbles is 7
Total number of marbles is 12

(i) $P(red \;\; from \;\; P \;\; and \;\; yellow \;\; from \;\; Q) = P(R_p \cap Y_q) = P(R_p) \times P(Y_q)$

$$= \frac{2}{6} \times \frac{7}{12} = \frac{1}{3} \times \frac{5}{12} = \frac{5}{36}$$

(Since only box P has red marbles and box Q has yellow marbles)

(ii) $P(same \;\; colour) = P(green \;\; from\, P \;\; and \;\; green \;\; from \;\; Q) = \frac{4}{6} \times \frac{7}{12} = \frac{7}{18}$

(Since both boxes have green marbles)

Example 25.10

A bag contains 12 balls of which x are red. When 18 more red balls are added, the probability of selecting a red ball is $\frac{9}{10}$. Find the number of red balls that were originally in the bag.

Solution

Total number of balls is 12
Number of red balls is x
Adding 18 red balls to x red balls becomes (x + 18) red balls
Thus, after the addition, the probability of selecting a red ball is $\frac{x+18}{30}$

But from question, $P(red \;\; balls) = \frac{9}{10}$

Implies, $\frac{x+18}{30} = \frac{9}{10}$

x = 9

Hence, the original number of red balls was 9.

25.3.2 **Case 2**

Tossing Of A Coin

When a coin is tossed n times, the number of elements in the sample space is obtained by using 2^n, where n is the number of times the coin is tossed. The faces of the coin are labeled ***head*** (H) and ***tail*** (T). Tossing a coin twice is the same as tossing two coins once and tossing a coin thrice is the same as tossing three coins together once. We get the sample space by perming the ***H*** and ***T*** the number times observed.

Example 25.11

A fair coin is tossed twice.
(a) Write down the set of all possible outcomes
(b) Find the probability of obtaining (i) No tail (ii) A tail and a head
(iii) Exactly 2 tails (iv) At most one head

Solution

(a) Since $2^n = 2^2 = 4$, sample space is $\{HH, HT, TH, TT\}$
(b) (i) Set or event of no tail is $\{HH\}$

Hence, $P(no\ tail) = \frac{1}{4}$

(ii) event of a tail and a head is $\{HT, TH\}$

Hence, $P(a\ tail\ and\ a\ head) = \frac{2}{4} = \frac{1}{2}$

(iii) event of exactly 2 tails is $\{TT\}$

Hence, $P(exactly\ two\ tails) = \frac{1}{4}$

(iv) event of at most one head is {HT, TH, TT}

Hence, $P(at\ most\ two\ heads) = \frac{3}{4}$

Example 25.12

Three fair coins are tossed together once.
(a) Write down the set of all possible outcomes
(b) Find the probability of obtaining
(i) No tail (ii) Not more than two tails (iii) Exactly two tails

Solution

(a) Sample space is $\{HHH, HHT, THH, THT, HTH, TTH, HTT, TTT\}$

(b) (i) event of no tail is $\{HHH\}$ Hence, $P(no\ tail) = \frac{1}{8}$

(ii) event of not more than two tails is
{HHH, HHT, THH, THT, HTH, TTH, HTT}

Hence, $P(not\ more\ than\ two\ tails) = \frac{7}{8}$

(iii) event of exactly two tails is $\{THT, TTH, HTT\}$

Hence, $P(exactly\ two\ tails) = \frac{3}{8}$

25.3.3 **Case 3**

Throwing Or Casting A Die

Here, we consider the throw of a die. Either a once throw of the die or twice throw. When it is thrown once the sample space is just one (1) to six (6). But when it is thrown twice or two dice are together once we obtain 36 different pairs of possible outcomes (sample space). For convenience, we advise that a table is used to illustrate the set of possible outcomes or the sample space for throwing two dice.

Example 25.13

A boy tossed a fair die once and he obtained a score of 6. Find the probability that he will obtain a score of 6 in the next toss.

Solution

Sample space is $\{1,2,3,4,5,6\}$

$P(obtaining\ a\ 6) = \frac{1}{6}$

Example 25.14

Two fair dice A and B each with faces numbered 1 to 6 are thrown together

(a) Construct a table showing all the 36 equally likely outcomes

(b) From your table, list the pair of numbers on the two dice for which the sum is

(i) 5 (ii) More than 10

(c) Find the probability that the sum of the numbers on the two dice is

(i) 5 (ii) More than 10

Solution (a)

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

(b) (i) From the table above, the event (pairs) of numbers with the sum of 5 are:

(1,4), (2,3), (3,2), (4,1)

(ii) From the table above, the event (pairs) of numbers with the sum of more than 10 are: (5,6), (6,5), (6,6)

(c) (i) From table, the total number of outcomes is 36

Number of outcomes whose sum is 5 is 4

$P(sum\ is\ 5) = \frac{4}{36} = \frac{1}{9}$

(ii) Number of outcomes whose sum is more than 10 is 3

$P(sum\ is\ more\ than\ 10) = \frac{3}{36} = \frac{1}{12}$

Example 25.14

Two fair dice are tossed at the same time.

(a) Draw the sample space for the possible outcomes

(b) Find the probability of

(i) a total of 6 or 8

(ii) the same number on the two dice

(iii) a total of not less than 5

Solution

(a)

	1	2	3	4	5	6
1	(1, 1)	(1,2)	(1,3)	(1,4)	(1, 5)	(1, 6)
2	(2, 1)	(2,2)	(2,3)	(2,4)	(2, 5)	(2, 6)
3	(3, 1)	(3,2)	(3,3)	(3,4)	(3, 5)	(3, 6)
4	(4, 1)	(4,2)	(4,3)	(4,4)	(4, 5)	(4, 6)
5	(5, 1)	(5,2)	(5,3)	(5,4)	(5, 5)	(5, 6)
6	(6, 1)	(6,2)	(6,3)	(6,4)	(6, 5)	(6, 6)

(b)

(i) Total number of elements in the sample space is 36

A total of 6 or 8 ***means*** the total number of sample points in the sample space that will add up or sum up to 6 or 8

Implies, a total number of 6 or 8 from the table is 10

Hence,

$$P(Total\ of\ 6\ or\ 8) = \frac{10}{36} = \frac{5}{18}$$

(ii) Total number of elements of the same number is 6

Implies,

$$P(Same\ number) = \frac{6}{36} = \frac{1}{6}$$

(iii) Totals not less than 5 means the total number of sample points in the sample space whose sum is less than 5.

Implies, number of totals not less 5 is 30

Hence,

$P(Not\ less\ than\ 5) = \frac{30}{36} = \frac{5}{6}$

Example 25.15

A boy tossed a fair die once and he obtained a score of 6. Find the probability that he will obtain a score of 6 in the next toss.

Solution

$P(Score\ is\ 6) = \frac{1}{36}$

25.3.4 **Case 4**

Here, the probabilities of certain events are given and asked to use the values to solve questions. We mainly apply the multiplication and addition laws here.

Example 25.15

In a class, the probability that a student passes a test is $\frac{2}{5}$.
What is the probability that if 2 students are chosen at random from the class, one would pass and the other will fail?

Solution

Probability that a student fails = 1 – probability that a student passes
Implies, $P(Student\ fails) = 1 - P(Student\ passes) = 1 - \frac{2}{5} = \frac{3}{5}$

$Since\ two\ students\ are\ chosen,\ we\ assume\ the\ 2\ students\ are\ X\ and\ Y$

$\Rightarrow P(X\ \ passes\ and\ \ Y\ \ fails) = P(A\ \ student\ \ passes) \times P(A\ \ student\ \ fails)$

$$= \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$$

$Again,\ P(Y\ \ passes\ and\ \ X\ \ fails) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$

$Hence,\ P(One\ \ passes\ and\ the\ other\ \ fails)$

$= P(X\ \ passes\ and\ \ Y\ \ fails)\ or\ P(Y\ \ passes\ and\ \ X\ \ fails)$

$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25} = 0.48$

Example 25.16

The probabilities of two candidates, A and B passing an examination are $\frac{2}{3}\ and\ \frac{3}{4}\ respectively.$. Find the probability that:

(i) only one candidate will pass

(ii) at least one candidate will pass

Solution

(i)

$P(A) = \frac{2}{3} \quad \Rightarrow P(\bar{A}) = 1 - \frac{2}{3} = \frac{1}{3}$

$P(B) = \frac{3}{4} \quad \Rightarrow P(\bar{B}) = 1 - \frac{3}{4} = \frac{1}{4}$

Probability that only one candidate will pass is "probability that A will pass and B will fail or probability that B will pass and A will fail"

$\Rightarrow P(only\ one\ will\ \ pass) = P(A \cap \bar{B})\ or\ P(\bar{A} \cap B)$

$$= \frac{2}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{3}{4} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

(ii) Probability that at least one candidate will pass is equal to 1 minus the probability that none will pass.

Thus, $P(at\ least\ one\ will\ \ pass) = 1 - P(none\ will\ \ pass)$

But $P(\bar{A}) = \frac{1}{3}\ \ and\ \ P(\bar{B}) = \frac{1}{4}$

$$P(none\ will\ pass) = P(A\ will\ not\ pass)\ and\ P(B\ will\ not\ pass) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

$$P(at\ least\ one\ will\ pass) = 1 - P(none\ will\ pass) = 1 - \frac{1}{12} = \frac{11}{12}$$

Example 25.16

Three blue balls, five green balls and a number of red balls are put together in a sack. One ball is picked at random from the sack. If the probability of picking a blue ball is $\frac{1}{6}$, find (i) the number of red balls in the sack (ii) the probability of picking a green ball.

Solution

(i) Number of blue balls is 3
Number of green balls is 5
Let number of red balls be x
Therefore, total number of balls in sack is $3 + 5 + x = 8 + x$

$$P(blue\ balls) = \frac{3}{8 + x}$$

$$\Rightarrow \frac{3}{8 + x} = \frac{1}{6}$$

(Since probability of blue balls is $\frac{1}{6}$)

$$\Rightarrow x = 10$$

Therefore, the number of red balls is 10

(ii) Number of green balls is 5
Total number of balls is $8 + x = 8 + 10 = 18$

$$P(green\ balls) = \frac{5}{18}$$

25.3.5 **Case 5**

Here, a table is used to represent the given question.

Example 25.17

Class	A	B	C
Boys	16	13	13
Girls	14	22	18

The table shows three classes A, B and C in a certain school. The three classes are put together to select a prefect. What is the probability that the prefect will be (i) a boy?
(ii) a girl in class B?

Solution

Total number of students is $16+13+13+14+22+18=96$
(i) Number of boys is $16+13+13=42$

$$P(a\ boy)=\frac{42}{96}=\frac{7}{16}$$

(ii) number of girls in class B is 22

$$P(a\ girl\ in\ class\ C)=\frac{22}{96}=\frac{11}{48}$$

25.3.6 **Case 6**

This is usually called drawing '***with replacement'*** from a bag or box. Here, the number of items do not change or reduce after drawing an item from the bag or box.

Example 25.18

A bag contains 20 marbles of which 7 are green and 9 are white. Two balls are picked one after the other with replacement. What is the probability that:
(i) Both are green
(ii) Both are of the same colour

Solution

Number of green marbles is 7
Number of white marbles is 9
Total number of marbles is 20

$(i)\ P(both\ are\ green) = first\ is\ green\ and\ second\ is\ green = P(G_1 \cap G_2)$

$$= \frac{7}{20} \times \frac{7}{20} = \frac{49}{400} = 0.1225$$

$(ii)\ P(both\ are\ of\ same\ colour) = P(both\ green)\ or\ P(both\ white) = \frac{49}{400} + \frac{81}{400} = \frac{13}{40}$

25.3.7 **Case 7**

This type of probability is usually called drawing '***without replacement'***. Thus, a subsequent withdrawal will depend on the previous draw. i.e. a second withdrawal will reduce the number of that item by one and reduce the total number also by one.

Example 25.19

A bag contains 12 blue and 8 red balls. If 2 balls are picked at random from the bag
without replacement, what is the probability that they are both blue?

Solution

Number of blue balls is 12
Number of red balls is 8
Total is 8 + 12 = 20

$P(both\ are\ blue) = P(first\ blue) \times P(second\ is\ blue) = \frac{12}{20} \times \frac{11}{19} = \frac{33}{95}$

(Since after first draw, the number of blue reduces by 1 and total also by 1)

Example 25.20

A bag contains 6 red, 8 black and 10 yellow identical beads. Two beads are picked at random, one after the other without replacement. Find the probability that:
(a) **both** are red
(b) **one** is black and the other yellow

Solution

Number of red beads is 6
Number of black beads is 8
Number of yellow beads is 10
Total number of beads is 6 + 8 + 10 = 24

$$(a)\ P(both\ red) = P(first\ red\ and\ \sec ond\ red) = \frac{6}{24} \times \frac{5}{23} = \frac{5}{92}$$

$$(b)\ P(1\ black\ and\ other\ yellow) = P(1st\ black\ \alpha\ 2nd\ yellow)\ or\ P(1st\ yellow\ \alpha\ 2nd\ black)$$

$$= \frac{8}{24} \times \frac{10}{23} + \frac{10}{24} \times \frac{8}{23} = \frac{10}{69} + \frac{10}{69} = \frac{20}{69}$$

EXERCISE

QUE. A

A box contains 3 black and 5 red balls of the same size. If a ball is picked at random, what is the probability that it is black?

QUE. B

In a race, 3 students completed from the green house, 2 from the red house, 4 from the white house and 1 from the yellow house. What is the probability that a student from the white house will come first, if there is no tie?

QUE. C

There are 8 boys and 4 girls in a lift. What is the probability that the first person who steps out of the lift will be a boy?

QUE. D

If X and Y are independent outcomes of an event, then the probability P_r (X or Y) is given by what?

QUE. E

The probabilities that two boys pass an examination are $\frac{2}{3}$ *and* $\frac{5}{8}$.

Find the probability that:

(a) The two boys pass the examination

(b) Only one of the boys passes the examination

QUE. F

When a die is thrown, what is the probability of obtaining

(a) a 7?

(b) a whole number between 1 and 6 inclusive?

(c) an even number?

QUE. G

A basket contains 50 oranges, 12 of which are rotten. If an orange is selected at random, what is the probability that it will not be rotten?

QUE. H

What is the probability of selecting a boy from a class of 19 boys and 17 girls if a pupil is chosen at random?

QUE. I

In a 'silver chance' raffle, 250 tickets were sold. What is the probability of Madam Akos winning the first prize if she bought five tickets?

QUE. J

A survey of 150 families with four children was carried out in Navrongo to find the number of girls in each family. The results were as follows.

Number	0	1	2	3	4
Frequency	8	41	55	33	13

Estimate the probability that a family with four children will have
(a) two girls (b) no girls (c) one boy (d) four boys (e) at least three girls

QUE. K

Select at random a number from the first 30 whole numbers
1, 2, …30. What is the probability that the selected number is

(a) divisible by five?
(b) not divisible by five?

QUE. L

In a class of 36 girls, 28 are wearing earrings. A girl is selected at random. What is the probability that the selected girl

(a) is wearing earrings
(b) is not wearing earrings

CHAPTER 26
STATISTICS
(PIE & BAR CHARTS, HISTOGRAM, OGIVE, MEASURE OF CENTRAL TENDENCY AND DISPERSION, STANDARD DEVIATION)

Statistics is an aspect that collects information called data, represents it graphically and interprets the graph mathematically.

26.1 Some Basic Terminologies

26.1.1 Frequency

This is simply the number of times each item occurs in a classification.

26.1.2 Frequency Distribution Table

This is a table of frequencies with corresponding classes of data.

Illustration

Grouped data freq. table

Age (years)	Frequency
1- 9	5
10 – 19	1
20 - 29	12
30 – 31	3

Ungrouped data freq. table

Age (years)	Frequency
1	7
2	3
3	1
4	9

26.1.3 **Class Interval**

The class interval is simple the range of values in each class grouping. The first and last numbers in an interval are called the ***class limits*** where the first is described as the lower limit and the second number is the upper limit. Eg. in the interval 1 – 9, 1 is the lower limit and 9 is the upper limit.

26.1.4 **Class Midpoint**

This is the middle value obtained by finding the average of the sum of the class limits.
Thus, class midpoint = ½ (lower limit + upper limit)

Illustration

Age(years)	Frequency	Class midpoint
0 – 9	2	4. 5
10 – 19	5	14. 5
20 – 29	1	24. 5

i.e. for the interval 10 – 19, class midpoint
= ½ (10 + 19) = ½ (29) = 14 . 5

26.1.5 **Class boundaries**

This is a range obtained by subtracting **0.5** from the lower class limit and adding **0.5** to the upper class limit.

Illustration

Age(years)	Frequency	Class boundaries
5 – 9	1	4.5 – 9.5
10 - 14	2	9.5 - 14.5
15 – 19	3	14.5 – 19.5

NB

The lower class boundary of a successor is the upper class boundary of preceding boundary.

26.1.6 **Class Size**

This is the difference between the lower and upper class boundaries.

NB

For equal class interval data, the class size will be equal throughout.
Illustration

Age(yrs)	Frequency	Class size
5 – 9	1	5
10 – 14	2	5
15 – 19	3	5

26.2 **The Pie Chart**

The ***Pie Chart*** is a pictorial representation of data with the help of the sectors of a circle drawn using the protractor.

The following are the ***steps*** in drawing a pie chart:

1. Compute the total of the items (values)
2. Calculate the angle of each item by dividing the value of the item by the total value and multiplying the result by 360

$$Angle\ of\ item = \frac{item\ value}{Total\ values} \times 360$$

3. Draw a circle of any radius using the pair of compasses.
4. Locate the centre of the circle drawn and draw a radius from it to any point on the circle.
5. With the black horizontal line of the protractor on the radius drawn and the 90^{o} mark at the center, measure and draw first angle labeling the sector.

NB

The next angle is measured starting from that drawn in step 5. Hence, subsequent angles are drawn or measured from preceding ones.

Example 26.1

The table below shows the no. of cars sold by a company from January to June 1990.

JAN	FEB	MARCH	APRIL	MAY	JUNE
7100	7668	10366	9940	8236	7810

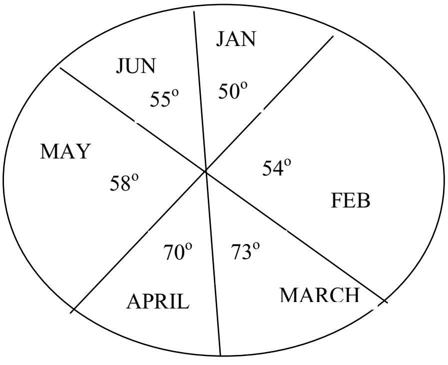
(a) Draw a pie chart to illustrate the information
(b) What is the percentage of cars sold in February?

Solution

(a) Total angles in a circle = 360^{o}
Total no. of cars = 7100 + 7668 + 10366 + 9940 + 8236 + 7810
= 51120

Month	No. of Cars	Angle (in degree)
JAN	7100	$\frac{7100}{51120}\times 360 = 50$
FEB	7668	$\frac{7667}{51120}\times 360 = 54$
MARCH	10366	$\frac{10366}{51120}\times 360 = 73$
APRIL	9940	$\frac{9940}{51120}\times 360 = 70$
MAY	8236	$\frac{8236}{51120}\times 360 = 58$
JUNE	7810	$\frac{78}{51120}\times 360 = 55$

The Pie Chart



(b) no. of cars sold in FEB = 7668
Total no. of cars sold = 51120

$$percentage\ of\ cars\ sold\ in\ FEB = \frac{No\ of\ cars\ in\ FEB}{Total\ no.\ of\ cars}\times 100\%$$

$$= \frac{7668}{51120}\times 100\% = 15\%$$

OR

$$percentage\ of\ cars\ sold\ in\ FEB = \frac{Angle\ for\ FEB}{Total\ angle} \times 100\%$$

$$= \frac{54}{360} \times 100\% = 15\%$$

Example 26.2

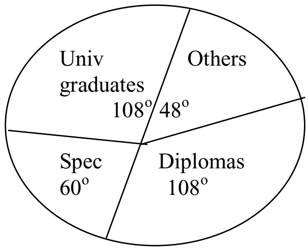
A group of mathematics teachers were classified as follows:
University graduate 120
Diplomas 90
Specialists 50
Others y
(a) Calculate the value of y if the teacher altogether were 300.
(b) Draw a pie chart to illustrate the above information

Solution

(a) total no. of teachers = 300
But sum of all teachers = 120 + 90 + 50 + y = 260 + y
Implies, 260 + y = 300
Therefore, y = 300 – 260 = 40
(b)

Teachers	No. of Teachers	Angle(in degree)
University graduates	120	$\frac{120}{300} \times 360 = 144$
Diplomas	90	$\frac{90}{300} \times 360 = 108$
Specialists	50	$\frac{50}{300} \times 360 = 60$
Others	40	$\frac{40}{300} \times 360 = 48$

The Pie Chart



Not drawn to scale

Example 26.3

A pie chart was drawn to represent the monthly car sales for the period January to June 2003. The angles of the six sectors of the pie chart are: 50°, 54°, 73°, 70°, 59°, 54°.
The total number of cars sold during the given period was 511200.
Calculate the no. of cars sold in January and what was the highest no. of cars sold during the period?

Solution

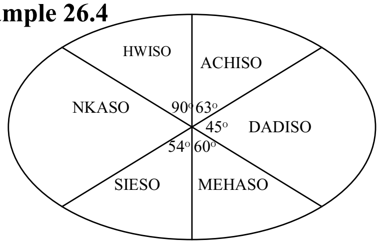
Total angles = 360°
Total cars sold = 511200
Angle of cars sold in January = 50°

$$No\ of\ cars\ sold\ in\ Jan = \frac{50}{360} \times 511200 = 71000$$

Highest angle for cars sold = 73°

$$Highest\ no\ of\ cars\ sold = \frac{73}{360} \times 511200 = 103660$$

Example 26.4



Not drawn to scale

The pie chart represents the votes Cecilia obtained in six villages during an election. Cecilia obtained 4200 votes at Achiso. Use the information to answer question
31 to 33.

QUE. 31

How many votes did Cecilia obtain altogether in the six villages?

QUE.32

In which village were Cecilia's votes equal to her mean votes?

QUE . 33

How many votes did Cecilia obtain at NKaso?

Solution

QUE . 31

Sum of all angles = 360^o
Implies, 90 + 63 + 45 + 60 + 54 + NKaso = 360
312 + Nkaso = 360
Hence, angle for Nkaso = 360 – 312 = 48^o
Total votes at Achiso = 4200
Angle for Achiso = 63^o
Let total votes altogether = y

$$No.\ of\ votes\ at\ Achiso = \frac{63}{360}y$$

$$Hence, \frac{63}{360}y = 4200$$

$$\Rightarrow y = 24000$$

Therefore, total votes obtained by Cecilia is 24000

QUE. 32

Total votes = 24000

$$mean\ votes = \frac{24000}{6} = 4000$$

$$And\ cecilia's\ votes\ at\ Mehaso = \frac{60}{360} \times 24000 = 4000$$

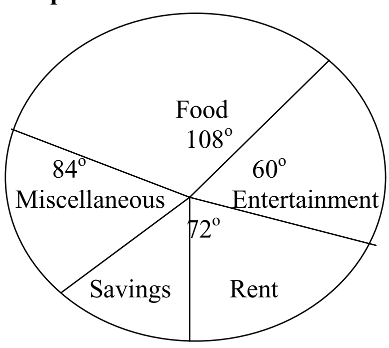
Hence, Cecilia's votes at Mehaso equal her mean votes

QUE 33

Angle for Nkaso = 48^o
Total angle = 360^o
Total votes = 24000

$$no.\ of\ votes\ at\ Nkaso = \frac{48}{360} \times 24000 = 3200$$

Example 26. 5



Not drawn to scale

The pie chart shows how a worker spends his salary in a month.
If he spends
¢180, 000.00 on rent, find
(a) how much he earns in a month
(b) how much he saves in a month
(c) the percentage of his salary that he spends on food and entertainment together

Solution

(a) Amount spent on rent = ¢108,000.00 let total amount earned in a month = ¢y
But

$$amount\ spent\ on\ rent = \frac{angle\ for\ rent}{total\ angle} \times total\ amount$$

$$180{,}000 = \frac{72}{360}y$$

$$\Rightarrow y = 900{,}000$$

Hence, he earns ¢900,000.00 in a month
(b) Angle for saving = 360 – (72 + 60 + 108 + 84) =360 - 324 = 36°

$$Amount\ he\ saves\ in\ a\ month = \frac{angle\ on\ savings}{total\ angles} \times total\ amount$$

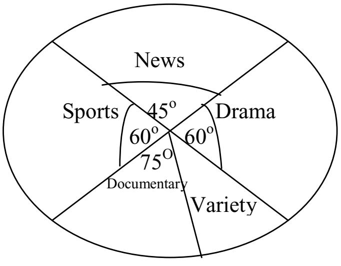
$$= \frac{36}{360} \times 900000 = 90{,}000$$

(c) Angle for food and entertainment together = 108 + 60 = 168°

$$percentage\ on\ food\ \&\ entertainment = \frac{168}{360} \times 100 = 47\%$$

Example 26.6

The pie chart below shows the program analysis of a television station. The station telecasts 6 hours each day. Use it to answer question 16 and 17.



Not drawn to scale

QUE.16

What is the size of the angle for variety?

QUE. 17

How many hours in a day does the station telecast documentary?

Solution

QUE. 16

Sum of angles including variety = 360^{o}
Implies, 45 + 60 + 75 + 60 + variety = 360
240 + variety = 360
Variety = 360 – 240 = 120^{o}
The angle for variety = 120^{o}

QUE . 17

Total no. of hours = 6 hrs.
Total angle of circle = 360^{o}
Angle for documentary = 75^{o}

$$If\ 360^{o} \equiv 6hrs$$

$$then,\ 75^{o} \equiv xhrs$$

$$\Rightarrow x = \frac{75 \times 6}{360} = 1\tfrac{1}{2}\,hrs$$

Hence, the station uses 1½ hrs to telecast documentary.

26.3 **The Bar Chart**

The bar chart is a graph representing frequency distribution of data. We represent the frequencies using vertical bars where the height becomes the frequency of each item.
NB
The bars of a bar chart do not touch each other.

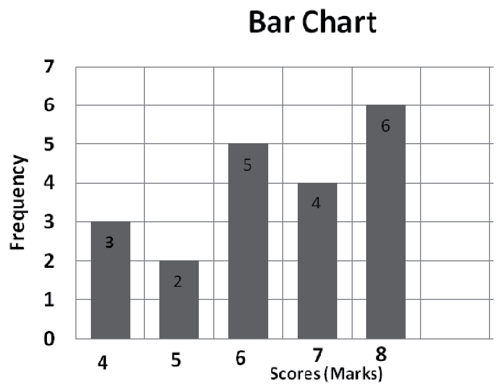
Example 26.7

The table below shows the scores obtained by 20 students in a test

Score	4	5	6	7	8
Freq	3	2	5	4	6

(b) Draw a bar chart for the distribution

Solution



26.4 **The Histogram**

The ***histogram*** is also a graphical representation of a frequency distribution. It is represented with vertical standing bars touching each other where the y-axis takes the frequencies label and the x-axis takes the class midpoint values or sometimes the upper-class boundaries and called “marks less than” values.

NB

In using the midpoints values as x-axis values, we mark exactly the middle of each bar and label it with its corresponding class midpoint value whiles the upper class boundaries are written at the starting of each bar starting from the lower limit of the first class boundary.

Example 26.8

The table below shows the distribution of marks by students in an examination.

Marks	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Freq	5	21	15	43	10	14	7	3	2

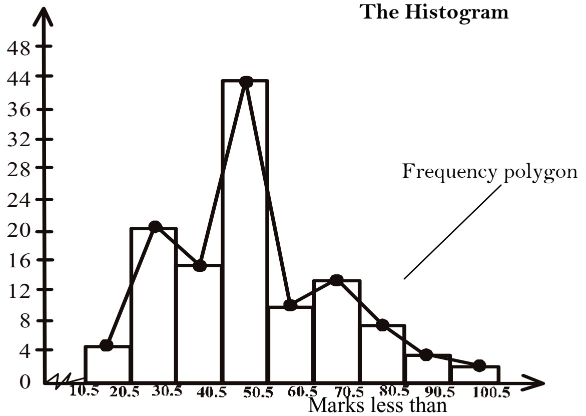
Draw a histogram to represent the above data.

Solution

Redrawing, the table becomes;

Marks	Frequency (f)	Class boundaries
11 – 20	5	10.5 - 20.5
21 – 30	21	20.5 - 30.5
31 – 40	15	30.5 – 40.5
41 – 50	43	40.5 – 50.5
51 – 60	10	50.5 – 60.5
61 – 70	14	60.5 – 70.5
71 – 80	7	70.5 – 80.5
81 – 90	3	80.5 – 90.5
91 - 100	2	90.5 -100.5

The Histogram



NB

The frequency polygon is not included in the question

26.4.1 **Frequency Polygons**

This is a straight line graph drawn through the top middle parts of each bar using a rule.
An illustration is on the histogram of example 26.8 above.

26.4.2 **Cummulative Frequency Table**

This is a table containing the cumulative frequencies obtained by maintaining the first frequency value and adding the second frequency value to it to obtain the second cumulative frequency value. Like wise we add the third frequency value to the second cumulative frequency value to obtain the third cumulative frequency value and so on till the last frequency value is added.

Example 26.9

The table below shows the heights, measured to the nearest meter of 300 tress

Heights (m)	2	3	4	5	6	7	8	9
No. of trees	14	21	42	83	118	12	7	3

Form a cumulative frequency table with the data

Solution

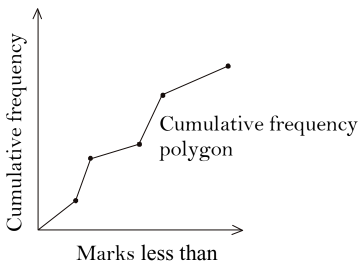
Heights (m)	No. of tree	Cumulative frequency
2	14	14
3	21	35
4	42	77
5	83	160
6	118	278
7	12	290
8	7	297
9	3	300

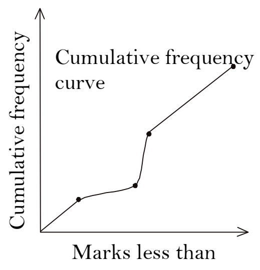
NB

The last cumulative frequency value must always correspond to the sum of all the frequencies.

26.4.3 **Cumulative Frequency Curve**

This is also called the '***Ogive***' (for points joined with free hand or 'frequency polygon' (for points joined by a straight line).
We use the upper class boundary values for the x-axis and the cumulative frequency values for the y-axis.





Example 26.10

The table below shows the heights in millimeters of a sample of 250 seedlings on an experimental farm.

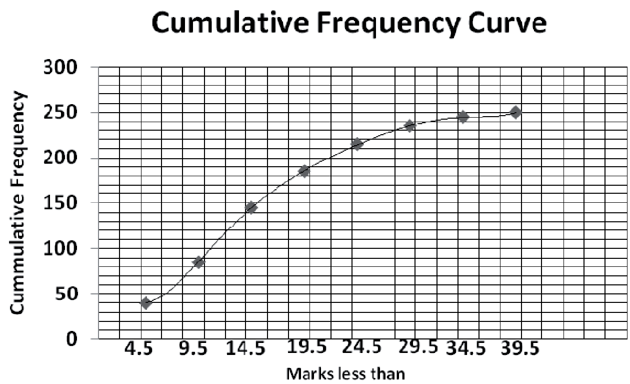
Heights of seedling	0 – 4	5 – 9	10 – 14	15– 19	20–24	25–29	30-34	35-39
No. of seedlings	40	45	60	40	30	20	10	5

Construct a cumulative frequency table and use it to draw a cumulative frequency curve for the sample

Solution

Cumulative frequency table

Heights (mm)	Freq.	Less than values	Cumulative freq.
0 – 4	40	4.5	40
5 – 9	45	9.5	85
10 – 14	60	14.5	145
15 – 19	40	19.5	185
20 – 24	30	24.5	215
25 – 29	20	29.5	235
30 – 34	10	34.5	245
35 – 39	5	39.5	250



26.5 **Measures Of Central Tendencies**

The main measures of central tendencies are: the mean, mode and median

26.5.1 **Mean**

The mean of a set of numbers is obtained by finding the average of the numbers. i.e. We sum all the given numbers and divide the result by 2. Mean is denoted by π.

Example 26.11

Find the mean of 306, 308, 304, and 314

Solution

$$Mean = \frac{sum\ of\ all\ numbers}{Total\ no.\ of\ \ values} = \frac{306+308+304+314}{4} = \frac{1232}{4} = 308$$

Example 26.12

The mean of the numbers 3, 5, 3x and 2 is 4. Find x.

Solution

$$Mean(\pi) = \frac{Sum\ of\ nos.}{Total\ no.\ of\ values} = \frac{3+5+3x+2}{4} = \frac{10+3x}{4}$$

$But\ Mean\ is\ 4\ from\ question$

$$\Rightarrow \frac{10+3x}{4} = 4 \Rightarrow 10+3x = 16 \Rightarrow x = 2$$

Example 26.13

The mean of the numbers, 22, 18, (2y+1), 10 and 20 is 15. Find the median.

Solution

$$\frac{22+18+(2y+1)+10+20}{5} = 15$$

$$\Rightarrow \frac{2y+71}{5} = 15 \Rightarrow 2y+71 = 75$$

$$\Rightarrow 2y = 75-71$$

$$\Rightarrow \frac{2y}{2} = \frac{4}{2} \Rightarrow y = 2$$

Hence, the numbers are: 22, 18, 5, 10, 20
Rearranging becomes: 5, 10, 18, 20, 22
Therefore, Median = 18 (The middle number when arranged)

Example 26 . 14

If the mean of p, q, r, s, and t is 10, calculate the mean of p+ 1, q + 2, r + 3, s + 4 and t + 5.

Solution

If the mean of p, q, r, s, t is 10

$$\Rightarrow \frac{p+q+r+s+t}{5} = 10$$

$$p+q+r+s = 50 - - - - - - - (1)$$

$Now, the\ mean of\ p+1, q+2, r+3, s+4, t+5\ is$

$$mean=\frac{p+1+q+2+r+3+s+4+t+5}{5}=\frac{(p+q+r+s+t)+15}{5}---------(2)$$

Substitute (1) into (2)

$$\Rightarrow mean=\frac{50+15}{5}=\frac{65}{5}=13$$

Example 26. 14

The mean of seven numbers is 15. When three numbers are added, the mean of the ten numbers becomes 12. Find the mean of the three numbers

Solution

Mean of seven numbers = 15

$$\Rightarrow mean=\frac{sum\ of\ seven\ nos.}{7}=15$$

$$\Rightarrow sum\ of\ seven\ nos.=7\times 15=105$$

$Now,$

$$mean\ of\ 10\ nos.=12$$

$$\Rightarrow mean=\frac{sum\ of\ 10\ nos.}{10}=12$$

$$\Rightarrow sum\ of\ 10\ nos.=10\times 12=120$$

Sum of three nos. added = sum of 10 nos. – sum of 7 nos.

= 120 – 105 = 15

$$\therefore mean\ of\ three\ nos.=\frac{sum of\ three\ nos.=15}{3}\frac{15}{3}=\ 5$$

Therefore, the mean of the three numbers added is 5.

Example 26. 15

The mean of marks of 16 girls in a class test is 20 and that of the 14 boys in the class is 25. Find the mean mark of the whole class.

Solution

The mean mark of 16 girls in class = 20

$\Rightarrow \frac{sum\ of\ marks\ of\ 16\, girls}{16} = 20$

$\Rightarrow sum\ of\ marks\ of\ 16\ girls = 20 \times 16 = 320$

Again, if the mean mark of 14 boys = 25

$\Rightarrow \frac{sum\ of\ marks\ of\ 14\ boys}{14} = 25$

$sum\ of\ marks\ of\ 14\ boys = 25 \times 14 = 350$

Total no of students in class = girls + boys = 16 + 14 = 30

$mean\ of\ 30\ students\ in\ class = \frac{320 + 350}{30} = 22.33$

26.5.2 **Finding Mean From Frequency Distribution Table**

In finding the mean from a frequency distribution table, we create a column for the product of frequencies and marks (fx). We then use the relation:

$$Mean\,(\pi) = \frac{\sum fx}{\sum f}$$

Example 26.16

The table below shows the distribution of marks obtained by twenty pupils in a test

Marks(x)	1	2	3	4	5	6	7
No. of pupils	1	3	5	6	2	1	2

Find the mean of the distribution.

Solution

Marks(x)	1	2	3	4	5	6	7
Frequency(f)	1	3	5	6	2	1	2
fx	1	6	15	24	10	6	14

$\sum f = 20, \sum fx = 76$

$mean = \frac{\sum fx}{\sum f} = \frac{76}{20} = 3.8$

NB

Mean from a frequency distribution could be calculated using the relation:

$mean = A + \frac{\sum fd}{\sum f}$

Where A = assumed mean usually given as one of the midpoint values, conveniently the midpoint that corresponds to the highest frequency.
d = deviation = x - A, i.e. Difference between each midpoint and assumed mean value. This Approach is appropriate for grouped data.

Example 26.17

The table below shows the frequency distribution of marks scored by 80 candidates in an examination

.

Marks	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Freq.	2	5	8	18	20	15	5	4	2	1

Find the mean using an assumed mean of 44.5

Solution

Marks	freq	Midpoint(x)	D = x - A	fd
0 – 9	2	4.5	-40	-80
10 – 19	5	14.5	-30	-150
20 – 29	8	24.5	-20	-160
30 – 39	18	34.5	-10	-180
40 - 49	20	44.5	0	0
50 – 59	15	54.5	10	150
60 – 69	5	64.5	20	100
70 – 79	4	74.5	30	120
80 – 89	2	84.5	40	80
90 - 99	1	94.5	50	50

$A = 44.5, \sum fd = -70, \sum f = 80$

$mean = A + \frac{\sum fd}{\sum f} = 44.5 + \frac{-70}{80} = 43.625$

Example 26.18

The marks scored by 40 candidates in aptitude test are as follows:

85	77	87	74	77
78	79	89	85	90
78	73	86	83	91
74	84	81	83	75
77	70	81	69	75
63	76	87	61	78
69	96	65	80	84
80	77	74	88	72

(a) Using a class interval of 60-64, 65-69, 70-74,…, prepare a frequency distribution table
(b) Calculate the mean mark of the candidates
(c) If 85 was the pass mark for the test, what percentage of the candidates passed the test?

Solution

(a)

Marks	f	x	fx
60-64	2	62	124
65-69	3	67	201
70-74	6	72	432
75-79	11	77	847
80-84	8	82	656
85-89	7	87	609
90-94	2	92	184
95-99	1	97	97

(b) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{3150}{40} = 78.75$

(c) Percentage of the candidates who passed the test

$= \frac{10}{40} \times 100 = 25\%$

26.5.3 **The Mode**

To find the mode from a set of numbers, we look for number that occurs most frequently.

Example 26. 18

Find the mode of the following set of numbers:

(i) {4, 5, 1, 3, 2, 1, 5, 7, 1}
(ii) {2, 1, 3, 3, 5, 0, 2, 3}

Solution

(i) from set of numbers, 1 occurs 3 times. Hence, the mode is 1 (since occurs most frequently)
(ii) 3 occurs three times (the highest occurrence). Hence, mode is 3.

NB

The mode of an ungrouped distribution table is the item/mark with the highest frequency.
Similarly, for grouped data distribution, mode is simply the interval with highest frequency.

Example 26.19

The heights in cm of 15 mango seedlings are as follows:
23, 21, 20, 18, 23, 21, 24, 21, 20, 19, 25, 21, 22, 18, 21

(i) What is the modal height of the seedlings?

(ii) Calculate the probability that a seedling selected at random from the group has a height of 21cm.

Solution

(i) Modal height of the seedlings is 21 since it occurs most frequently

(ii) $21 \text{ occurs } 5 \text{ times and total number of seedlings} = 15$

$$\Rightarrow P(\text{Seedling of height } 21cm) = \frac{5}{15} = \frac{1}{3}$$

Example 26.19

The frequency table shows the age distribution of students which offer mathematics at a college.

Age(in years)	17	18	19	20	21
No. of students	7	15	9	5	4

What is the modal age of the students?

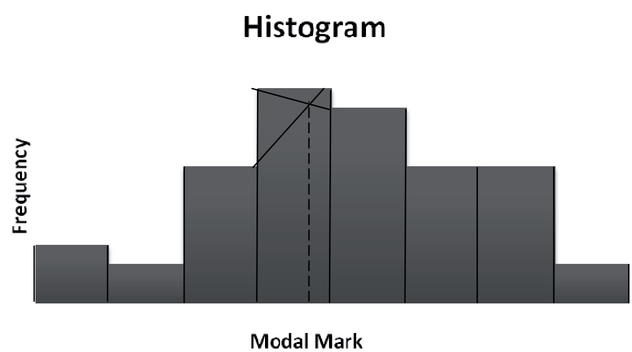
Solution

The modal age is 18 years since it corresponds to the highest frequency on the table.

NB

Again, we can calculate the mode from a histogram by Joining the upper right Corner of the highest bar to the upper right Corner of the immediate bar before it and also joining the upper left Corner of the highest bar to that of the immediate bar after it and tracing the intersection of these straight lines to the horizontal axis for the modal mark.

Illustration



26.5.4 **The Median**

In the finding the median of any set of numbers, we locate the middle value when the numbers are rearranged in ascending order regardless of whether some numbers are repeated or not. When exactly one number appears as the middle number, that number turns to be the median of the set. But when we arrive at two numbers being at the middle, we find their arithmetic mean for the median value.

Example 26.20

Find the median of the following set of numbers
(i) {2, 2, 1, 3, 5} (ii) {3, 2, 1, 0, 3, 1, 2, 5}

Solution

(i) for the set {2, 2, 1, 3, 5}
Rearranging becomes: {1, 2, 2, 3, 5}
Hence, the median is 2 since 2 is the middle number when arranged in ascending order
(ii) for the set {3, 2, 1, 0, 3, 1, 2, 5}
rearranging gives 0, 1, 1, 2, 2, 3, 3, 5
$median = \frac{2+2}{2} = 2 \text{ (}Since\ there\ are\ two\ middle\ nos.\text{)}$

Similarly, we find the median from a frequency distribution by considering the sum of its frequencies i.e. $\sum f$. Thus, if $\sum f$ is an add number, we add 1 to it and divide the result by 2.
i.e. $\frac{1}{2}(\sum f + 1)th\ number$. After which we start adding the frequencies from the first frequency until we get to the value obtained and its corresponding mark gives the median.
Again, when the $\sum f$ is even, we divide it by 2 and also add 1 to the result of dividing to obtain two values.
i.e. $\sum f\ \ and\ (\sum f + 1)th\ number$. We then start adding from the first frequency until we arrive at a frequency where both numbers fall and that corresponding mark becomes the median. Cases where the values fall under two different frequencies, the mean of their corresponding marks are evaluated to obtain the median.

Example 26.21

The table shows the distribution by age of the inhabitant in a certain village.

Age(yrs)	1	2	3	4	5	6	7	8	9	10
No. of people	2	1	3	3	4	1	3	1	0	1

Find the median age of the distribution

Solution

$\sum f = 19, \frac{\sum f + 1}{2} = 10$

Hence, by adding, 2 + 1 + 3 + 3 = 9. Thus, we need only 1 from the next frequentcy to make 10. Hence, the 10^{th} falls under that frequency. So the median age is its corresponding age. i.e. Median age = 5

Example 26.22

The table below shows the distribution of marks obtained by twenty pupils in a test.

Mark	1	2	3	4	5	6	7
No of pupils	1	3	5	6	2	1	2

Find the median mark.

Solution

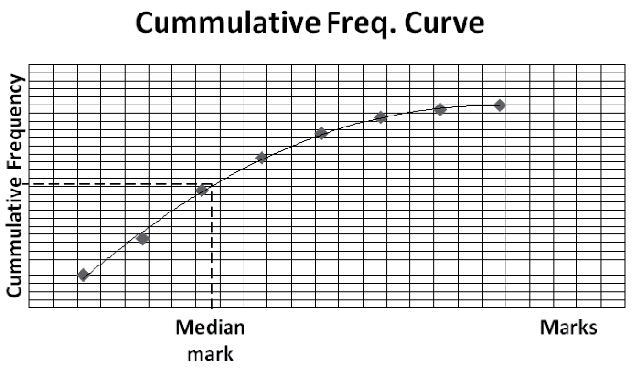
$\sum f = 20, \Rightarrow \frac{\sum f}{2} = \frac{20}{2} = 10th \;\; and \;\; (10+1)th = 11th$

By adding frequencies till the 10^{th} and the 11^{th} mark becomes:
1 + 3 + 5 = 9. Thus, we need 1 from the next frequency to make 10 and 2 from the same frequency to make 11. So we take the mean of that mark. i.e. $\frac{4+4}{2} = 4$

NB

We could also calculate the median mark from a cumulative frequency cure. This is done by tracing a horizontal line through $\frac{1}{2}\sum f$ mark on the y - axis to the curve and then a vertical line to the x – axis for the median mark usually obtained on the x – axis.

Illustration



26.6 **Finding Quartiles, Percentiles And Deciles**

These are usually asked of in grouped data distribution where a cumulative frequency curve is drawn.

26.6.1 **The Quartile**

Quartiles divide a data into four (4) equal divisions. We have two types of quartiles. ***The lower quartile*** obtained by using the relation: $\frac{1}{4}\sum f$ traced on the cumulative Frequency axis to the curve and to the horizontal axis.

The upper quartile obtained by using the relation: $\frac{3}{4}\sum f$ traced on the cumulative Frequency axis to the curve and to the horizontal axis.

NB

We can find the two quartiles in a similar way as median from a distribution table. Thus, the lower quartile = $\frac{1}{4}\sum f +$
$\frac{1}{4}\sum f + 1 (for\ even\ \sum f)$
and lower quartile = $\frac{1}{4}(\sum f + 1)\ value\ for\ odd\ \sum f$
upper quartile = $\frac{3}{4}\Sigma f + \frac{3}{4}\Sigma f + 1\ \ for\ even\ \Sigma f$
and upper quartile = $\frac{1}{4}(\sum f + 1)\ for\ odd\ \sum f$

Example 26.23

The heights of seedlings are given below

Height of seedlings	33	34	35	36	37	38	39
No. of seedlings	5	3	1	5	2	3	0

Find the lower and upper quartiles

Solution

$$\sum f = 5 + 3 + 1 + 5 + 2 + 3 + 0 = 19$$
$$Lower\ quartile = \tfrac{1}{4}(\sum f + 1) = \tfrac{1}{4}(19 + 1) = \tfrac{1}{4}(20) = 5$$
$$Upper\ quartile = \tfrac{3}{4}(\sum f + 1) = \tfrac{3}{4}(19 + 1) = \tfrac{3}{4}(20) = 3 \times 5 = 15$$

26.6.2 **The Percentile**

Percentiles divide a data into 100 equal parts. Percentiles are denoted by P_1, P_2, P_3, ------- P_{99}. Thus, the 60^{th} percentile is P_{60} and the 75^{th} percentile is P_{75}.
Thus, percentiles are obtained by dividing them by 100 and multiplying by $\sum f$ and traced on the Cumulate frequency axis to the curve and then to the horizontal axis.
Thus, 60^{th} percentile = $\frac{60}{100}\sum f$
And the 75^{th} percentile = $\frac{75}{100}\sum f$

26.6.3 **The Decile**

Deciles divide a data into 10 equal points and denoted by:
D_1, D_2, D_3, -------- D_9.
Thus, the nth Decile = $\frac{n}{10}\sum f$ *on the cummulative frequency axis*
Implies, 9^{th} decile = $\frac{9}{10}\sum f$ *on the cummulative frequency axis*

Example 26.23

The marks obtained by 40 students in an examination are:
63 79 87 61 78 85 77 87 74
80 77 74 88 72 78 79 89 85
77 70 81 69 75 78 73 86 83
69 96 65 88 84 74 84 81 83
77 90 91 75

(a) Copy and complete the table below

Class Boundary	Tally	Freq (f)	Class midpoint (x)	fx
59.5 – 64.5			62	
64.5 – 69.5			67	
69.5 – 74.5				
74.5 – 79.5				
79.5 – 84.5				
84.5 – 89.5				
89.5 – 94.5				
94.5 – 99.5		1	97	97
		$\sum f = 40$		$\sum fx =$

(b) (i) Using the relation $\pi = \frac{\sum fx}{\sum f}$ *or otherwise find the mean,* π

(ii) Calculate the probability that a student chosen at random obtained at least 75 marks.
(c) Draw a histogram using the frequency table

Solution

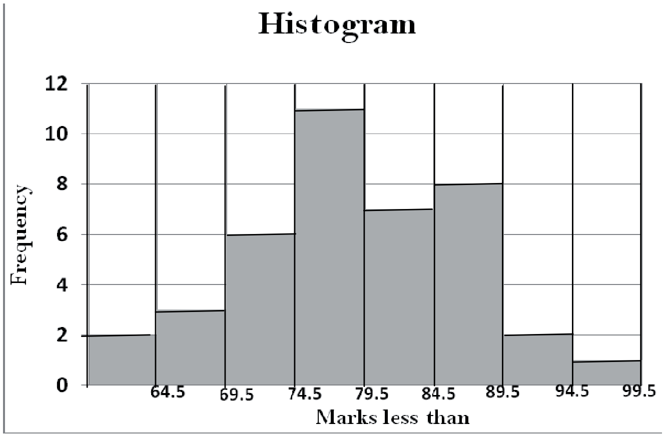
(a)

Class Boundary	Tally	Freq (f)	Class midpoint (x)	fx
59.5 – 64.5	II	2	62	124
64.5 – 69.5	III	3	67	201
69.5 – 74.5	~~IIII~~ I	6	72	432
74.5 – 79.5	~~IIII~~ ~~IIII~~ I	11	77	847
79.5 – 84.5	~~IIII~~ II	7	82	574
84.5 – 89.5	~~IIII~~ III	8	87	696
89.5 – 94.5	II	2	92	184
94.5 – 99.5	I	1	97	97
		$\sum f = 40$		$\Sigma f = 3155$

(b) (i) $\pi = \dfrac{\sum fx}{\sum f} = \dfrac{3155}{40} = 78.875$

(ii) number of students who obtained at least 75 marks = 29

$\Rightarrow \Pr obability = \dfrac{29}{40}$



Example 26.24

The table shows the frequency distribution of marks scored by 80 candidates in an examination.

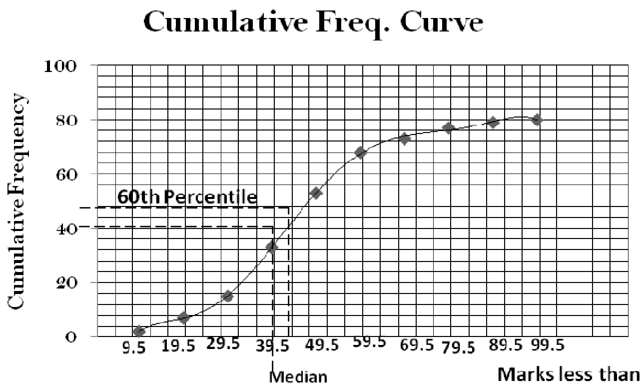
Marks (%)	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Freq	2	5	8	18	20	15	5	4	2	1

(a) Draw the Cummulative frequency curve for the distribution
(b) Use your curve to estimate
(i) The median (ii) the 60^{th} percentile mark
(c) If the mark for distinction was 70%, what percentage of students passed with distinction?

Solution

Marks (%)	Frequency	Cum. frequency	Marks less than
0 - 9	2	2	9.5
10 -19	5	7	19.5
20 – 29	8	15	29.5
30 – 39	18	33	39.5
40 – 49	20	53	49.5
50 – 59	15	68	59.5
60 – 69	5	73	69.5
70 – 79	4	77	79.5
80 - 89	2	79	89.5
90 - 99	1	80	99.5

(a)



(b) (i) Median mark = $\frac{1}{2}\sum f = \frac{1}{2}\times 80 = 40th\ candidate's\ mark$

Hence, from the graph, median mark is 42.5

(ii) 60^{th} Percentile = $60th\ Pecentile = \frac{60}{100}\times 80 = 48th\ candidate's\ mark$

Hence, from the graph, 60^{th} percentile is 47.5

(c) Distinction mark = 70%

Number of student who had distinction = 4 + 2 + 1 = 7

Total number of students = 80

Percentage who had distinction = $\frac{7}{80}\times 100 = 8.8\%$

NB: Question does not require the use of the graph

Example 26.25

The marks obtained by 40 students in an examination are as follows:

51	46	38	68	21	51	58	64
72	33	86	48	67	93	71	63
44	50	22	91	78	66	52	81
43	64	53	82	45	58	57	72
62	77	61	74	88	35	43	56

(a) Using a class interval of 20 - 29, 30 – 39, 40 - 49 etc, construct a frequency distribution table for the data.

(b) Draw a cumulative frequency cumulative for the distribution

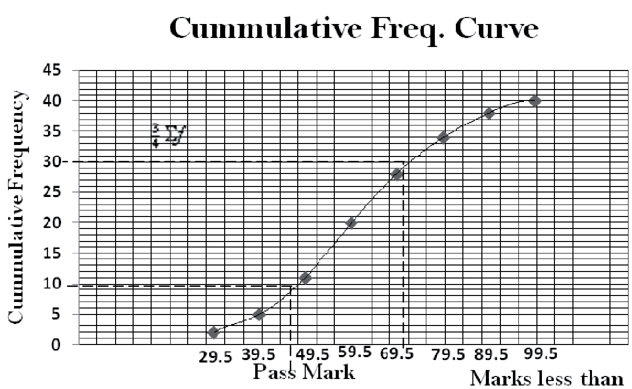
(c) Use your curve to estimate

(i) the upper and lower quartiles
(ii) the pass mark , if 31 students passed

Solution

Marks	Frequency (f)	Cumulative frequency	Marks less than
20 - 29	2	2	29.5
30 - 39	3	5	39.5
40 - 49	6	11	49.5
50 - 59	9	20	59.5
60 - 69	8	28	69.5
70 - 79	6	34	79.5
80 - 89	4	38	89.5
90 - 99	2	40	99.5

(b)



(c) (i) $Lower\ quartile = \frac{1}{4}\sum f = \frac{1}{4}(40) = 10th\ student's\ mark$

From graph, lower quartile = 47.5

$Upper\ quartile = \frac{3}{4}\sum f = \frac{3}{4}(40) = 30th\ student's\ mark$

From graph, upper quartile = 71.5

(ii) If 31 students passed means, 9 students failed.

i.e. 40 – 31 = 9. Thus, the pass mark is the mark that corresponds to 9. i.e. The pass mark from graph is 46.5

Example 26.26

The data below shows the distribution of marks by some students in a test

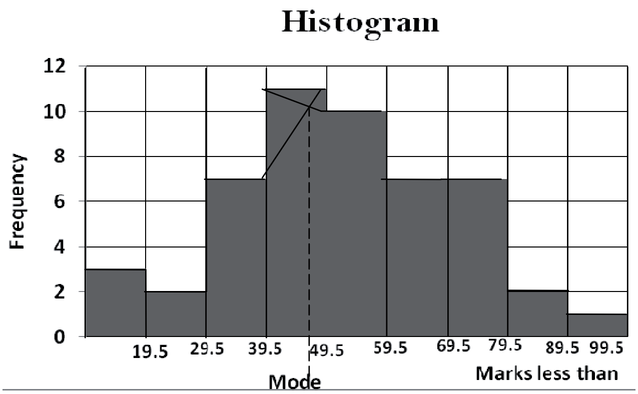
10	73	19	78	24	78	34	34	35	35
37	59	41	63	41	65	45	55	55	58
48	49	49	53	53	55	45	55	65	67
58	38	59	43	61	44	48	68	85	90
14	74	29	76	31	48	68	85	90	70

(a) Using a class interval of 10 – 19, 20 – 29 etc. construct a frequency table for the distribution.
(b) (i) Draw a histogram for the distribution
(ii) Use your histogram to estimate the mode
(c) Calculate the mean of the distribution.

Solution
(a)

Marks	Freq (f)	Class boundaries	Midpt(x)	fx
10 - 19	3	9.5 -19.5	14.5	43.3
20 – 29	2	19.5 – 29.5	24.5	49.0
30 - 39	7	29.5 – 39.5	34.5	241.5
40 - 49	11	39.5 – 49.5	44.5	489.5
50 – 59	10	49.5 - 59.5	54.5	545.5
60 – 69	7	59.5 -69.5	64.5	451.5
70 – 79	7	69.5 – 79.5	74.5	521.5
80 – 89	2	79.5 – 89.5	84.5	169.5
90 – 99	1	89.5 – 99.5	94.5	94.5

(b)



(ii) from graph, mode is 46.5

(c) $\sum f = 50 \ \ and \ \sum fx = 2605$

$$\Rightarrow Mean = \frac{\sum fx}{\sum f} = \frac{2605}{50} = 52.1$$

Example 26.27

The table below shows the marks obtained by 60 candidates in a test

Marks	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45	46-50
Freq	2	3	7	9	11	13	7	5	2	1

(a) Construct a cumulative frequency table and use it to draw the Cummulative freq curve.

(b) Use your Cummulative frequency curve to determine:

(i) the median mark

(ii) the number of candidates who passed, if the pass mark was 18

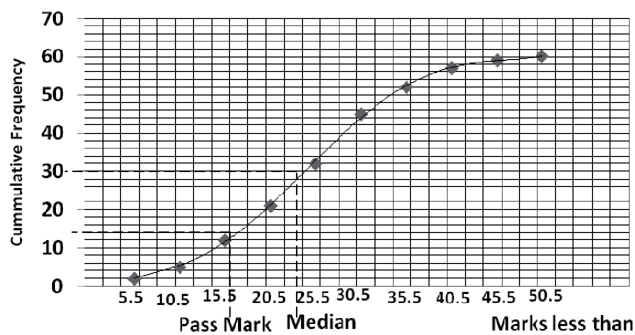
(iii) the lowest mark for distinction, if 10% of the candidates obtained distinction.

Solution

(a)

Marks	Freq	Cumm. freq	Marks less than
1 - 5	2	2	5.5
6 - 10	3	5	10.5
11 – 5	7	12	15.5
16 – 20	9	21	20.5
21 – 25	11	32	25. 5
26 – 30	13	45	30.5
31 - 35	7	52	35.5
36 – 40	5	57	40.5
41 – 45	2	59	45.5
46 – 50	1	60	50.5

Cummulative Frequency Curve



(b) (i) From graph, median mark is 24.5

(ii) The number of candidates who obtained less than 18 marks = 15

Hence, number who passed = 60 – 15 = 45

(iii) 10 % of total of 60 candidates = $\frac{10}{100}\times 60 = 6$

Therefore, 6 candidates obtained distinction.

Implies, 60 – 6 = 54 candidates did not obtain distinction

Therefore, from graph, 54 on y – axis corresponds to 37.5 on x axis.

Hence, the lowest mark for distinction is 37.5

NB

When given number who passed, subtract it from the total number to obtain number who failed and find its corresponding value and vice versa. Again, when given a number of distinctions, we subtract it from the total number for non-distinctions and then find its corresponding value.

Example 26.28

The following are the marks obtained by students in and an achievement test.

90	25	31	35	52	48	15
19	40	60	93	23	40	70
55	65	43	68	58	62	83
46	33	9	75	59	71	5
63	19	50	47	42	28	21

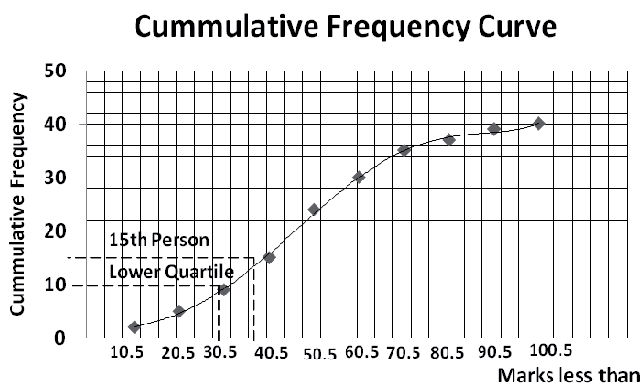
(a) Contract a cumulative frequency table for the distribution using the class intervals
1 – 10, 11- 20, 21 – 30, -----
(b) Draw a cumulative frequency curve for the distribution
(c) Use your graph to find (i) the lower quartile
(ii) the marks obtained by the student who was 15^{th} in the test.

Solution

(a)

Marks	Freq	Cum. freq	Marks less than
1 – 10	2	2	10.5
11 - 20	3	5	20.5
21 – 30	4	9	30.5
31 – 40	6	15	40.5
41 - 50	9	24	50.5
51 – 60	6	30	60.5
61 – 70	5	35	70.5
71 – 80	2	37	80.5
81 – 90	2	39	90.5
91 - 100	1	40	100.5

(b)



(c) (i) lower quartile = $\frac{1}{4}\sum f = \frac{1}{4}(40) = 10$

From graph, 10 correspond to 32.5 marks. Hence, lower quartile = 32. 5

(ii) 15 on cum. freq. axis corresponds to 40.5 on the horizontal axis.

Therefore, the mark obtained by 15th person in the test was 40.5

26.7 **Measures Of Dispersion**

The different categories of measures of dispersion are: Range, Inter quartile range, Semi-Inter quartile range, Mean deviation and Standard deviation.

26.7.1 **The Range**

This is the difference between the highest and lowest values
E.g. find the range of the following data:
(i) 11, 20, 31, 53, 30
(ii) 2, 7, 15, 39

Solution

Range = 53 – 11 = 42
Range = 39 – 2 = 37

26.7.2 **Interquartile Range**

This is the difference between the upper and lower quartiles.
i.e. Interquartile range = upper quartile - lower quartile

26.7.3 **Semi-Interquartile Range**

Semi- Interquartile Range is obtained by talking half the value of the Interquartile range
i.e. Semi- Interquartile Range = ½ (upper quartile – lower quartile)

Example 26.29

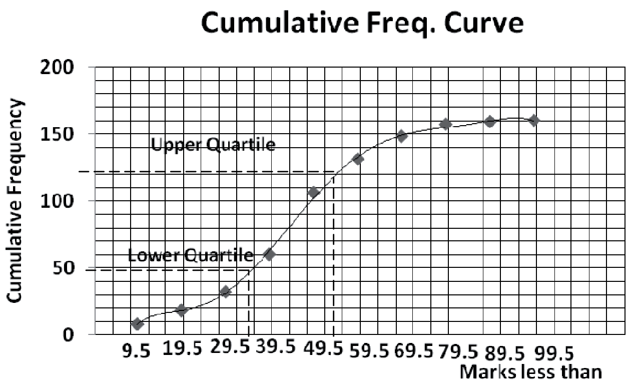
The table gives the frequency distribution of the marks scored by some candidates in an examination.

Marks	Frequency
0 - 9	8
10 -19	10
20 - 29	14
30 – 39	28
40 - 49	46
50 – 59	25
60 – 69	17
70 - 79	9
80 – 89	2
90 – 99	1

(a) Construct a cumulative frequency table and use it to draw a cumulative frequency curve.
(b) Use your graph to estimate the
(i) Interquartile range
(ii) Percentage of candidates who scored at least 65 marks

Solution
(a)

Marks	Freq	Cum. freq	Class boundaries	Marks less than
0 – 9	8	8	-0.5 – 9.5	9.5
10 – 19	10	18	9.5 – 19.5	19.5
20 – 29	14	32	19.5 - 29.5	29.5
30 – 39	28	60	29.5 – 39.5	39.5
40 – 49	46	106	39.5 – 49.5	49.5
50 – 59	25	131	49.5 – 59.5	59.5
60 – 69	17	148	59.5 – 69.5	69.5
70 – 79	9	15 7	69.5 – 79.5	79.5
80 – 89	2	159	79.5 – 89.5	89.5
90 – 99	1	160	89.5 – 99.5	99.5



(b) (i) From graph, the lower quartile is 32.5
From graph, upper quartile 55.5
Hence, Interquartile range = Upper – Lower = 55. 5 – 32. 5 = 23
(ii) From graph, 65 corresponds to 141(on x - axis).
Implies, number of candidates who scored at least 65 marks
= 160 – 141 = 19
Implies, Percentage of those who had at least 65 marks

$$= \frac{19}{160} \times 100 = 11.875 \equiv 12\%$$

Example 26.30

The table below shows the number of children per family in a village.

No. of children	1	2	3	4	5	6
No. of families	3	5	7	4	3	2

(a) Find the
(i) Mode
(ii) Third quartile
(iii) The probability that a family has at **least** 3 children

(b) If a pie chat is to be drawn for the data, what would be the sectorial angle representing the families with 2 children?

Solution

(a)

(i) Mode = 3 (Since 3 has the highest frequency)

(ii) Third quartile $= \frac{3}{4} \times 24th\ number = 18th\ number = 4$

(iii) Family with at least 3 children = 16
Hence, probability that a family has at least 3 children
$= \frac{16}{24}\frac{2}{3} = 0.67$

(b) Families with 2 children are 5, hence, $\frac{5}{24} \times 360 = 75^o$

26.8 **Mean Deviation**

26.8.1 **Finding Mean Deviation From A Set Of Numbers**

The mean deviation of the set: x_1, x_2, x_3, ------- x_n is given by:

$$Mean\ deviation = \frac{\sum|x-\pi|}{n}\ where\ \pi = \frac{\sum x}{n},\ n = number\ of\ values$$

NB: $|-5| = 5\ and\ |5| = 5$

Example 26.30

Find the mean deviation from the following set of numbers:
2, 7, 9, 29

Solution

$$\pi = \frac{\sum x}{n} = \frac{2+7+9+29}{4} = \frac{47}{4} = 11.75$$

Now;

$For\ x = 2, |x-\pi| = |2-11.75| = |-9.75| = 9.75$

$For\ \ x = 7,\ |x - \pi| = |7 - 11.75| = |-4.75| = 4.75$

$For\ \ x = 9,\ |x - \pi| = |9 - 11.75| = |-2.75| = 2.75$

$For\ \ x = 29,\ |x - \pi| = |29 - 11.75| = |17.25| = 17.25$

$\Rightarrow \sum |x - \pi| = 9.75 + 4.75 + 2.75 + 17.25 = 34.5$

$$Mean\ deviation = \frac{\sum |x - \pi|}{n} = \frac{34.5}{4} = 8.625$$

26.8.2 **Finding Mean Deviation From Frequency Distribution Table**

In finding the mean deviation from a distribution table, we use the relation:

$$Mean\ deviation = \frac{\sum f|x - \pi|}{\sum f}\ \ where\ \pi = \frac{\sum fx}{\sum f}\ \ and\ \ f = frequency$$

Example 26.31

Find the mean deviation from the distribution table below:

Marks	1	2	3	4	5	6	7	8	9	10
Freq	2	1	3	3	4	1	3	1	0	2

Solution

Marks (x)	Freq (f)	fx	$\lvert x-\pi\rvert$	$f\lvert x-\pi\rvert$
1	2	2	4	8
2	1	2	3	3
3	3	9	2	6
4	3	12	1	3
5	4	20	0	0
6	1	6	1	1
7	3	21	2	6
8	1	8	3	3
9	0	0	4	0
10	2	20	5	10
				$\sum f\lvert x-\pi\rvert = 40$

$$Mean\ deviation = \frac{\sum f\lvert x-\pi\rvert}{\sum f} = \frac{40}{20} = 2$$

26.9 **Standard Deviation**

This is a measure of dispersion used to compare two or more given sets of data

NB

Variance is obtained by taking the square of the standard deviation value.

i.e. variance = (standard deviation)^2

In finding the standard deviation from a set of numbers, we use the relation:

$$SD = \sqrt{\tfrac{1}{n}\sum (x-\pi)^2} \quad OR \quad SD = \sqrt{\frac{\sum x^2}{n} - \pi^2}$$

Example 26.32

The scores obtained by students in a test are:
21, 25, 27, 25, 27, 21, 24, 23, 23 and 24.
Calculate the standard deviation.

Solution

The Set is: 21, 25, 27, 25, 27, 21, 24, 23, 23, 24

$$\pi = \frac{\sum x}{n} = \frac{240}{10} = 24$$

$$\therefore \pi = 24 \ \ and \ \ n = 10$$

$$\Rightarrow (x-\pi)^2 = 9,1,9,1,9,9,0,1,1,0$$

$$\Rightarrow \sum (x-\pi)^2 = 40$$

$$\Rightarrow SD = \sqrt{\tfrac{1}{10} \times 40} = 2$$

NB

Could also form a distribution table with the given set.
By so doing, we use any of the relations:

$$SD = \sqrt{\frac{\sum fx^2}{\sum f} - \pi^2} \ \ OR \ \ SD = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \ \ For \ grouped \ data$$

Example 26.33

The marks scored by 15 pupils in a test are as follows:
14, 14, 11, 13, 17, 14, 11, 13, 20, 19, 17, 11, 20, 14, 17
Find (a) the mean mark
(b) the standard deviation

Solution

Marks (x)	Freq (f)	fx	x^2	fx^2
11	3	33	121	363
13	2	26	169	338
14	4	56	196	784
17	3	51	289	867
19	1	19	361	361
20	2	40	400	800

$$(a)\ \pi = \frac{225}{15} = 15$$

$$(b)\ SD = \sqrt{\frac{\sum fx^2}{\sum f} - \pi^2} = \sqrt{\frac{3513}{15} - 225} = \sqrt{9.2} = 3.033 \equiv 3$$

Example 26.34

The table below gives the frequency distribution of marks scored by 400 students in a test.

Marks (x)	Frequency (f)	fx
1	14	14
2	30	60
3	32	96
4	40	160
5	52	260
6	80	480
7	59	413
8	56	448
9	21	189
10	16	160

Find correct to **one** decimal place,

(a) The mean mark of the distribution
(b) The standard deviation of the distribution

Solution

Marks (x)	Frequency (f)	fx	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
1	14	14	22.09	309.26
2	30	60	13.69	410.70
3	32	96	7.29	233.28
4	40	160	2.89	115.60
5	52	260	0.49	25.48
6	80	480	0.09	7.20
7	59	413	1.69	99.71
8	56	448	5.29	296.24
9	21	189	10.89	220.29
10	16	160	18.49	295.84
	$\sum f = 400$	$\sum fx =$	$\sum(x-\bar{x})^2 = 82.9$	$\sum f(x-\bar{x})^2 = 2013.6$

(a) Mean, $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{2280}{400} = 5.7$

(b) Standard deviation $= \sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\dfrac{2013.6}{400}} = 2.24$

EXERCISE

QUE. A

The table gives the distribution of the ages in years of all persons (to the nearest thousand) in a town who were under the ages of 40 years on 30th June 1990.

Age	0-4	5-9	10-14	15-19	20 - 24	25-29	30 - 34	35 - 39
No. in 100s	2	3	6	15	12	7	4	1

(a) Using the table, calculate
(i) the total number of persons under 20 years
(ii) The total number of persons between 15 and 30 years.

(b) (i)Prepare the cumulative frequency table and use it to draw the cumulative frequency curve.
(ii) Use your curve to find the median age of the distribution.

QUE.B

The table below shows the heights, measured to the nearest of 300 trees

Heights (m)	2	3	4	5	6	7	8	9
No of trees	14	21	42	83	118	12	7	3

(a) Draw a histogram to represent the above data
(b) Calculate the mean of the distribution correct to the nearest meter.
(c) If a tree is selected at random, find the probability that it is at least 6 meters tall.

QUE. C

The table shows the height, in millimeter of a sample of 250 seedlings on an experimental farm

Heights of Seedlings	0 – 4	5 – 9	10 -14	15-19	20- 24	25 – 29	30 – 30	35- 39
No. of seedlings	40	45	60	40	30	20	10	5

(a) Construct a cumulative frequency table and use it to draw a cumulative frequency curve for the sample.
(b) Using your cumulative frequency curve, find
(i) The first and third quartiles (ii) the percentage of the sample of seedlings whose heights are above 22mm

QUE. D

The marks obtained by 40 pupils are as follows

78	42	30	33	66	39	72	60
54	44	42	24	45	60	72	30
39	36	66	33	36	48	18	45
27	78	33	42	67	63	30	63
60	27	30	81	27	33	72	18

(a) Construct a frequency table, using class intervals of
(b) Draw a histogram for the distribution taking 200
for the width of each bar and 2cm to present 2 pupils
(c) (i) Use your histogram to estimate the mode
(ii) Calculate the mean of the distribution

QUE . E

The table shows the scores obtained by 20 students in a test

Score	4	5	6	7	8
Freq	3	2	5	4	6

(a) Calculate the mean score for the test
(b) Draw a bar chart for the distribution

QUE. F

The score obtained by students in a test are:
21, 25, 27, 25, 27, 21, 24, 23, 23 and 24
Draw a bar chart to illustrate the information

QUE.G

The mean of the numbers 3, 5, 3x and 2 is 4. Find x.

QUE. H

The median of five numbers is 15. The ratio of the numbers is: 2 : 4 : 5 : 7 : 8 respectively. Find the first number.
(***Hint***: total ratio = 26, ratio in ascending order gives 5 as the middle number. Implies, 5 corresponds to 15

$$\Rightarrow Sum = \frac{26}{5} \times 15 = 78$$

$$\therefore first\ number = \frac{2}{26} \times 78 = 6$$

QUE. I

The mean of 40, 50 and 90 is p. If the mean of 40, 50, 90 and p is q, what is the value of q?

QUE. J

The table gives the scores of a group of students in a test. Use it to answer questions 32 and 33.

Score	0	1	2	3	4	5
Freq	2	3	4	2	7	2

QUE. 32

How many students scored above the mean score?

QUE. 33

Find the median

QUE. K

The mean of five numbers is 10. If a new number N is added, the mean becomes 12. Find the value of N.

QUE. L

The table shows the number of students who offer certain subjects in a school.

Subject	Number of students
Mathematics	45
Physics	39
Chemistry	28
Biology	14
Economics	36
History	18

Draw a pie chart to illustrate the data

QUE. M

In a class, there are 20 boys and their total score in a test is 810 marks. If the average score of the girls is 37 and that of the whole class is 39, calculate the
(a) Number of students in the class
(b) Ratio of the total marks of the girls to that of the boys

QUE. N

The table shows the distribution of marks scored by 50 students in a test.

Marks(%)	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90
Frequency	1	3	5	8	12	10	5	4	2

(a) Construct a cumulative frequency table for the distribution
(b) Draw a cumulative frequency curve for the distribution
(c) Use the curve to estimate the
  (i) interquartile range
  (ii) percentage of students who scored more than 66%

QUE. O

The ages of 50 taxi drivers in Damongo are given in the table below:

Ages (in years)	Frequency
24-29	3
30-34	7
35-39	11
40-44	12
45-49	8
50-54	5
55-59	3
60-64	1

(a) draw a cumulative frequency graph for this data
(b) use the cumulative frequency graph to estimate the median age of the taxi drivers

QUE. P

The following table shows the frequency distribution of the marks of 200 candidates in a Chemistry examination

Marks (%)	Number of Candidates
0-9	4
10-19	20
20-29	48
30-39	39
40-49	37
50-59	36
60-69	13
70-79	3

(a) draw up a cumulative frequency table for the data
(b) draw a cumulative frequency curve and use it to determine

(i) the lowest mark for a distinction if 6% of the candidates had distinction
(ii) the percentage of candidates passing, if the mark for a pass is 40
(iii) the lowest mark for a credit if a quarter of the candidates had a credit

QUE. Q

The table gives the masses, to the nearest kg, of 100 clients of an EL-AK SERIES PUBLICATION.

Mass (to nearest kg)	Frequency
55-59	3
60-64	5
65-69	10
70-74	18
75-79	20
80-84	17
85-89	11
90-94	7
95-99	4
100-104	2
105-509	2
110-114	1

Draw a cumulative frequency graph for the data and use it to find:
(a) the lower and upper quartile masses
(b) the 43^{rd} and 85^{th} percentiles
(c) the percentage of clients who had mass of less than 62kg
(d) the percentage of clients who had mass of more than 87kg.

QUE. R

Calculate the mean and the standard deviation of these sets of data.
(a) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

(b) 16, 14, 17, 12, 18, 13
(c) 1, 2, 5, 3, 2, 5, 2, 4, 3, 3
(d) 0, 9, 4, 6, 6
(e) 185, 185, 187, 187, 187, 189, 189
(f) 44, 42, 51, 47, 46

CHAPTER 27

VECTORS

A ***vector*** is a quantity which has both ***magnitude*** and ***direction***. Examples are: Momentum, acceleration, velocity, displacement etc.

Graphically, we represent vectors by directed line segments using upper case letters to denote them together with ***arrows*** on them describing ***direction***.

For instance, the "vector AB" is dented by $\vec{AB}$.

27.1 **Vector Representations**

The different ways vectors can be represented are:

1. The column or component form. i.e. the x-value is written ***up*** and the y-value ***below*** it. Egs. $\begin{pmatrix} a \\ b \end{pmatrix}$ *or* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
2. The ***row*** or ***coordinate*** form. i.e. the x-value is written first on the left whiles the y-value is written on its right with a '***comma***' separating the values. Egs. (a, b) or (2, 3)
3. The ***Cartesian*** vector form. i.e. The x-value is multiplied by 'i' and the y-value by 'j' and the two results summed. Egs. ai + bj, or 2i + 3j
4. The magnitude bearing form. This type is usually written in column form with the magnitude taking the place of x-value and the bearing that of the y-value. Egs. (xkm, 000^o) or (5m, 030^o)

NB

Bearings are taken or measured from geographical north.

27.2 **The Magnitude Of A Vector**

This is also referred to as the distance, length or modulus of a given vector.

In finding the magnitude of any given vector, we take the square root of the sum of squares of the x and y- values. We denote magnitude of a vector by two straight lines enclosing the vector. Thus, if $\vec{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ is a vector, then the magnitude is given by:

$$|\vec{AB}| = \sqrt{x^2 + y^2}$$

Example 27.1

If $\vec{PQ} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$, *find* $|\vec{PQ}|$

Solution

$$\vec{PQ} = \begin{pmatrix} -8 \\ 6 \end{pmatrix},$$

$$|\vec{PQ}| = \sqrt{-8^2 + 6^2} = 10 units$$

27.3 **Addition Of Vectors**

In performing addition on two or more vectors, we add corresponding components. Thus, if

$$p = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } q = \begin{pmatrix} a \\ b \end{pmatrix} \text{ then } p + q = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

Example 27.2

Given that $a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and* $c = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, *find* $(i)\ a + c\ (ii)\ b + c$

Solution

$$(i)\ a + c = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \quad (ii)\ b + c = \begin{pmatrix} 1 + 4 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

27.4 **Subtraction Of Vectors**

Here, we subtract corresponding components.

Example 27.3

If $p=\begin{pmatrix}1\\1\end{pmatrix}, q=\begin{pmatrix}2\\4\end{pmatrix}$ $and\ r=\begin{pmatrix}1\\2\end{pmatrix}$ $Find\ (i)\ p-q\ \ (ii)\ q-r$

Solution

$(i)\ p-q=\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}-1\\-3\end{pmatrix}\ (ii)\ q-r=\begin{pmatrix}2-1\\4-2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$

27.5 **Multiplication Of A Scalar By A Vector**

A scalar is simply any number (real numbers). In multiplying a scalar by a vector, we multiply the scalar by each component of the vector. Thus, if k is a scalar and

$\vec{AB}=\begin{pmatrix}x\\y\end{pmatrix}, then\ k\ \vec{AB}=k\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}kx\\ky\end{pmatrix}$

Example 27.4

(i) If $a=\begin{pmatrix}2\\4\end{pmatrix}$ $and\ b=\begin{pmatrix}-2\\6\end{pmatrix}$ $Find\ c\ such\ that\ c=2a+b$

(ii) If $x=\begin{pmatrix}-3\\2\end{pmatrix}$ $and\ y=\begin{pmatrix}4\\-1\end{pmatrix}, calculate\ 3x-y$

Solution

(i) $c=2a+b=2\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}-2\\6\end{pmatrix}=\begin{pmatrix}4\\8\end{pmatrix}+\begin{pmatrix}-2\\6\end{pmatrix}=\begin{pmatrix}2\\14\end{pmatrix}$

(ii) $3x-y=3\begin{pmatrix}-3\\2\end{pmatrix}-\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}-13\\7\end{pmatrix}$

NB

We multiply two vectors by multiplying their corresponding components and summing the result.
For instance, if

$$a = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \text{ then } a.b = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 6 \end{pmatrix} = 2(-2) + 4(6) = 20$$

Example 27.5

$$If\ \ p = \begin{pmatrix} 2 \\ -1 \end{pmatrix},\ q = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } r = \begin{pmatrix} 5 \\ -6 \end{pmatrix},\ find$$

(a) m and n, such that $r = mp + nq$, where m and n are scalars

(b) (i) $|g|$, if $g = 3q + r$ $\qquad (ii)$ the bearing of g

Solution

$$(a)\ p = \begin{pmatrix} 2 \\ -1 \end{pmatrix},\ q = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } r = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$$

$\Rightarrow r = mp + nq$

$$\Rightarrow \begin{pmatrix} 5 \\ -6 \end{pmatrix} = m\begin{pmatrix} 2 \\ -1 \end{pmatrix} + n\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 2m \\ -m \end{pmatrix} + \begin{pmatrix} n \\ 3n \end{pmatrix}$$

By equality of vectors, we have;

$5 = 2m + n ---------- (1)$

$-6 = -m + 3n -------- (2)$

Solve simultaneously,

From (1), $n = 5 - 2m -------- (3)$

Put (3) *into* (2)

$\Rightarrow -6 = -m + 3(5 - 2m) = -m + 15 - 6m$

$\Rightarrow -6 = -7m + 15 \Rightarrow 7m = 15 + 6 = 21$

$$\Rightarrow \frac{7m}{7} = \frac{21}{7} = 3 \Rightarrow m = 3$$

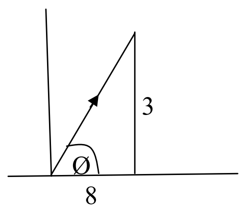
Put $m = 3$ *into* (3)

$\Rightarrow n = 5 - 2(3) = 5 - 6 = -1 \therefore n = -1$

$(b)\ (i)\ g = 3q + r$

$$\Rightarrow g = 3\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}5\\-6\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} + \begin{pmatrix}5\\-6\end{pmatrix} = \begin{pmatrix}3+5\\9-6\end{pmatrix} = \begin{pmatrix}8\\3\end{pmatrix}$$

$$|g| = \sqrt{8^2 + 3^2} = \sqrt{73} = 8.54$$



(ii) *From above figure,*

$$\tan\theta = \frac{Opp}{Adj} = \frac{3}{8} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{8}\right) = 20.56^o$$

Hence, bearing of g = 90-20.56 = 69.44° = 069°

27.6 Expressing Two Points As A Single Vector

NB Generally, we denote the position vector of any given vector by simply attaching the origin, '**O**" to the letter or just denoting it by the lower case of the vector. For example, we denote the position vector of A by $\vec{OA}$ or '***a***'.

In expressing two points as a single vector we subtract the first point (the vector of the first letter) from the second point (the vector of the second letter).

Thus, $\vec{AB} = b - a \ \ and \ \ \vec{PQ} = q - p$

Example 27.6

Given that $A(2,3),\ B(5,7)\ and\ C(-2,-1)\ find\ \vec{AB}\ and\ \vec{BC}$

Solution

$\vec{AB} = b - a = \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\vec{BC} = c - b = \begin{pmatrix} -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix}$

NB

1. A zero vector is a vector with both components being zeros (0). It is denoted by '***O***'. Thus, $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2. The negative or inverse vector is obtained by negating the two components of the vector. Thus, the inverse of

$\vec{AB} \text{ is } \vec{BA} \text{ and vice versa}$

$\Rightarrow \vec{BA} = -\vec{AB}$

For instance, if $\vec{AB} = \begin{pmatrix} -13 \\ 7 \end{pmatrix}$, $\text{then } \vec{BA} = -\vec{AB} = -\begin{pmatrix} -13 \\ 7 \end{pmatrix} = \begin{pmatrix} 13 \\ -7 \end{pmatrix}$

Example 27.7

In triangle PQR, $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\text{and } \overrightarrow{RQ} = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$, $\text{find } \overrightarrow{PR}$.

Solution

We note that,

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$\text{But } \overrightarrow{QR} = -\overrightarrow{RQ} = -\begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

$\Rightarrow \overrightarrow{PR} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

27.7 **Equality Of Vectors**

Two vectors are said to be equal if they have equal corresponding components.

Thus, if $a = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} p \\ q \end{pmatrix}$ and $a = b,$ $then$ $x = p$ and $y = q$

In short, when proving that two vectors are equal, we equate the corresponding components and solve for resulting variables.

Example 27.8

(i) If $r = \begin{pmatrix} x-2 \\ y+1 \end{pmatrix}$ and $s = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $find$ x and y if $r = s$

(ii) If $p = \begin{pmatrix} a+5 \\ b \end{pmatrix}$ and $q = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $find$ a and b if $p = q$

Solution

(i) If $r = s \Rightarrow \begin{pmatrix} x-2 \\ y+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

By equality of vectors,

$x - 2 = 3 \Rightarrow x = 5$

and

$y + 1 = 5 \Rightarrow y = 4$

(ii) If $p = q \Rightarrow \begin{pmatrix} a+5 \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

Similarly, $a + 5 = 7 \Rightarrow a = 2$ and $b = 3$

27.8 **Finding The Mid-Point Of Line Segments**

In finding the midpoint of two points, we find the average of the sum of the two points.

Thus, if r is the mid point of $\overrightarrow{AB}$ $then$ $r = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(a + b)$

Example 27.9

M is the mid point of JK. If the coordinates of J is (-5, 4) and M is (-2, 1), find the coordinates of K.

Solution

$$J = \begin{pmatrix} -5 \\ 4 \end{pmatrix},\ M = \begin{pmatrix} -2 \\ 1 \end{pmatrix}\ \textit{and let}\ K = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow M = \tfrac{1}{2} JK$$

$$\Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \tfrac{1}{2}\left[\begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \tfrac{1}{2}\begin{bmatrix} -5+x \\ 4+y \end{bmatrix}$$

$$\Rightarrow 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{bmatrix} -5+x \\ 4+y \end{bmatrix} \quad \Rightarrow \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -5+x \\ 4+y \end{pmatrix}$$

By equality of vectors, $-4 = -5 + x \Rightarrow x = 1\ \textit{and}\ 2 = 4 + y \Rightarrow y = -2$

Hence, K = (1, -2)

Example 27.10

P(7, 4) is a vertex of triangle PQR. If $\overrightarrow{QP} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ *and* $\overrightarrow{PR} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, *find* :

(i) The co-ordinates of Q

(ii) $\overrightarrow{PN}$ *where N is the midpoint of* $\overrightarrow{QR}$.

Solution

$$(i)\ From\ \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$$

$$\Rightarrow \overrightarrow{OQ} = \overrightarrow{OP} - \overrightarrow{QP} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \therefore Q = (4, -1)$$

$$(ii)\ From\ \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$$

$$\Rightarrow \overrightarrow{PR} + \overrightarrow{OP} = \overrightarrow{OR}$$

$$\Rightarrow \overrightarrow{OR} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

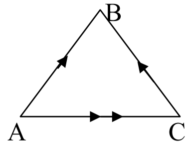
$$Again,\ N = Midpo\text{int}\ of\ \overrightarrow{QR} = \frac{1}{2}\left[\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{7}{2} \end{pmatrix}$$

$$\Rightarrow N = (4, \tfrac{7}{2})$$

$$Hence,\ \overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP} = \begin{pmatrix} 4 \\ \frac{7}{2} \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -\frac{1}{2} \end{pmatrix}$$

27.9 **The Resultant Vector**

This aspect describes several ways of getting to the same point. The shortest route is called the ***resultant***. Consider the diagram below:



Form the diagram above, $\vec{AC} = \vec{AB} + \vec{BC}$
i.e. In getting to C from A, one can either choose to go from A to B and then from B to C or can choose to go along A to C straight away.

NB

In finding resultant when given two vectors, we ensure that the repeated letter (or point) in the two vectors comes second in the first vector and then comes first in the second vector. Instances where the arrangement is not so in one of the vectors given, we apply inversing to it.

Example 27.11

(a) If $\vec{BA} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ *and* $\vec{AC} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$, *calculate the magnitude of* $\vec{BC}$

(b) If $\vec{PQ} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ *and* $\vec{RQ} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, *find* $\vec{RP}$

P, Q and R are points in the Cartesian plane.

(c)

$\vec{QP} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ *and* $\vec{PR} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$, *Find vector* $\vec{QR}$

(d) In a quadrilateral, PQRS

$\vec{PQ} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \vec{RS} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ *and* $\vec{PS} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, *calculate* $\vec{QR}$

Solution

(a)

$\vec{BA} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ *and* $\vec{AC} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

NB: from $\vec{BA}$ *and* $\vec{AC}$, A is the repeated letter, and hence must come second in first vector and first in second vector.

Implies,

$\vec{BA} + \vec{AC} = \vec{BC}$

$\Rightarrow \vec{BC} = \vec{BA} + \vec{AC} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

(b) $\vec{PQ} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ *and* $\vec{RQ} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **NB:** Q is repeated. Hence, we find the inverse of $\vec{RQ}$ *to get* $\vec{QR}$

$\Rightarrow \vec{QR} = -\vec{RQ} = -\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$\Rightarrow \vec{PR} = \vec{PQ} + \vec{QR} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

But we want $\vec{RP}$ *so we invert* $\vec{PR}$

$\Rightarrow \vec{RP} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c)

$\vec{QP} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ *and* $\vec{PR} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

$\vec{QR} = \vec{QP} + \vec{PR} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

(d) $\vec{PS} = \vec{PQ} + \vec{QR} + \vec{RS}$

$\Rightarrow \vec{QR} = \vec{PS} - \vec{PQ} - \vec{RS} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

NB

Here, we arranged the letters in all vectors together in alphabetic order.

27.10 **Parallel Vectors**

Two vectors, usually non-zero vectors are said to be parallel if one is a scalar
multiple of the other.

NB

Vectors in the same direction have positive scalar multiple and vectors in opposite directions have negative scalar multiple.
Again, we can say that, two vectors are parallel if the ratios of their corresponding components are equal.

Thus, if $p = \begin{pmatrix} a \\ b \end{pmatrix}$ *is parallel to* $q = \begin{pmatrix} x \\ y \end{pmatrix}$ *then* $a : x = b : y \Rightarrow \frac{a}{x} = \frac{b}{y}$

Example 27.12

(a) Which of the following vectors is parallel to $\begin{pmatrix}5\\2\end{pmatrix}$?

(b) Which of the following vectors is parallel to $\begin{pmatrix}20\\8\end{pmatrix}$?

(c) If $\begin{pmatrix}4\\x\end{pmatrix}$ *is parallel to* $\begin{pmatrix}12\\9\end{pmatrix}$, *find the value of* x.

Solution

(a) $\begin{pmatrix}25\\10\end{pmatrix}$ *is parallel to* $\begin{pmatrix}5\\2\end{pmatrix}$ *with scalar multiple of* $\frac{1}{5}$ *and vicely* 5

(b) $\begin{pmatrix}5\\2\end{pmatrix}$ *is parallel to* $\begin{pmatrix}20\\8\end{pmatrix}$ *with a scalar multiple of* 4 *and vicely* $\frac{1}{4}$.

(c) Here, it is appropriate to use the ratio approach.
Thus, $4:12 = x:9$

$\Rightarrow \frac{4}{12} = \frac{x}{9} \Rightarrow x = 3$

27.11 **Perpendicular Vectors**

Two vectors are said to be perpendicular to each other if one is a rotation of either 90^o or 270^o anticlockwise about the origin of the other. i.e. for the vector

$\begin{pmatrix}x\\y\end{pmatrix}$, *its anticlockwise rotaions of* 90^o *and* 270^o *respectively* $\begin{pmatrix}-y\\x\end{pmatrix}$ *and* $\begin{pmatrix}y\\-x\end{pmatrix}$ *are*

its perpendicular vectors.
Again, the scalar multiples of the rotations of 90^o and 270^o anticlockwise are also perpendicular to $\begin{pmatrix}x\\y\end{pmatrix}$.

i.e. $\begin{pmatrix}ky\\-kx\end{pmatrix}$ *and* $\begin{pmatrix}-ky\\kx\end{pmatrix}$ *are perpendicular to* $\begin{pmatrix}x\\y\end{pmatrix}$.

NB

For a rotation of 90^o anticlockwise about the origin, we interchange x and y values and then negate the resulting up (x – value) value. And rotation of 270^o anticlockwise about the origin, we interchange x and y values and then negate the resulting down value (y-value)

Example 27.13

(i) Which of the following vectors is perpendicular to the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

(ii) Which vector is perpendicular to $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$?

Solution

(i) Rotating

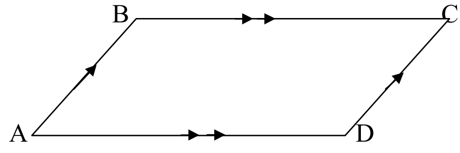
$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \textit{ through } 90^o \textit{ becomes} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \textit{ and through } 270^o \textit{ becomes} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence, $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ *and* $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ *are the two vectors perpendicular to* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(ii) $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$ *and* $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ *are perpendicular to* $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$

27.12 **The Parallelogram**

A parallelogram is considered as a plane figure in which the opposite sides are equal. Example: Squares, rectangles, rhombus etc. Consider the figure below:



Generally, in the parallelogram above;

$\vec{AB} = \vec{DC}$ *and* $\vec{AD} = \vec{BC}$ (*Similar arrow sides are equal*)

NB

We label plane figures such as the parallelogram in the clockwise or in the anticlockwise direction.

Example 27.14

The points P, Q, R and S are vertices of a parallelogram in the Cartesian plane. The coordinates of P and R are (-8, 2) and (5 – 2) respectively and $\vec{QR} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. find

(i) The coordinates of Q and S

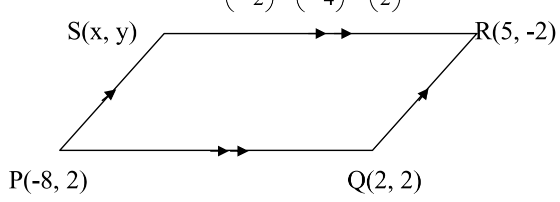
(ii) The magnitude of $\vec{PR}$

Solution

(i)

$P(-8,2),\ R(5,-2), \vec{QR} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ *and let* $S = (x, y)$

$\vec{QR} = r - q \Rightarrow q = r - \vec{QR} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow Q = (2, 2)$



From parallelogram laws,

$\vec{PQ} = \vec{SR}$

$\Rightarrow q - p = r - s$

$\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -8 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} \quad \therefore S = (-5, -2)$

(ii) $\vec{PR} = r - p = \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -8 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ -4 \end{pmatrix}$

$|\vec{PR}| = \sqrt{13^2 + (-4)^2} = 13.6\,units$

Example 27.15

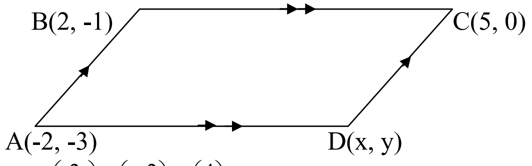
A(-2, -3), B(2, -1), c (5, 0) and D(x , y) are the vertices of the parallelogram ABCD.

(a) Find $\vec{AB}$, $\vec{DC}$ hence find the coordinates of D.

(b) Calculate correct to one decimal place, $|\vec{DB}|$.

Solution

(a)



$\vec{AB} = b - a = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\vec{DC} = c - d = \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 - x \\ -y \end{pmatrix}$

But $\vec{AB} = \vec{DC} \Rightarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 - x \\ -y \end{pmatrix} \Rightarrow x = 1 \;\; and \;\; y = -2 \;\therefore D = (1, -2)$

(b) $\vec{DB} = b - d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow |\vec{DB}| = \sqrt{1^2 + 1^2} = 1.412 \; units$

Example 27.16

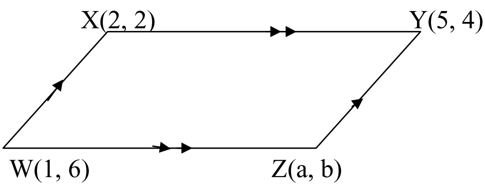
The coordinates of the vertices of a parallelogram WXYZ are W(1, 6), X(2, 2),
Y(5, 4) and Z(a, b). Find

(i) $\overrightarrow{WX}$

(ii) $\overrightarrow{ZY}$

(iii) The coordinates of Z.

Solution



(i) $\overrightarrow{WX} = x - w = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

(ii) $\overrightarrow{ZY} = y - z = \begin{pmatrix} 5-a \\ 4-b \end{pmatrix}$

(iii) From

$\overrightarrow{WX} = \overrightarrow{ZY} \Rightarrow \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 5-a \\ 4-b \end{pmatrix} \Rightarrow a = 4 \;\; and \;\; b = 8 \Rightarrow Z(4,8)$

Example 27.17

(a) The vectors $p = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, q = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \;\; and \;\; r = \frac{1}{2}(q - p)$ Find

(i) the vector r

(ii) $m \;\; and \;\; n \;\; if \;\; mp + nq = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \;\; where \;\; m \;\; and \;\; n \;\; are \;\; scalars$

(b) $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \;\; and \;\; \overrightarrow{AC} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$ are vectors in the same plane.

A is the point (1, 2)

(i) find the coordinates of B and C

(ii) If D is the mid-point of BC, show that $\vec{AB}+\vec{AC}=2\vec{AD}$

Solution

(a) (i) $r=\frac{1}{2}(q-p)=\frac{1}{2}\left[\begin{pmatrix}2\\5\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\therefore r=(0,1)$

(ii) $mp+nq=\begin{pmatrix}4\\3\end{pmatrix}$

$m\begin{pmatrix}2\\3\end{pmatrix}+n\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$

$\Rightarrow\begin{pmatrix}2m\\3m\end{pmatrix}+\begin{pmatrix}2n\\5n\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$

$\Rightarrow 2m+2n=4$ ----------- (1)

$3m+5n=3$ ----------- (2)

From (1), we get

$m=\dfrac{4-2n}{2}$ ----------- (3)

From (2), we get

$m=\dfrac{3-5n}{3}$ ----------- (4)

Equate (3) and (4)

$\dfrac{4-2n}{2}=\dfrac{3-5n}{3}$

$2(3-5n)=3(4-2n)$

$\Rightarrow n=-\dfrac{3}{2}$

Put $n=-\dfrac{3}{2}$ into (3)

$m=\dfrac{4-2\left(-\frac{3}{2}\right)}{2}=\dfrac{4+3}{2}=\dfrac{7}{2}$ *Or* $3\frac{1}{2}$

(b) (i) $\vec{AB}=\begin{pmatrix}4\\-2\end{pmatrix}$ *and* $\vec{AC}=\begin{pmatrix}-1\\6\end{pmatrix}$

$But \;\; A = (1, 2)$

$\Rightarrow \vec{AB} = b - a$

$\Rightarrow b = \vec{AB} + a = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \;\; \therefore B = (5, 0)$

$Again, \; \vec{AC} = c - a$

$\Rightarrow c = \vec{AC} + a = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

$(ii) \;\; D = \frac{1}{2}BC \, (Since \, D \; is \; midpo\text{int} \; of \;\; BC)$

$\Rightarrow D = \frac{1}{2}(c - d) = \frac{1}{2}\left[\begin{pmatrix} 0 \\ 8 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 4 \end{pmatrix} \;\; \therefore D = (\frac{5}{4}, 4)$

$But \;\; \vec{AD} = d - a = \begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}$

$\Rightarrow \vec{AB} + \vec{AC} = 2\,\vec{AD}$

$\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$Since \;\; LHS = RHS, \Rightarrow \vec{AB} + \vec{AC} = 2\,\vec{AD}$

Example 27.18

The vertices of a triangle are (1, -3), Q(7, 5) and R(-3, 5)

(i) Express $\vec{PQ}, \vec{QR} \; and \; \vec{PR}$ as column vectors

(ii) Show that triangle PQR is isosceles.

Solution

(i) $\vec{PQ} = q - p = \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \vec{QR} = r - q = \begin{pmatrix} -10 \\ 0 \end{pmatrix}, \vec{PR} = r - p = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$

(ii) If isosceles, two sides must be equal:

$|\vec{PQ}| = \sqrt{6^2 + 8^2} = 10$

$|\vec{QR}| = \sqrt{-10^2 + 0^2} = 10$

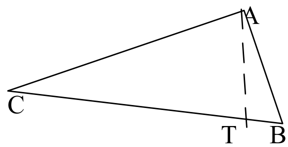
$|\vec{AD}| = \sqrt{-4^2 + 8^2} = 8.9$

Hence, triangle PQR is isosceles since $|\vec{PQ}| = |\vec{QR}|$

Example 27.19

(a) In the diagram,

$\vec{AB} = m,\ \vec{AC} = n \text{ and } T \text{ is such that} |CT| : |TB| = 2:1$



(i) Find in terms of $m \text{ and } n, \vec{BC}$

(ii) Show that $\vec{TA} = -\frac{1}{3}(2m + n)$

(b) If $P = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \text{ and } q = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

(i) Calculate $|\vec{q - p}|$, correct to one decimal place

(ii) Find the vector r, such that p + r = 2q

Solution

(i) from diagram, $\vec{BC} = \vec{BA} + \vec{AC}$

$\text{But } \vec{BA} = -\vec{AB} = -m \text{ and } \vec{CA} = -\vec{AC} = -n$

$\Rightarrow \vec{BC} = -m + n = n - m$

(ii) Again, $\vec{TA} = \vec{TC} + \vec{CA} \text{ But } |CT| : |TB| = 2:1$

$|TC| = \frac{2}{3}\vec{BC} \,(Since\ total\ ratio\ is\ 3)$

$$\vec{TA} = \vec{TC} + \vec{CA} = \frac{2}{3}\vec{BC} + \vec{CA} = \frac{2}{3}(n-m) + (-n) = \frac{2}{3}n - \frac{2}{3}m - n = \frac{1}{3}n - \frac{2}{3}m$$

$$= \frac{1}{3}(-n-2m) = -\frac{1}{3}(n+2m)$$

$$\therefore \vec{TA} = -\frac{1}{3}(2m+n)$$

(b) (i) $p = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $q = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

$$q - p = \begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -9 \\ 10 \end{pmatrix}$$

$$\Rightarrow |q-p| = \sqrt{-9^2 + 10^2} = 13.5\,units$$

(ii) p + r = 2q

Implies, r = 2q – p

Hence, $r = 2\begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ 17 \end{pmatrix}$

EXERCISE

QUE . A

Given that $p = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, q = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $r = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $find$ $2p + 3q - 4r$

QUE. B

If $q = \begin{pmatrix} -4 \\ -8 \end{pmatrix}, r = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $s = r - q$ $find$ $|s|$

QUE. C

Two vectors p and are defined by $p = \begin{pmatrix} 5\cos x \\ 5\sin x \end{pmatrix}$ and $q = \begin{pmatrix} 2\cos x \\ 2\sin x \end{pmatrix}$

(a) If $p + q = \begin{pmatrix} 4.690 \\ 5.208 \end{pmatrix}$, find to the nearest whole number, the value of x, where x is acute.

(b) Find the value of the vector c, if c = 2p + q

QUE. D

A, B, C and D are four points such that A=(-3, 2), C=(6,3), $\vec{AB} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ *and* $\vec{CD} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$

(a) Calculate

(i) the coordinates of B and D

(ii) the vectors $\vec{BC}$ *and* $\vec{AD}$

(b) What is the relationship between $\vec{BC}$ *and* $\vec{AD}$?

QUE. E

PQRS is a quadrilateral with P(2, 2), S(4, 4), R(6, 4). If $\vec{PQ} = 4\vec{SR}$, Find the coordinates of Q.

QUE. F

(a) The coordinates of the vertices of a parallelogram QRST are Q(1, 6), R(2, 2), S(5, 4) and T(x, y)

(i) Find the vector $\vec{QR}$ *and* $\vec{TS}$ and hence determine the values of x and y.

(ii) Calculate the magnitude of $\vec{RS}$

(iii) Express $\vec{RS}$ in the form (k, θ^o) where k is the magnitude and θ the bearing.

(b) Find the values of x and y in the equation:

$$\begin{pmatrix} x+3 \\ 2 \end{pmatrix} - \begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

QUE. G

If $a = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$, *find*

(i) m and n such that $c = ma + nb$ where m, n are scalars.

(ii) $|d|, \, if \;\; d = c - 2a$

QUE. H

The vector $a = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, b = \begin{pmatrix} x \\ y \end{pmatrix}, c = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ are in the same plane.

If 3a – 2b = b, find

(i) the vector b

(ii) $|d|$ and express your answer in the form

$p\sqrt{q}, \; where \;\; p \;\; and \;\; q$ are integers and $d = b - c$

CHAPTER 28

BUSINESS MATHEMATICS

(PERCENTAGES, PROFIT & LOSS, DISCOUNT, DEPRECIATION, COMMISSION, SIMPLE & COMPOUND INTERESTS, HIRE PURCHASES, RATIOS AND PROPORTIONS)

28.1 **Percentages**

In general, the word ***percentage*** means out of ***a hundred*** or ***divided by a hundred*** (100). It is denoted by the symbol %. Thus, any number with the symbol % attached to it on its right means divide that number by 100. For instance, 25% means $\frac{25}{100} = \frac{1}{4}$

NB

1. From the above, it can be deduced that every ***percentage*** is a ***fraction***.
2. The word '***Of***' is used to denote ***'multiplication'***
3. Again, we express fractions as percentages by ***multiplying*** the given fraction by a ***hundred*** and attach the symbol % to the result. For instance, in expressing $\frac{1}{5}$ as a percentage, we get $\frac{1}{5} \times 100 = 20\%$
4. Further, we can find ***the percentage*** of a certain ***quantity*** by expressing the percentage as a fraction and multiplying the result by the given quantity. For instance,
   $30\% \ of \ \ 600 \ Ghana \ \ cedis \ becomes \ \frac{30}{100} \times 600 = 180 \ Ghana \ \ cedis$

Example 28.1

Three friends **K**, **L** and **M** were voted into office as school prefects. **K** secured 45% of the votes, **L** had 33% of the votes and **M** had the rest of the votes. If **M** secured 1,430 votes, calculate
(i) The total number of votes cast;
(ii) How many more votes **K** received than **L**?

Solution

Method 1
Percentage of K is 45%
Percentage of L is 33%
Total percentage = 100% = Sum of percentages of K, L and M
Implies,
100% = percentage of K + percentage of L + percentage of M
100% = 45% + 33% + percentage of M
Implies, percentage of M = 100 – (45 + 33) = 100 – 78 = 22%
Let the total number of votes be y
From question, M secured 1,430 votes means 22% of the total votes is 1,430
Implies, $\frac{22}{100}\times y = 1430 \Rightarrow \frac{11}{50}y = 1430 \Rightarrow y = 6{,}500$
Hence, the total number of votes cast was 6,500

Method 2

Considering the percentage of M, we say;
$\text{If } 22\% \equiv 1430$

$\text{Then } 100\% \equiv \frac{100\times 1430}{22} = 6500$
(i) Hence, the total number of votes cast was 6,500
(ii) **K** had 45% of the total votes means **K** had
$\frac{45}{100}\times 6500 = 2925 \text{ votes}$
L had 33% of the total votes means **L** had
$\frac{33}{100}\times 6500 = 2145 \text{ votes}$

Therefore, 2925 – 2145 = 780
Hence, **K** had 780 more votes than **L**

NB

1. In finding the ***fraction of any quantity***, we just multiply the fraction by the given quantity. For instance, $\frac{1}{5}$ of Gh ¢ 200 becomes $\frac{1}{5}\times 200 =$ Gh ¢ 50
2. Again, in expressing a ***quantity as a fraction of another quantity***, we divide the first quantity by the second quantity. Thus, $20cm$ as a fraction of $300cm\ is\ \frac{20}{300}=\frac{1}{15}$
3. Similarly, when expressing a ***quantity as a percentage of another quantity***, we divide the first quantity by the second and multiply the result by 100%.
For instance, when asked to express $20cm$ as a percentage of $80cm,\ we\ have\ \frac{20}{80}\times 100 = 25\%$
Hint: The quantity that comes after the word 'Of' is taken as the denominator.

Example 28.2

In a class of 40 students, 30 are tall and 10 are short.
What percentage of the students are short?

Solution

Number of short students is 10
Percentage of short students is $\frac{10}{40}\times 100 = 25\%$

Example 28.3

What percentage of $60cm\ is\ 20cm$?

Solution

$60cm$ is taken as the denominator since it follows the word '***of***'.
Hence,

$$\frac{20}{60}\times 100 = 33.33\%$$

NB

1. $Percentage\ Increase = \dfrac{New\ value - original\ value}{original\ value}\times 100$

Example 28.4

The length of a metal increased from 34mm to 55mm. what is the percentage increase of the metal?

Solution

$$Percentage\ \ Increase\ = \frac{New\ value - original\ \ value}{original\ \ value}\times 100$$

$$= \frac{55-34}{34}\times 100 = \frac{21}{34}\times 100 = 61.76\%$$

2. $Percentage\ Decrease = \dfrac{Original\ value - New\ value}{original\ value}\times 100$

Example 28.5

The cost of a wireless set was Gh ¢ 20. The set now sells at Gh ¢18. What is the percentage decrease of the set?

Solution

$$Percentage\ \ decrease\ = \frac{Original\ \ value - New\ value}{original\ \ value}\times 100$$

$$= \frac{20-18}{20}\times 100 = \frac{2}{20}\times 100 = 10\%$$

Example 28.6

If 15% of a number m is equal to r, what is y% of m?

Solution

15% of a number m is equal to r means

$$\frac{15}{100}m = r \Rightarrow 15m = 100r \Rightarrow m = \frac{100r}{15}$$

Similarly,

$$y\% \text{ of } m = \frac{y}{100}m = \frac{y}{100} \times \frac{100r}{15} = \frac{ry}{15}$$

28.2 **Increasing And Decreasing Quantities By A percentage**

Example 28.6

Increase 350 by 20%

Solution

Method 1

First,

20% of 350 is $\frac{20}{100} \times 350 = 70$

Therefore, new value is 350 + 70 = 420

(since involves increasing, we add)

Method 2

Always assume the original quantity given to be equivalent to 100% and then add the given percentage to it since its increased.

i.e. 100% + 20% = 120%

Now, **if** $100\% \equiv 350$ **Then,** $120\% \equiv x$

Implies, 100% = 350 then 120% = x

Cross and multiply to make x the subject

$$\frac{100x}{100} = 120 \times 350 = \frac{42000}{100}$$
$$\Rightarrow x = 420$$
Therefore, the new value is 420

Example 28.7

Decrease 300 by 15%

Solution

Method 1

First,
15% of 300 is $\frac{15}{100} \times 300 = 45$
Hence, new value is 300 – 45 = 255
Method 2

Always assume the original quantity given to be equivalent to 100% and then subtract the given percentage from it since its decreased. i.e. 100% **-** 15% = 85%
Now, **if** $100\% \equiv 300$
Then, $85\% \equiv x$
Implies,
$$100\% \equiv 300$$
$$85\% \equiv x$$
Cross and multiply to make x the subject
$$\frac{100x}{100} = 85 \times 300 = \frac{25500}{100} = 255$$
$$\Rightarrow x = 255$$
Therefore, the new value is 255

Example 28.8

The table below shows the expenditure pattern of a family in 1993. The net incone of the family for 1993 was 1.8million cedis.

ITEM	%
Food	60
Clothing and Footwear	22
Plant, Transport & Household Equipment	7
Education, Entertainment & Recreation	7
Miscellaneous goods and services	4

Calculate the amount the family spent on (i) food (ii) Miscellaneous goods and services
The family's income for 1994 increased by 35%. At the same time expenditure for food recorded an increase of 25% while all the other items recorded an increase of 20%. Find the amount the family saved at the end of 1994.

Solution

(i) Family's net income is ¢ 1,800,000
Percentage on food is 60%

$$\therefore Amount\ on\ food\ = \frac{60}{100} \times 1800000 = 1{,}080{,}000$$

Therefore, the amount the family spent on food is ¢ 1, 080,000
(ii) Net income is ¢ 1,800,000
Percentage spent on miscellaneous goods and services is 4%

$$\therefore Amount\ on\ miscellaneous\ goods\ services\ = \frac{4}{100} \times 1800000 = 72{,}000$$

Percentage increase in income is 35%

$$\therefore Amount\ increased\ on\ income\ in\ 1994 = \frac{35}{100} \times 1800000 = 630{,}000$$

Hence, income for 1994 is ¢630,000 + ¢1,800,000 = ¢2,430,000
Now, amount spent on food is ¢1,080,000 in 1993 and
Percentage increase on food is 25% in 1994

$$\therefore Amount\ increased\ on\ food\ in\ 1994 = \frac{25}{100} \times 1{,}080{,}000 = 270{,}000$$

Hence, amount spent on food in 1994 is:
¢1,080,000 + ¢270,000 = ¢1,350,000

$\therefore amount\ for\ remaining\ items\ in\ 1993 = Net\ in\ 1993 - amount\ on\ food\ in\ 1993$

$= 18000000 - 1080000 = 720,000$

Percentage increase on remaining items is 20%

$\therefore Amount\ increased\ on\ remaining\ items\ in\ 1994 = \frac{20}{100} \times 720,,000 = 144,000$

Hence, amount spent on remaining items is
¢144,000 + ¢720,000 = ¢864,000
Therefore, total amount spent in 1994
= amount spent on food + amount spent on remaining items
= ¢1,350,000 + ¢864,000 = ¢2,214,000
Hence, amount saved in 1994 = net income in 1994–total amount spent in 1994 = ¢2,430,000 – ¢2,214,000 = ¢216,000

28.3 **Profit And Loss**

28.3.1 **Profit**

Generally, in the market serene, when the price of selling (***selling price)*** a particular commodity is higher than the amount used to buy the commodity (***cost price***) then we say a ***profit*** is made.
Mathematically, $\Pr ofit = Selling\ price - Cost\ price$

Similarly, $\Pr ofit\ percent = \frac{\Pr ofit}{\cos t\ price} \times 100$

28.3.2 **Loss**

Similarly, when the price of buying (***cost price)*** a particular commodity is higher than the amount of sale of the commodity (***selling price***) then we say a ***loss*** is made.
Mathematically, $Loss = Cost\ price - Selling\ price$

Similarly, $Loss\ percent = \frac{Loss}{\cos t\ price} \times 100$

NB

In each of the relations stated above, either quantity could be made the subject to obtain another relation.

Example 28.9

A trader sold 1,750 articles for ¢525,000 and made a profit of 20%.
(i) Calculate the cost price of each article
(ii) If he wanted 45% profit on the cost price, how much should he have sold each of the articles?

Solution

(i) Number of articles is 1,750
Selling price is ¢525,000
Profit percent is 20% (whenever profit is given in percentage, its taken as the profit percent)
From the relation,

$$\text{Profit percent} = \frac{\text{Profit}}{\text{cost price}} \times 100$$

$$\Rightarrow \text{Profit percent} = \frac{\text{Selling Price} - \text{Cost Price}}{\text{cost price}} \times 100$$

$$20 = \frac{525000 - \text{cost price}}{\text{cost price}} \times 100$$

Divide through by 100

$$\frac{20}{100} = \frac{525000 - \text{cost price}}{\text{cost price}}$$

$$\frac{1}{5} = \frac{525000 - \text{cost price}}{\text{cost price}}$$

$$5(525000 - \text{cost price}) = \text{cost price}$$

$$2625000 - 5\,\text{cost price} = \text{cost price}$$

$$2625000 = \text{cost price} + 5\,\text{cost price}$$

$$\frac{2625000}{6} = \frac{6\,\text{cost price}}{6}$$

437500 = cost price
Hence, the cost price of all the items is ¢437500

Therefore, the cost price of each article is $= \dfrac{437500}{1750} =$ ¢250

(ii) Cost price is ¢437,500 and Profit percent is 45%

But $\text{Profit percent} = \frac{\text{Selling Price} - \text{Cost Price}}{\text{cost price}} \times 100$

$$\Rightarrow 45 = \frac{\text{Selling Price} - 437500}{437500} \times 100$$

$$\frac{45}{100} = \frac{\text{selling price} - 437500}{437500}$$

$$\frac{45}{100} \times 437500 = \text{selling price} - 437500$$

$$196875 + 437500 = \text{selling price}$$

$$634{,}375 = \text{selling price}$$

Hence, selling price for each article is $= \frac{634{,}375}{1750} =$ ¢362.50

Example 28.10

A woman bought 130kg of tomatoes for ¢52,000.00. She sold half of them at a profit of 30%. The rest of the tomatoes began to go bad. She then reduced the selling price per kg by 12%. Calculate
(a) the new selling price per kg
(b) the percentage profit on the whole transaction if she threw away 5kg of bad tomatoes.

Solution

(a) Cost price is ¢52,000
Profit percent on 65 (half) is 30%
Then, cost price on 65 (half) is ¢26,000

$$\text{Profit percent} = \frac{\text{Selling Price} - \text{Cost Price}}{\text{cost price}} \times 100$$

$$30 = \frac{\text{selling price} - 26000}{26000} \times 100$$

$$\frac{30}{100} \times 26000 = \text{selling price} - 26000$$

$$7800 = \text{selling price} - 26000$$

$$\Rightarrow \text{Selling price} = 33{,}800$$

Therefore, selling price per kg is $= \frac{33800}{65} =$ ¢520

Price per kg reduced by 12%
Implies, 100% - 12% = 88%

$$new\ selling\ \ price = \frac{88}{100} \times 520 = 457.60$$

(b) Quantity of tomatoes thrown away is 5kg
Quantity of tomatoes sold is 60kg
Selling price per kg of the 60kg left is ¢457.60
Implies, total selling price of the 60kg is $= 457.60 \times 60 =$ ¢27,456
Total selling price of the whole transaction
$= 33{,}800 + 27{,}456 =$ ¢61256
Cost price of wholesale transaction is ¢52,000
Implies, profit = ¢61256 - ¢52,000 = ¢9256

$$\text{Profit}\ \ percent = \frac{profit}{\cos t\ \ price} \times 100 = \frac{9256}{52000} \times 100 = 17.8\%$$

Example 28.11

A dealer made a profit of 22.5% by selling a car for 5.88million cedis. Find the percentage profit he would have realized if he had sold it for 6.12million cedis.

Solution

Selling price is 5.88million cedis
Percentage profit is 22.5%

But $\text{Profit}\ \ percent = \dfrac{Selling\ \text{Price} - Cost\ \text{Price}}{\cos t\ \ price} \times 100$

$$22.5 = \frac{5{,}880{,}000 - Cost\ \ \text{Price}}{\cos t\ \ price} \times 100$$

$$\frac{22.5}{100} = \frac{5880000 - \cos t\ \ price}{\cos t\ \ price}$$

$$100(5880000 - \cos t\ \ price) = 22.5 \cos t\ \ price$$

$$588000000 - 100 \cos t\ \ price = 22.5 \cos t\ \ price$$

$$\frac{588000000}{122.5} = \frac{122.5 \cos t\ \ price}{122.5}$$

$$\therefore \cos t\ \ price = 4{,}800{,}000$$

Now, if he had sold the car for ¢6,120,000 then,

$\Pr ofit = 6,120,000 - 4,800,000 = ¢1,320,000$

Therefore, $\Pr ofit\ percent = \frac{1,320,000}{4,800,000} \times 100 = 27.5\%$

Example 28.12

A trader sold 100 boxes of fruit at ¢8,000.00 per box, 800 boxes at ¢6,000.00 per box and 600 boxes at ¢4,000 per box. Find the average selling price per box.

Solution

Cost per box of the 100 boxes is ¢8000
Implies, total cost for the 100 boxes is $100 \times 8000 = 800000$
Cost per box of the 800 boxes is ¢6000
Total cost of the 800 boxes is 800(¢6000) = ¢4,800,000
Cost per box of the 600 boxes is ¢4000
Total cost of the 600 boxes is 600(¢4000) = ¢2,400,000
Total number of boxes of fruits sold = 100 + 800 + 600 = 1,500 and
Total cost of the boxes altogether is
¢4,800,000 + ¢2,400,000 + 800, 000 = ¢8million
Hence, average selling price per box is

$$= \frac{8000000}{1500} = 5,333.33$$

28.4 **Discount**

In trading, when goods are sold to very ***regular customers*** or customers who usually buy goods in very ***large quantities*** at a reduced price less than the actual marked price or original price of the commodity calculated as a percentage of the marked price of the commodity or article is termed as ***discount***.

Mathematically, we calculate discount on commodities using:

$$Discount = \frac{\mathrm{Re}duction\ in\ price}{Original\ price} \times 100$$

Similarly, we use the relation below to find the new value after discounting:

$$New\ value = \frac{(100-x)}{100} \times Original\ \ price$$

Where x is the discount percent

Example 28.13

Prof. Elvis was given a discount of 15% for an item which cost 25,000 cedis in a shop. Find the new value of the item.

Solution

Method 1

Original or marked price is ¢25,000
Percentage discount is 15%
Implies, x is 15

$$New\ value = \frac{(100-x)}{100} \times Original\ \ price$$

$$New\ value = \frac{(100-15)}{100} \times 25000$$

$$= \frac{85}{100} \times 25000 = 21{,}250$$

Therefore, the new value of the item is ¢21,250

Method 2

$$Amount\ \ Discount = \frac{Percentage\ \ discount}{100} \times Original\ \ price$$

$$Amount\ \ Discount = \frac{15}{100} \times 25000 = 3750$$

$$New\ value = original\ \ price - Amount\ Discount = 25000 - 3750 = 21{,}250$$

Example 28.14

The marked price for a tin of milk is ¢20. But for a cash purchase, Madam Akos was allowed a discount of 10%. How much did she pay for it?

Solution

Marked price is ¢20

Percentage Discount is 10%

$$New\ value = \frac{(100-10)}{100}\times 20 = \frac{90}{100}\times 20 = 18$$

Hence, madam Akos paid ¢18

Example 28.15

If the marked price for a radio set is Gh¢ 250 and a discount of 15% was allowed on purchase by Dr. Eva. How much did she save by buying the set?

Solution

Method 1

Marked price is Gh¢250

Percentage Discount is 15%

$$New\ value = \frac{(100-x)}{100}\times Original\ \ price$$

$$New\ value = \frac{(100-15)}{100}\times 250 = \frac{85}{100}\times 250 = 212.5$$

Dr. Eva bought the set at Gh¢212.5

Hence, amount she saved is Gh¢250 – Gh¢212.5 = Gh¢37.5

Method 2

$$\Rightarrow Amount\ \ saved = 15\%\ of\ \ marked\ \ price$$

$$= \frac{15}{100}\times 250 = 37.5$$

Hence, amount saved is ¢37.5

Example 28.16

Kwame pays GH¢45 for a tv set marked at GH¢65.

Find his discount allowed.

Solution

Original price is GH¢65
New value is GH¢45
Reduction in price is GH¢65 - GH¢45 = GH¢20
Hence, using;

$$Discount = \frac{\text{Re}\,duction\ in\ \ price}{Original\ \ price} \times 100$$

$$Discount = \frac{20}{65} \times 100 = 30.769\%$$

Example 28.17

Dr. Erwin paid GH¢ 25 for a bottle of Voltic mineral water after being allowed a discount of 12%. Find the ***marked price***.

Solution

Original percentage is assumed to be 100%
Percentage Discount is 12%
New percentage (value) is: Original – Discount
= 100% - 12% = 88%

$$Now,\ 88\% \equiv 25$$

$$\Rightarrow 100\% \equiv x$$

Crossing and multiplying gives

$$x = \frac{100 \times 25}{88} = 28.41$$

Therefore, the marked price is GH¢28.41

Example 28.18

A man pays ¢20,000 for an item in a shop after a discount of 15% was allowed. Find the discount allowed.

Solution

Method 1

New price is ¢20,000
Percentage Discount allowed is 15%

$\Rightarrow 85\% \equiv 20,000$

$then\ 15\% = x$

$\Rightarrow x = \frac{15 \times 20000}{85} = 3,529.4118$

Hence, he was allowed a discount of ¢3,529.4

Method 2

$\Rightarrow 85\% \equiv 20,000$

$then\ 100\% = x$

$\Rightarrow x = \frac{100 \times 20000}{85} = 23529.412$

Therefore, discount allowed is original price – new price
= ¢23,529.412 – 20,000 = ¢3,529.4

Example 28.19

Mr. Austin bought a mobile phone at a discount of 20%.
What is the original amount of the phone if he paid ¢250,000?

Solution

Percentage Discount is 20%
New price is ¢250,000 which is equivalent to (100 – 20)% = 80%
Original price is equivalent to 100%

$\Rightarrow 80\% \equiv 250,000$

$then\ 100\% = x$

$\Rightarrow x = \frac{100 \times 250000}{80} = 312,500$

Hence, the original price is ¢312,500

Example 28.20

The cost of production of a wireless set was ¢40,000.
The manufacturer sold it to a wholesaler at a profit of 20%.
The wholesaler sold it to a retailer at a profit of 25%.
The retailer marked the set to be sold at a price 50% above what he paid for it.
(i) Find the marked price
(ii) to a customer who paid cash, the retailer reduced the price to ¢75,000. Find the percentage discount.

Solution
(i)
For manufacturer
Cost price is ¢40,000
Profit percent is 20%

$$\Pr ofit\ \ percent = \frac{\Pr ofit}{Cost\ \ price} \times 100$$

$$\Rightarrow 20 = \frac{\Pr ofit}{40000} \times 100$$

$$\Rightarrow profit = \frac{20 \times 40000}{100} = 8{,}000$$

Implies, the manufacturer's selling price was
¢40,000 + ¢8,000 = ¢48,000

For wholesaler
Cost price is ¢48,000
Profit percent is 25%

$$\Pr ofit\ \ percent = \frac{\Pr ofit}{Cost\ \ price} \times 100$$

$$25 = \frac{\Pr ofit}{48000} \times 100$$

$$\Rightarrow profit = 12{,}000$$

The wholesaler's selling price was ¢48,000 + ¢12,000
= ¢60,000

For retailer

Cost price is ¢60,000

Profit percent is 50%

$$\text{Profit percent} = \frac{\text{Profit}}{\text{Cost price}} \times 100$$

$$50 = \frac{\text{Profit}}{60000} \times 100$$

$$\Rightarrow \text{profit} = 30{,}000$$

Hence, retailer's selling price is ¢60,000 + ¢30,000 = ¢90,000

Therefore, the marked price is ¢90,000

(ii) Marked or original price is ¢90, 000

Selling price is ¢75,000

Discount is ¢90,000 – 75,000 = ¢15,000

$$\text{percentage discount} = \frac{\text{discount}}{\text{original price}} \times 100$$

$$= \frac{15000}{90000} \times 100 = 16.67\%$$

Example 28.21

A shopkeeper bought 150 articles and paid ¢40,000 for transporting them to her shop. The articles were marked for sale at ¢10,000 each. The shopkeeper sold 90 of them this price and the remainder on discount of 30% on the marked price. Altogether she made a profit of 32% on the total amount she spent. Calculate the amount she paid for each article.

Solution

Transportation cost is ¢40,000

Amount for selling 90 of them is 90(¢10,000) = ¢900,000

Implies, remaining 60 of them would have been sold for 60(¢10,000) = ¢600,000

But with the 30% discount means the remaining 60 will be sold altogether for:

$$\frac{70}{100} \times 600{,}000 = 420{,}000$$

Total amount for selling all 150 is ¢900,000 + ¢420,000
= ¢1,320,000
Now, let the total amount she spent be x
Profit percent is 32% on all 150 articles

$$profit\ \ percent = \frac{selling\ \ price - \cos t\ \ price}{\cos t\ \ price} \times 100$$

$$\Rightarrow 32 = \frac{1320000 - \cos t\ \ price}{\cos t\ \ price} \times 100$$

$$\Rightarrow \frac{32}{100} \times \cos t\ \ price = 1320000 - \cos t\ \ pprice$$

$$\Rightarrow \cos t\ \ price = 1{,}000{,}000$$

But she spent ¢40,000 on transport
Implies, amount spent on the 150 articles is
¢1,000,000 – ¢40,000 = ¢960,000

$$she\ \ paid\ \ \frac{960000}{150} = 6{,}400\ \ on\ each\ article$$

28.5 **Depreciation**

When the value of an item reduces or falls as the item is being used for a certain period of time, we say, that item has ***depreciated***.
The methods of compound interest apply to depreciation, but we need to remember that the value of the object is decreasing (depreciating) and not increasing (appreciating). Instead of adding on the compound interest, we need to subtract the amount of depreciation.
We use the following relations in calculating depreciation:

$$Depreciation = \frac{\mathrm{Re}duction\ in\ value}{Original\ \ value} \times 100$$

and

$$New\ value = \frac{100 - x}{100} \times Original\ \ value\ where\ x\ is\ rate\ of\ \ depreciation$$

Example 28.22

The value of a printing machine depreciated each year by 8% of its value at the beginning of that year. If the value of a new machine is 54million cedis, find its value at the end of the third year.

Solution

Method 1

Rate of depreciation is 8%
Value of new machine is 54million cedis
Depreciation at the end of 1st year is

$$= \frac{8}{100} \times 54{,}000 = 4{,}320{,}000$$

Hence, value of machine at the end of the 1st year is

$$= 54{,}000{,}000 - 4{,}320{,}000 = 49{,}680{,}000$$

Depreciation at the end of the 2nd year is

$$\frac{8}{100} \times 49{,}680{,}000 = 3{,}974{,}400$$

Hence, value of machine at the end of the 2nd year is

$$= 49{,}680{,}000 - 3{,}974{,}400 = 45{,}705{,}600$$

Depreciation at the end of the 3rd year is

$$= \frac{8}{100} \times 45{,}705{,}600 = 3{,}656{,}448$$

Hence, value of machine at the end of the 3rd year is

$$= 45{,}705{,}600 - 3{,}656{,}448 = 42{,}049{,}152$$

Therefore, the value of the machine at the end of the third year is ¢42,049,152

Method 2

Rate of depreciation is 8%
Value of new machine is 54million cedis
Value of machine at the end of 1st year is

$$= \frac{92}{100} \times 54{,}000{,}000 = 49{,}680{,}000$$

Value of machine at the end of 2^{nd} year is

$$= \frac{92}{100} \times 49{,}680{,}000 = 45{,}705{,}600$$

Value of machine at the end of 3^{rd} year is

$$= \frac{92}{100} \times 45{,}705{,}600 = 42{,}049{,}152$$

Example 28.23

A factory installs a new machine costing ¢90,000,000. In its operation it depreciates at the rate of 15% in the first year and 10% yearly thereafter, calculate the estimated value of the machine at the end of the third year.

Solution

Value of machine at the end of first year at 15% depreciation is

$$= \frac{85}{100} \times 90000000 = 76{,}500{,}000$$

Value of machine at the end of second year at 10% depreciation is

$$= \frac{90}{100} \times 76{,}500{,}000 = 68{,}850{,}000$$

Value of machine at the end of third year still at 10% depreciation is

$$= \frac{90}{100} \times 68{,}850{,}000 = 61{,}965{,}000$$

28.6 **Commission**

Usually, for some workplaces, salesmen and women are paid a certain percentage of their daily, weekly, monthly or yearly sales in addition to their salary. This additional payment is called ***commission.***

Example 28.24

A house agent's commission on the sale of a house is 7.5%. Calculate his commission on a house sold for ¢14,000,000.

Solution

Agent's commission is 7.5%
Cost of the house is ¢14,000,000
The agent's commission is therefore

$$=\frac{7.5}{100}\times 14{,}000{,}000 = 1{,}050{,}000$$

Example 28.25

A bag of rice cost GH¢ 300. The commission paid on one bag is 15% of the cost price. Find the commission on 20 bags.

Solution

Rate of commission is 15%
Commission on a bag is

$$=\frac{15}{100}\times 300 = 45$$

Commission on all 20 bags is

$$= 20\times 45 = 900$$

Example 28.26

A salesman received a commission of 10% on sales made in a month. His commission was GH¢60. Find the man's total sales for the month.

Solution

Rate of commission is 10%

$$\textit{If}\ \ 10\% \equiv 60$$

$$\textit{then}\ 100\% \equiv x$$

$\Rightarrow x = \frac{60 \times 100}{10} = 600$

Hence, total sales for the month was GH¢ 600

28.7 **Interest**

In the banking sector, customers save money in the banks to be used by the banks. Again, customers also pick money as loans from the bank. Money saved by customers in the bank attracts a percentage increase in the amount saved payable by the bank to the customer and similarly, loans taken attracts a percentage increase in the amount taken and paid in addition to the total amount borrowed or loaned payable by the customer to the bank. These addition percentages paid by a bank to the customer for savings or paid by the customer to the bank for loans services is referred to as ***interest***. The two forms of interest are: ***simple interest*** and ***compound interest***.

28.7.1 **Simple Interest**

Simple interest is simply the amount of money borrowed or invested at a particular time for a given rate.
Mathematically, we calculate simple interest using the relation:

$$I = \frac{P \times R \times T}{100}$$

Where I is the interest, P is the principal, R is the rate and T is the time

Example 28.27

A woman takes a loan of ¢16,200,000 for one year at an interest rate of 4.5% per annum. The loan together with the interest is to be paid in twelve equal monthly installments. Calculate the interest on the loan and how much she pays each month.

Solution

From question,

Principal, P is ¢16,200,000, Rate is 4.5% and Number of years is 1

$$I = \frac{P \times R \times T}{100} = \frac{16,200,000 \times 1 \times 4.5}{100} = 729,000$$

Hence, interest is ¢729,000

Amount to be paid is principal + interest = ¢16,200,000 + ¢729,000 = ¢16,929,000

Hence, amount paid each month is

$$= \frac{16,929,000}{12} = 1410750$$

Example 28.28

The simple interest on ¢360,000 for 2 years is ¢90,000.
Find the rate percentage per annum.

Solution

From question,

Principal is ¢360, Time is 2 years and interest is ¢90,000

$$I = \frac{P \times R \times T}{100} \Rightarrow R = \frac{I \times 100}{P \times T} = \frac{90000 \times 100}{360000 \times 2} = 12.5\%$$

Example 28.29

What principal will amount to ¢1,160,000 in 3 years at 15% per annum simple interest?

Solution

Amount is ¢1160000, time is 3 years, rate is 15%

$$Interest = amount - principal = 1160000 - P$$

Substituting into the relation:

$$I = \frac{P \times R \times T}{100}$$

$$1160000 - P = \frac{P \times 15 \times 3}{100}$$

$$\Rightarrow P = \frac{116000000}{145} = 800{,}000$$

Hence, principal is ¢800,000

Example 28.30

In how many years will GH¢20 amount to GH¢40 at 10% per annum simple interest?

Solution

Principal = GH¢20, amount = GH¢40 and rate = 10%
Interest = Amount – Principal = GH¢40 **-** GH¢20 = GH¢20

$$T = \frac{100\times 20}{20\times 10} = 10\,years$$

Example 28.31

A man borrowed a sum of money from bank at an interest rate of 12%. After 1 year he paid ¢896,000 to settle the loan and the interest. How much did he borrowed from the bank?

Solution

Amount = ¢896,000, rate = 12%
Time = 1 year

NB

Amount borrowed from the bank is the principal

$$I = \frac{P\times R\times T}{100}$$

$$But\;\; Amount = \text{int}\,erest + principal$$

$$\Rightarrow P = A - I$$

$$I = \frac{(A-I)\times R\times T}{100} = \frac{(896000-I)\times 12\times 1}{100} = \frac{10752000-12I}{100}$$

$100I + 12I = 10752000$

$\therefore I = 96000$

$But \ Amount = \text{int}erest + principal$

$\Rightarrow P = A - I = 896000 - 96000 = 800{,}000$

Hence he borrowed ¢800,000 from the bank

28.7.2 **Compound Interest**

Compound interest involves a continuous earning of interest on a principal over a certain period of time. This means that, after the first interest is earned on the principal, the interest together with the principal now becomes the outstanding principal and the new interest is calculated on this new principal. Thus, in calculating compound interest, we calculate interest for each year singly or one by one denoting number of years in each case as 1.

Example 28.32

Calculate the compound interest on ¢2,000,000 for 2 years at 3.5% per annum.

Solution

Principal = ¢2,000,000, time = 2 years, rate = 3.5%
First, interest for first year i.e. T = 1 year

$$I = \frac{P \times R \times T}{100} = \frac{2000{,}000 \times 1 \times 3.5}{100} = 70{,}000$$

Hence, interest at the end of first year is ¢70,000
Implies, amount at the end of first year
= ¢2,000,000 + ¢70,000 = ¢2,070,000
Secondly, interest for second year i.e. T = 1 year

NB

Amount for first year becomes principal for second year

$$I = \frac{P \times R \times T}{100} = \frac{2{,}070{,}000 \times 1 \times 3.5}{100} = 72{,}450$$

Hence, interest at the end of first year is ¢72,450
Implies, amount at the end of second year = ¢2,070,000 + ¢72,450
= ¢2,142,450
Hence, compound interest = amount at end of 2^{nd} year – principal
= ¢2,142,450 – ¢2000,000 = ¢142,450

Example 28.33

A man deposited ¢80,000 in a bank at 12% compound interest per annum. Find his total amount at the end of the third year.

Solution

Principal = ¢80,000, rate = 12%
Interest at end of first year is

$$I = \frac{P \times R \times T}{100} = \frac{80{,}000 \times 1 \times 12}{100} = 9{,}600$$

Amount at end of first year = ¢80,000 + ¢9,600 =¢89,600
Interest at end of second year is

$$I = \frac{P \times R \times T}{100} = \frac{89600 \times 1 \times 12}{100} = 10{,}752$$

Amount at end of second year = ¢89,600 + ¢10,752 = ¢100,352
Interest at end of third year is

$$I = \frac{P \times R \times T}{100} = \frac{100352 \times 1 \times 12}{100} = 12{,}042.24$$

Amount at end of third year = ¢100,352 + ¢12,042.24 = ¢112,394.24

28.8 **Hire Purchase**

Hire purchase is a system under which someone may buy and pay for goods by instalments. Usually, the buyer pays some deposit for the article he/she intends to buy. In addition, he undertakes to make a series of regular payments until the article is fully paid for. In some cases, this regular payment may take the form of monthly instalments. This system gives an opportunity to people who cannot afford immediate cash payment for goods to acquire some property.

Example 28.34

Jones bought a car for ¢6,800,000. He later put it up for sale at ¢8,800,000. He agreed to sell it to Ruby under the following hire purchase terms: An initial payment of 20% of the price and the balance paid at 15% simple interest per annum in twelve monthly equal installments. Calculate (a) the amount paid every month
(b) the total amount Ruby paid for the car
(c) the percentage profit Jones made on the cost price of the car

Solution

Cost price = ¢6,800,000, Selling price = ¢8,800,000,
Initial payment or deposit = 20% of selling price

$$= \frac{20}{100} \times 8{,}800{,}000 = 1{,}760{,}000$$

Amount left to be paid = ¢8,800,000 – ¢1,760,000 = ¢7,040,000
Interest on balance = 15% per annum

$$\Rightarrow I = \frac{P \times R \times T}{100} = \frac{7040000 \times 15 \times 1}{100} = 1056000$$

Implies, total amount to be paid = ¢7040000 + ¢1056000
= ¢8,096,000

$$Monthly\ amount = \frac{8096{,}000}{12} = 674{,}666.67$$

Therefore, Ruby pays ¢674,666.67 every month
Total amount paid = ¢1,760,000 + ¢8,096,000 = ¢9,856,000
Profit on the car = selling price – cost price
= ¢9,856,000 – ¢6,800,000
= ¢3,056,000

28.9 **Value Added Tax (VAT)**

Value added tax (VAT) is the tax charged on goods and services calculated as a percentage and added to the basic cost. Mathematically,

$$VAT = \frac{p}{100} \times Basic\ Cost \quad OR \quad VAT = \frac{p}{100 + p} \times VAT\ inclusive\ cost$$

Where p% is the VAT rate.

NB

$Basic\ \cos t = Cost\ of\ \ item\ \ without\ VAT = VAT\ \ Exclusive\ \cos t \equiv 100\%$

$VAT\ inclusive\ \cos t \equiv 100 + p$

Example 28.35

The vat rate of a state is 10%. Prof. Elvis bought a pressing iron for GH¢60 VAT inclusive. Calculate his basic cost of the iron and the VAT he paid.

Solution

VAT rate = 10%,
VAT inclusive cost = GH¢60

$If\ \ Basic\ \cos t \equiv 100\%$

$then\ GH60 \equiv 100 + 10 = 110\%$

$\Rightarrow Basic\ \cos t = \frac{100 \times 60}{110} = 54.545455$

Hence, basic cost of the iron is ¢54.55
VAT paid = 10% of basic cost

$\Rightarrow VAT = \frac{10}{100} \times 54.545455 = 5.45455$

Therefore, the VAT paid is GH¢5.45455

Example 28.36

The VAT rate of a country is 12%. A man bought an item with the price tag GH¢80 plus VAT. What did he pay for the item?

Solution

The VAT paid = 12% of basic cost

$\Rightarrow VAT\ \ paid = \frac{12}{100} \times 80 = 9.6$

Hence, VAT paid = GH¢9.6
Price paid = VAT inclusive cost = GH¢80 + GH¢9.6 = GH¢89.6

28.10 **Ratios And Proportions**

28.10.1 **Ratios**

Ratios are generally fractions that compare quantities or items such as numbers. Basically, the ratio of two numbers is written with a ***colon*** in between the numbers. Thus, the ratio of the numbers 7 and 9 is written as ***7:9*** and read as "***7 is to 9***".

NB

7:9 can be written as the fraction $\frac{7}{9}$.

Again, when ratios are not in their simplest forms, we first simplify them. Thus, the ratio ***12:8*** is simplified as ***3:2***.

Example 28.37

In a market of 50 women, 30 sell cassava and the rest sell yam.
Find the ratio of the women who sell yam to that of cassava sellers

Solution

Number of yam sellers = 50 – 30 =20
Ratio of yam sellers to cassava sellers = 20:30 = 2:3

Example 28.38

Find y in the ratio 9:27 = 4y:24

Solution

9:27 = 4y:24

$$\Rightarrow \frac{9}{27} = \frac{4y}{24} \Rightarrow \frac{1}{3} = \frac{y}{6}$$

$$\Rightarrow \frac{3y}{3} = \frac{6}{3} \Rightarrow y = 2$$

NB

Mathematically, we can share or divide quantities or numbers into any given ratio. We do this by:
First summing the proportions of the ratio for the total of the ratios
Finally, we find the corresponding share of each ratio by dividing the ratio by the total ratios found in step 1 above and multiplying the result by the quantity or amount given.

Example 28.39

Three men shared ¢480,000.00 in the ratio 7:8:9. Find the difference between the least and greatest shares.

Solution

Ratio = 7:8:9
Total ratio = 7 + 8 + 9 = 24
Amount shared = ¢480,000.00
Least ratio = 7
Hence, least share = $\frac{7}{24}\times 480000 = 140{,}000$
Greatest ratio = 9
Hence, greatest share = $\frac{9}{24}\times 480000 = 180{,}000$
Therefore, difference between least and greatest shares
= ¢180,000 – ¢140,000 = ¢ 40,000.00

Example 28.40

An amount of ¢300,000.00 was shared among Ama, Kojo and Esi. Ama received ¢60,000.00, Kojo received $\frac{5}{12}$ of the remainder, while the rest went to Esi. What ratio was the money shared?

Solution

Total amount shared = ¢300,000.00

Ama's share = ¢60,000.00
Remainder or amount left = ¢300,000 - ¢60,000 = ¢240,000
Kojo's share = $\frac{5}{12}$ $of\ \ remainder = \frac{5}{12} \times 240{,}000 = 100{,}000$
Esi's share = remainder after kojo's share = ¢240,000 - ¢100,000
= ¢140,000.00
Ratio of their shares = Ama **:** Kojo **:** Esi
= ¢60,000 : ¢100,000 : ¢140,000
= 3:5:7

Example 28.41

In his will, a father left an estate worth ¢76,000,000.00. out of this ¢16,000,000.00 was reserved for various purposes. The rest of the amount was shared among his three children. The eldest son received 20% of the amount. The remaining amount was shared between the other son and daughter in the ratio 3 **:** 2 respectively. Calculate
(i) the amount that the eldest son received
(ii) the amount that the daughter received
(iii) the difference between the amounts the two sons received

Solution

(i) Total amount father left = ¢76,000,000.00
Amount reserved = ¢16,000,000.00
Amount shared among three children = ¢76,000,000 - ¢16,000,000
= ¢60,000,000
Share of eldest son = 20% of ¢60,000,000
$= \frac{20}{100} \times 60{,}000{,}000 = 12{,}000{,}000$
(ii) Remaining amount = ¢60,000,000 - ¢12,000,000 = ¢48,000,000
Ratio of sharing among other son and daughter = 3 **:** 2
Total ratio = 3 + 2 = 5
Daughter's share = $\frac{2}{5} \times 48{,}000{,}000 = 19{,}200{,}000$
(iii) Other son's share = $\frac{3}{5} \times 48{,}000{,}000 = 28{,}800{,}000$

Difference in eldest son's share and other son's share
= ¢28,800,000 - ¢12,000,000 = ¢9,600,000

Example 28.42

The development budget of a district council includes expenditure on feeder roads, schools and water supply. The expenditure on roads, schools and water supply are in the ratio 7:15:2 respectively. If the expenditure on roads is ¢28million, find the expenditure on
(i) Schools (ii) Water supply (iii) What is the total budget for these three projects? The cost of maintaining libraries is ¢900,000.00 and this is met from the expenditure on schools. What percentage, correct to 3 significant figures, on the schools is spent on maintaining libraries?

Solution

Ratio of roads : schools : water supply = 7 : 15 : 2
Total ratio = 7 + 15 + 2 = 24
Let the total budget = ¢y

$$Amount\ on\ roads = \frac{7}{24} \times total\ budget = \frac{7}{24}y$$

But amount or expenditure on roads = ¢28,000,000.00

$$\Rightarrow 28{,}000{,}000 = \frac{7}{24} \times y$$

$$\Rightarrow y = 96{,}000{,}000$$

(i) Expenditure on schools = $\frac{15}{24} \times 96{,}000{,}000 = 60{,}000{,}000$

(ii) Expenditure on water supply = $\frac{2}{24} \times 96{,}000{,}000 = 8{,}000{,}000$

(iii) Total budget for three projects = ¢96,000,000.00
Expenditure on schools = ¢60,000,000
Cost of maintaining libraries = ¢900,000
Let percentage of school's expenditure used on maintaining libraries be x%.
Implies, x% of school's expenditure = ¢900,000.00

$$\Rightarrow \frac{x}{100} \times 60,000,000 = 900,000$$

$$\Rightarrow x = \frac{900,000}{600,000} = 1.50\%$$

Hence, percentage spent on maintaining libraries = 1.50%

NB

We can ***increase*** or ***decrease*** an amount in any given ratio by simply expressing the ratio as a ***fraction*** and multiplying the resulting fraction by the amount.

Example 28.43

(i) Increase the amount GH¢7,500 in the ratio 8:12
(ii) Decrease the amount GH¢2,800 in the ratio 5:7

Solution

(i) Ratio = 8:12

Expressing ratio as fraction = $\frac{8}{12} = \frac{2}{3}$

$$\Rightarrow 7500 \times \frac{2}{3} = 5,000$$

Hence, GH¢5,000

(ii) Ratio = 5:7

Ratio as fraction = $\frac{5}{7}$

$$\Rightarrow 2,800 \times \frac{5}{7} = 2,000$$

28.10.2 **Proportions**

Earlier, we learnt that a ***ratio*** is a ***proportion*** as well as a ***fraction***. Whenever two items or quantities vary but remain in the same ratio, they are described as ***direct proportions***. Similarly, whenever two items vary in a way that as one is increasing, the other decreases, they are described as ***inverse proportions***.

In solving questions under either type of proportion, we usually evaluate for the ***unit*** (single) ***amount*** or we use the ***equivalence approach***.

Example 28.44

If a man uses 4 hours to eat 8 balls of kenkey, find how many balls of kenkey he will eat in 7 hours.

Solution

Method 1 (equivalence approach)

$If\ \ 4hrs \equiv 8balls$

$Then\ \ 7hrs \equiv x\,balls$

$\Rightarrow x = \dfrac{8 \times 7}{4} = 14$

Method 2 (unitary approach)

If in 4 hrs 8 balls are eaten, then in 2 hrs $\frac{8}{2} = 4$ balls are eaten

Likewise in 1 hr $\frac{4}{2} = 2$ balls are eaten.

Hence, in 7 hrs $2 \times 7 = 14$ *balls are eaten.*

Example 28.45

If 24 men can use 45 days to weed a plot of land, how many men will be required to weed the same plot of land in 30 days?

Solution

$$\begin{aligned}&\text{If } 45\,days \equiv 24\,men\\&\text{Then } 30\,days \text{ will require more men to weed the same plot of land}\\&\Rightarrow \frac{45}{30}\times 24 = 36\,men\end{aligned}$$

NB

If more*, *less divide and ***if less, more divide***. i.e. always multiply the two corresponding values (in above, 24 and 45 correspond) given and divide the result by the figure that requires the unknown quantity.

Example 28.46

If 15 girls can do a piece of work in 50 minutes, how long will it take 25 girls to do the same piece of work?

Solution

$$\begin{aligned}&\text{If } 50\,\text{minutes} \equiv 15\,girls\\&\text{Then } 25\,girls \text{ will require less minutes to do the same piece of work}\\&\Rightarrow \frac{50}{25}\times 15 = 30\,\text{minutes}\end{aligned}$$

NB

If more, less divide

Example 28.47

If 3 ladies lodge in a guest house for 10 hours and pays GH¢ 60, what will be the cost if 5 ladies lodge for 8 hours?

Solution

3 ladies lodge for 10 hours and paid GH¢60

Implies, 1 lady lodges for 10 hours will pay $\frac{60}{3} = 20$

Again, 1 lady lodges for 1 hour will pay $\frac{60}{3 \times 10} = 2$

5 ladies lodge for 1 hour paid $\frac{60 \times 5}{3 \times 10} = 10$

Hence, 5 ladies lodge for 8 hours and paid $\frac{60 \times 5 \times 8}{3 \times 10} = 80$

28.11 **Financial Partnership**

Here, a number of persons, say two or more agree to invest in a specific business and profits realized shared in the ratio of their contributions or initial capitals.

Example 28.48

Three friends Ato, Oko and Edem entered into a business partnership. They contributed 3.0 million, 2.4 million and 3.6 million cedis respectively. It was agreed that profits will be shared in proportion to their contributions. After one year of operation, the profit was 2.7 million cedis.

(i) Find the amount received by each partner as his share of the profit

(ii) Express Edem's share of the profit as a percentage of his investment

Solution

(i) Ratio of investment = Ato : Oko : Edem
= 3,000,000 : 2,400,000 : 3,600,000 = 5:4:6
Total ratio = 5 + 4 + 6 = 15
Total profit realized = ¢2,700,000
Hence,

Ato's share = $\frac{5}{15} \times 2{,}700{,}000 = 900{,}000$

Oko's share = $\frac{4}{15} \times 2{,}700{,}000 = 720{,}000$

Edem's share = $\frac{6}{15} \times 2{,}700{,}000 = 1{,}080{,}000$

(ii) Edem's share = ¢1,080,000

Edem's investment = ¢3,600,000

Therefore, Edem's share of the profit as a percentage of his investment is

$$= \frac{1080000}{3{,}600000} \times 100 = 30\%$$

Example 28.49

Kofi and Yaw entered into business partnership with a total capital of ¢81 million. They agreed to contribute the capital in the ratio 2:1 respectively. The profit was shared as follows: Kofi was paid 5% of the total profit for his services as manager. Each partner was paid 3% of the capital he invested. The remainder of the profit was then shared between them in the ratio of their contributions to the capital. If Kofi's share of the total profit was ¢7.5 million, calculate

(a) the total profit for the year, to the nearest thousand cedis

(b) Yaw's share of the profit as a percentage of his contribution to the capital

Solution

(a) Ratio of contributions = Kofi : Yaw = 2 : 1

Total ratio = 2 + 1 = 3

Kofi's capital = $\frac{2}{3} \times 81{,}000{,}000 = 54{,}000{,}000$

Yaw's capital = $\frac{1}{3} \times 81{,}000{,}000 = 27{,}000{,}000$

Kofi had 3% of his capital = $\frac{3}{100} \times 54{,}000{,}000 = 1{,}620{,}000$

Yaw had 3% of his capital = $\frac{3}{100} \times 27{,}000{,}000 = 810{,}000$

Let total profit be ¢y

But Kofi was again given 5% of the profit as manager

$= \frac{5}{100} \times y = \frac{1}{20} y$

Hence, remainder of the profit =

$y - \left[\frac{1}{20} y + 1{,}620{,}000 + 81{,}000\right] = \frac{19}{20} y - 2{,}430{,}000$

Since the remainder of the profit was shared in the ratio of their contributions,

Kofi's share of remainder = $\frac{2}{3} \times \left[\frac{19}{20} y - 1{,}620{,}000\right] = \frac{19}{30} y - 1{,}620{,}000$

Hence, Kofi's total share = $\frac{1}{20} y + 1620000 + \frac{19}{30} y - 1{,}620{,}000 = \frac{41}{60} y$

But since Kofi's share of the profit = ¢7,500,000,

Implies, $\frac{41}{60} y = 7{,}500{,}000 \Rightarrow y = 10{,}976{,}000$

Therefore, the total profit = ¢10,976,000

(b) total profit = ¢10,976,000

Kofi's share of profit = ¢7,500,000

Yaw's share of profit = ¢10,976,000 – ¢7,500,000 = ¢3,476,000

Yaw's capital = ¢27,000,000

Yaw's share as a percentage of his capital = $\frac{3{,}476{,}000}{27{,}000000} \times 100 = 12.8\%$

EXERCISE

QUE. A

A man bought n articles for x cedis each. He sold p of them for (x + 2) cedis each and the remainder for (x + 1) cedis each.

(i) Find his profit in terms of p and n

(ii) If n = 800,000, p = 640,000 and x = 50, express his profit as a percentage of the cost.

QUE. B

Ama bought p cups of rice at x cedis per cup and q onions at 5 for ¢200. Find the total cost in cedis.

QUE. C

A woman bought 36 baskets of oranges each containing 150 oranges. Out of the total number, 72 went bad. She sold the rest at 2 for ¢300 and made a profit of ¢16,200. Find the cost of 1 basket of oranges.

QUE. D

The total age of two sisters is 108 years. One is 18 years older than the other. Find the ratio of the age of the older to that of the younger.

QUE. E

If it cost a manufacturer ¢60,000 to make a wireless set. He sold the wireless set to a wholesaler at a profit of 20%. The wholesaler also sold it to a retailer at a profit of 25%.

Q15

How much did the wholesaler pay for the wireless set?

Q16

How much did the retailer pay for the wireless set?

QUE. F

A meat seller bought 250kg of meat for ¢224,000. At what price per kg should he retail it in order to make a profit of 25%?

QUE. G

The value of a TV set depreciates according to the following table:

Year of manufacture	Depreciation on the original value
In the first year	Nil
In the second year	10%
In the third year	12%
In the fourth year	20%

The original value of the machine is ¢20,000,000. Find the value of the machine at the end of each of the first four years.

QUE. H

A trader buys a crate of 24 bottles of soft drinks at D54 and sells each bottle at D3. Find his percentage profit.

QUE. I

The simple interest on ¢12,000 for two and half years is ¢1,200. What is the rate per annum?

QUE. J

The sum of GH¢18,000 lent at a certain rate of simple interest amounted to GH¢18,600 at the end of 8 months. What was the rate of interest per annum?

QUE. K

The simple interest on ¢600,000 for 3.75 years is ¢56,250. Find the rate percent per annum.

QUE. L

Mr. Amos bought a car for 2.5 million cedis in 1985. He paid 40% of the cost and paid the rest in equal monthly installments. He took

8 years to make full payment for the car. Interest was charged at 18% simple interest. Calculate (i) the monthly installment;
(ii) the total amount he paid for the car; (iii) the percentage increase in the cost of the car.

QUE. M

Tickets for a film show were sold at ¢4,000 per adult and ¢2,000 per child. The total amount realized from the 500 tickets sold was ¢1,600,000.
Find the number of (a) adults who bought the tickets for the show
(b) Children who bought tickets for the show
12% of the amount realized was paid as tax and 2% of the amount was paid to the ticket sellers. Find the amount paid. (a) as tax (b) to the ticket sellers

QUE. N

Three business partners, Amos, Ben and Cudjoe, share a profit of ¢11,000,000.00. Amos takes 30% of the profit and the remainder is shared between Ben and Cudjoe in the ratio 4:3 respectively. How much did Ben receive?

QUE. O

Tickets for a film show were sold at ¢4,500 to the general public and ¢3,750 to students. 400 people attended the show and ¢1,680,000 was collected in ticket sales.
(a) How many tickets were sold to students?
(b) Mr. Mensah was issued with 25 tickets to be sold to the general public and 20 tickets to be sold to students. How much did Mr. Mensah collected after selling all the tickets issued to him?

QUE. P

Joe bought a small radio at N800 and sold it to Danli at a profit of 5%. Danli then sold it to Chike at a loss of 5%. How much did Chike pay for the radio?

QUE. Q

A man left the sum of ₦1,260,000 to be shared equally among all his surviving children. Three of the children died before their father and so each of the survivors received N70,000 more than they would have received if all had lived. How many children survived their father?

QUE. R

The cash price of a gas cooker was ¢60 000.00. A man paid 25% of the cash price as deposit. He then paid ¢8165.00 a month for six months.

(a) How much did he pay altogether for the cooker?
(b) Find the interest charged
(c) Find also the approximate rate of interest

QUE. S

Calculate the compound interest (payable yearly) on these loans.

(a) ¢50000.00 for 2years at 5% p. a
(b) ¢34000.00 for 3years at 3% p. a
(c) ¢80000.00 for 2years at 18% p. a
(d) ¢25500.00 for 3years at 15% p. a
(e) ¢45500.00 for 3years at 20% p. a

QUE. T

Find (a) the simple interest (b) the compound interest paid annually on ¢35000.00 for 3years at 18% p. a. What is the difference between the compound interest and the simple interest?

QUE. U

The value of a machine depreciated each year by $12\frac{1}{2}$ of its value at the beginning of that year. The value of the machine when new was ¢250000.00. Find its value when it was four years old.

QUE. V

A car which was bought for $9000.00 when new was valued at $7500.00 at the end of the first year. It then depreciated each year by $12\frac{1}{2}$ of its value at the beginning of that year. Calculate

(a) The depreciation at the end of the first year
(b) The value of the car at the end of the third year

CHAPTER 29
LINEAR INEQUALITIES II
(INEQUALITIES IN TWO VARIABLES)

The generally, solutions of inequalities in two variables are linear graphs which are illustrated by shading the region that satisfies the inequality.
Here, we shall consider (i) single inequalities in two variables and
(ii) simultaneous inequalities in two variables.

NB

In drawing graphs of inequalities, we use thick lines (————) when the inequality sign in the expression is either $\leq or \geq$ and we use dotted lines (-----) when the inequality sign in the expression is either < or >.

29.1 **Steps In Drawing Graphs Of Inequalities**

1. Use its equivalent equation to calculate the x and y intercepts of the resulting equation.

NB

For x – intercept, put y = 0 into the equation and solve for x and put x found and
y = 0 together as a point.
Again, for y - intercept, put x = 0 into the equation and solve for y, after which we put y found and x = 0 together as a point.
2. Plot both intercepts found and join them by a straight line,
Either ——— or ------- depending on whether the sign is $\leq, \geq, < \ or \ >$
3. Pick a point (an x and its corresponding y value) either above or below the graph and substitute into the inequality to see whether it will satisfy the given inequality or not.
4. Shade towards the side (either above or below) of the graph that satisfied the inequality.

NB

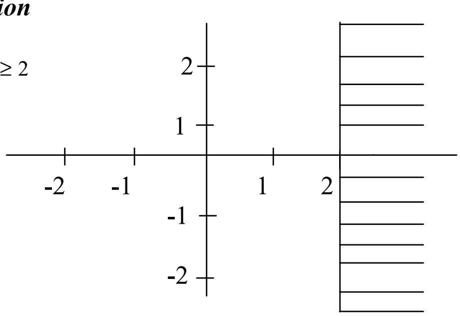
When the inequality is of only one variable, we locate the number on the graph and draw a straight line through it. When it is x, we draw a vertical line through the number on the x - axis and when it is y we draw a horizontal line through the number on the y - axis and shade according to the inequality sign.

Example 29.1

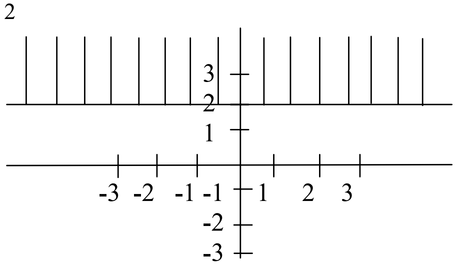
Shade on separate graphs the region representing the solution set of the following inequalities:
$(i)\ x \geq 2 \quad (ii)\ y \geq 2 \quad (iii)\ x < 1 \quad (iv)\ y \leq -1 \quad (v)\ x > 3 \quad (vi)\ y > 2 \quad (vii)\ y < -2$

Solution

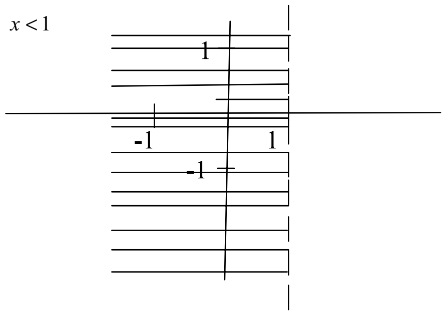
(i) $x \geq 2$



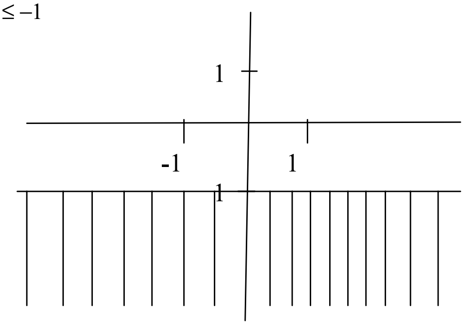
(ii) $y \geq 2$



(iii) $x < 1$



(iv) $y \leq -1$

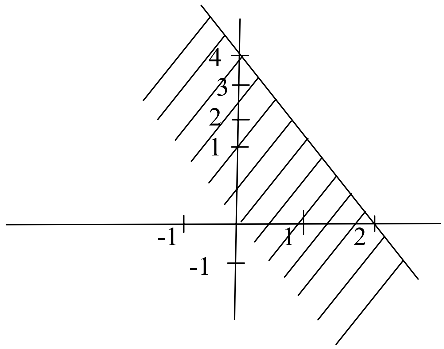


Example 29.2

Solve the inequality $y + 2x \leq 4$ graphically

Solution

Equivalently, we have y + 2x = 4
Put x = 0
Implies, y = 4
Hence, **(0, 4)**
Put y = 0
Implies, x = 2
Hence, **(2, 0)**



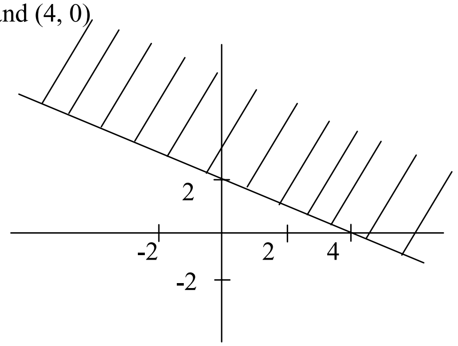
Now testing the region to shade; put x = 1 and y = 1
Implies, 1 + 2(1) = 1 + 2 = 3 < 4 satisfied.
Hence, we shade region below graph line where the point (1, 1) lie.

Example 29.3

Solve the inequality $x + 2y \geq 4$ graphically

Solution

Similarly, y + 2x = 4
Plot the points (0, 2) and (4, 0)



Testing, use (0, 0) which does not satisfy, hence shade above graph for solution set.

NB

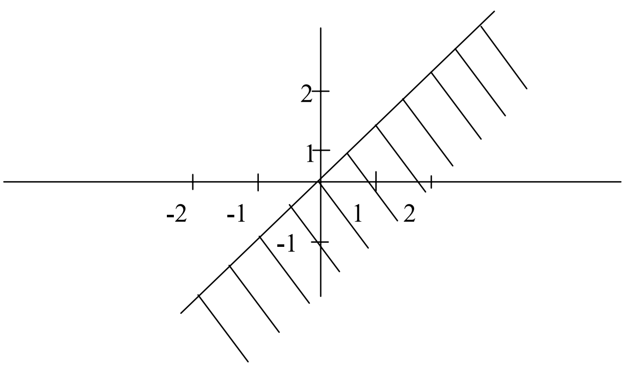
It is advisable to always use the point (0, 0) when testing for the shaded region when the graph does not pass through the origin and any other point when passing the origin

Example 29.4

Shade the region representing the solution set of the inequality
$y - 3x < 0$

Solution

Equivalently, $y - 3x = 0$
Put $x = 0$
Hence, **(0, 0)**
Put $y = 0$
Hence, **(0, 0)**



Testing we use (1, 0) which rightly satisfies the inequality.
Hence, we Shade towards that side.

29.2 **Drawing Graphs Of Simultaneous Inequalities**

We follow the same procedure for drawing that of a single inequality given above shading each graph (inequality) appropriately on the same graph. The region where the graphs intersect each other gives the solution set of the given inequalities.

Example 29.5

Which of the following inequalities represent the shaded portion?

Solution

$y + x \leq 3$

We have $y + x = 3$

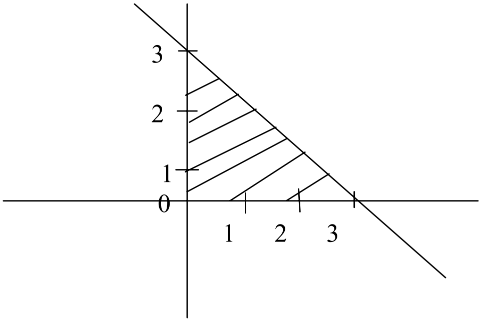
Put x = 0 implies, **(0, 3)**

Put y = 0 implies, **(3, 0)**

Now plot (0, 3) and (3, 0) and shade appropriate region.

Again, plot and shade $x \geq 0$ *and* $y \geq 0$ *on the same graph*

Locate the region where all shadings intersect for the solution set of the inequalities.



Example 29.6

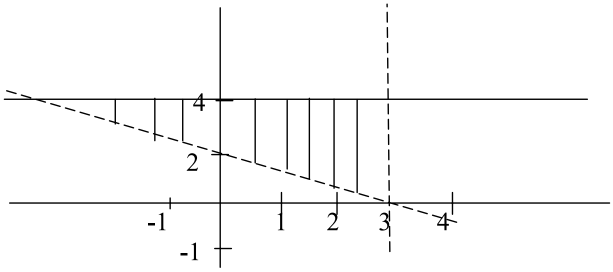
(a) Draw a graph to show the region where the inequalities x + y > 2, x < 3 and $y \leq 4$ are satisfied.

(b) List the integer solution (x, y) which satisfy these inequalities.

Solution

(a) For x + y > 2, we plot (0, 2) and (2, 0) with dotted lines, plot x < 3 with dotted lines and $y \le 4$ (with thick lines) and shade the intersecting region.

(b) The pair of integers satisfying these inequalities are:
(-1, 4), (0, 4), (1, 4), (2, 4), (0, 3), (1, 3), (2, 3), (1, 3), (2, 3), (1, 2), (2, 2), (2, 1)



NB

Students are advised to first draw, shading all the graphs on one graph sheet to see where they all intersect and then redraw the graph showing the intersecting shaded region of all the graphs.

29.3 **Finding The Maximum Or Minimum Values Of A Function**

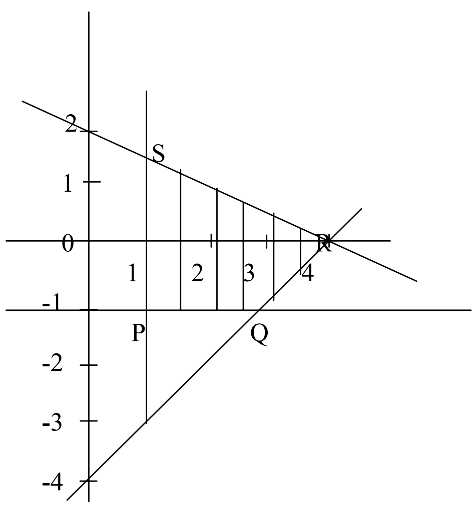
The maximum or minimum value of any function involving x and y of a system of inequalities is obtained by locating all coordinates of the vertices of the graph of the shaded region and substituting into the given function for the max or mini value.

Example 29.7

Indicate on a diagram by shading, the solution set of the following four inequalities:
$y \geq -1, x - y \leq 4,\ x + 2y \leq 4\ \ and\ x \geq 1.$ Using the diagram, find subject to these four conditions:
(i) The max. value of x and y
(ii) The max. value of x^2
(iii) The max. and min. value of 2x + y

Solution

Follow the same steps and shade the region of the intersection of all the inequalities.
Pick the coordinates of the vertices of the shaded region



(a) Pick the coordinates of the vertices of the shaded region:
i.e. P(1, -1), Q(3, 1), R(4, 0) and S(1, 1.5). Hence, the maximum value of x is 4 and that of y is 1.5 in the points of the vertices

(b) The max. value of x^2 is $4^2 = 16$. i.e. We picked the values of x in each vertices and substitute into x^2 for highest and lowest results. The max. value of 2x + y is 8 and min. value is 1. Also obtained by substituting vertices into 2x + y for highest and lowest results.

NB

We can graph sentences by first changing them into inequalities and graphed together on the same sheet shading the region of intersection of all the regions.

Example 29.8

A Tailor can make at most 20 male dresses and at most 30 female dresses a day. Each male dress requires 3 hours of labour and each female dress requires 2 hours of labour. The maximum total hours of labour that the Tailor has at his convenience is 96.
(a) Give three inequalities that expresses the condition
(b) Shade the common region that satisfies these inequalities
(c) Find the max. number of male and female dresses that he can make within the time frame.

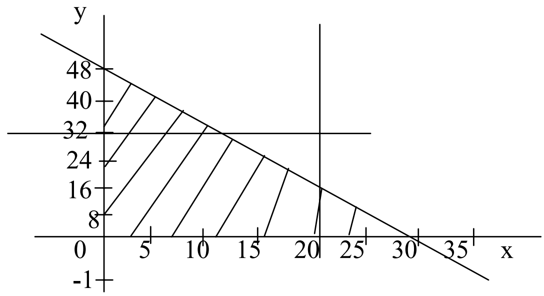
Solution

Let x and y represent the number of male and female dresses made per day respectively
Implies, number of male dresses constraint is : $x \le 20$ (for at most) and number of female dresses constraint is: $y \le 30$ (for at most)
Total hours required altogether = 3x + 2y
Hence, Time constraint is: $3x + 2y \le 96$
And using also: x > 0 and y > 0 (since no negative dresses)
For $3x + 2y \le 96$

Implies, 3x + 2y = 96
Put x = 0
Hence, **(0, 48)**
Put y = 0
Hence, **(32, 0)**
Plot and shade appropriately; (0, 48) and (32, 0), $y \leq 30$, $x \leq 20$ $x \geq 0$ *and* $y \geq 0$, on the same graph.

(a) The three inequalities are: $y \leq 30$, $x \leq 20$ *and* $3x + 2y \leq 96$

(b) Max. number of male dresses is 20 and max. number of female dresses are 15



EXERCISE

QUE. A

Shade the solution set of the following inequalities

1. $3x+4y-12\geq 0$ 2. $2y-3x-2<0$ 3. $2y-5x-4>0$

4. $y-5x-2\leq 0$ 5. $3y-4x+6\geq 0$ 6. $5x+2y+15<0$

QUE. B

Show by shading labeled A satisfies the three linear inequalities simultaneously

1. $x+y-1\geq 0$, $2x+y\leq 0$, $x\leq 4$
2. $3y+5x\leq 15$, $y\geq 3x$, $y=\geq 0$
3. $2x-3y\geq -6$, $3y+2x\leq 12$, $x\geq$
4. $y+3x-6\geq 0$, $x+y\leq 4$, $2x+5y\geq 10$
5. $4y+3x-12\geq 0$, $y-2x\geq -2$, $2x-2y+6\geq 0$
6. $2x-3y\geq -6$, $2y-x+2\geq 0$, $x\leq 3$

QUE. C

Shade in the x-y plane, the region that satisfies simultaneously the linear inequalities

$x+y\leq 6$

$3x-y\geq 2$

$3y-x\leq 2$

Find the members of the solution set which make

(a) 2y+x maximum

(b) 2+x minimum

Ans: a. 10 and occurs at (2, 4)   b. 3 and occurs at (1, 1)

QUE. D

Determine the maximum and minimum values of

(a) 2x + 3y

(b) 5x+4y

over the solution sets of the following system of linear inequalities:

$(a)\ y+x-8\leq 0$ $2y+5x-19\geq 0$ $4y+x-11\geq 0$

2. $3y+4x-26\leq 0$ $y+3x-12\geq 0$ $y+2x-9\geq 0$

3. $10y+x-34\geq 0$ $2y-x-2\leq 0$ $2y+x-22\leq 0$ $x\leq 14$

4. $y-4x+18\geq 0$ $y+x-12\leq 0$ $y\leq 2x$

5. $4x-y-2\geq 0$ $x+3y-7\geq 0$ $x+2y-14\leq 0$ $3x+2y-10\leq 0$

QUE. E

A farmer at El-Ak Series Farms engages two vehicles X and Y to carry three of his farm produce A, B and C to the next buying centre Tumokotaw. The farmer wishes to carry at least 14 tonnes of A, 10 tonnes of B and 18 tonnes of C. vehicle X can carry 2 tonnes of A, 1 tonne of B and 1 tonne of C per trip. Vehicle Ycan carry 1 tonne of A, 1 tonne of B and 3 tonnes of C per trip. The cost per trip vehicle X is ¢800.00 and that of vehicle Y is ¢850.00. How many trips should each vehicle make so that the farmer minimizes his cost?
Ans: 6 trips of X and 4 trips of Y.

QUE. F

A manufacturing company makes ¢2000.00 profit on each radio set it produces and ¢3000.00 on each colour TV. The production of radio set requires 1 hour of section A, 3 hours of section B, 2 hours of section C and 5 hours of section D of the company. The production of a colour TV requires 4 hours of A, 4 hours of B, 5 hours of C and 4 hours of D. in any given week section A, B, C and D work up to a maximum of 44, 52, 51 and 85 hours respectively. How many radio sets and colour TV's should the company produce per week to maximize the profit?

CHAPTER 30

PYTHAGORAS THEOREM AND TRIGONOMETRY

30.1 **The Pythagoras Theorem**

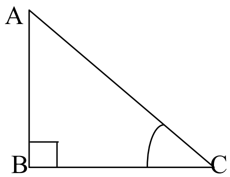
Basically, the Pythagoras theorem is an aspect that is useful in calculations involving the right angled triangle. A right angled triangle is a triangle with one of its angles being 90^o.

NB

In a right triangle, the longest side is called the "***hypotenuses***", the side facing the angle under consideration is called the "***Opposite***" and the remaining third side is called the "***Adjacent***".

Illustration

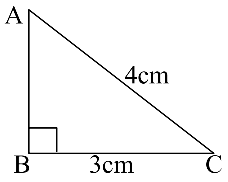
Consider the triangle below:



Assuming Ø is the angle at C, then the side AB is the 'opposite', the side BC is the 'adjacent' and the side AC is the 'hypotenuse'.
The Pythagoras theorem states that:
Square of hypotenuse = sum of squares of other sides.
i.e. $(\text{Hypotenuse})^2 = (\text{opposite})^2 + (\text{Adjacent})^2$
$i.e.\ |AC|^2 = |AB|^2 + |BC|^2$

Example 30.1

Find the length of the side AC in the figure below:



Solution

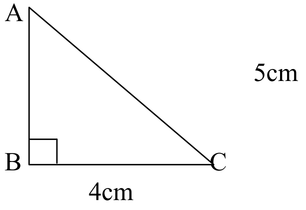
By Pythagoras theorem

$|AC|^2 = |AB|^2 + |BC|^2$

$\Rightarrow |AC|^2 = 4^2 + 3^2 = 25$

$\Rightarrow |AC| = 5cm$

Examples 30.2



In the diagram above, find the length of side AB

Solution

By Pythagoras theorem;

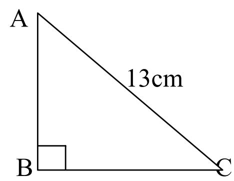
$|AC|^2 = |AB|^2 + |BC|^2$

$|AB|^2 = |AC|^2 - |BC|^2 = 25 - 16 = 9$

$\Rightarrow |AB| = 3cm$

Example 30.3

Find the length of the side BC in the diagram below:



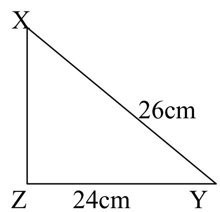
By Pythagoras theorem;

$$|AC|^2 = |AB|^2 + |BC|^2$$

$$|BC|^2 = |AC|^2 - |AB|^2 = 169 - 144 = 25$$

$$\Rightarrow |BC| = 5cm$$

Example 30.4



In the diagram above, XYZ is a right-angled triangle.
Calculate the area of the triangle.

Solution

By Pythagoras theorem,

$$|XY|^2 = |XZ|^2 + |YZ|^2$$

$$\Rightarrow 26^2 = |XZ|^2 + 24^2$$

$$\Rightarrow 676 = |XZ|^2 + 576$$

$$676 - 576 = |XZ|^2$$

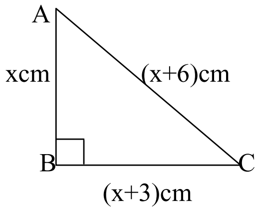
$$100 = |XZ|^2$$

$$\Rightarrow \sqrt{100} = |XZ| = 10cm$$

$$Area\ of\ triangle = \frac{1}{2}bh = \frac{1}{2} \times 24 \times 10 = 120cm^2$$

Example 30.5

Find the value of x in the figure below:



Solution

By Pythagoras theorem;

$$|AC|^2 = |AB|^2 + |BC|^2$$

$$(x+6)^2 = x^2 + (x+3)^2$$

$$x^2 + 12x + 36 = x^2 + x^2 + 6x + 9$$

$$\Rightarrow x^2 - 6x - 27 = 0$$

Solving quadratically,

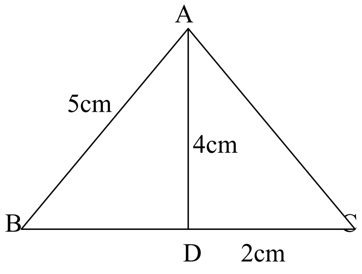
$$x = -3\ or\ x = 9$$

Hence, the value of x is 9cm

Example 30.6

In triangle ABC, find the values of

$|BD|$ *and* $|AC|$ *where* $|AB| = 5cm,\ |AD| = 4cm,\ |DC| = 2cm$



Solution

Considering triangle ABD;

$|AB|^2 = |AD|^2 + |BD|^2$

$\Rightarrow |BD|^2 = |AB|^2 - |AD|^2 = 25 - 16 = 9$

$\Rightarrow |BD| = 3cm$

Again, from triangle ACD;

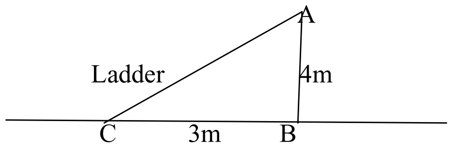
$|AC|^2 = |AD|^2 + |CD|^2 = 16 + 4 = 20$

$\Rightarrow |AC| = \sqrt{20} = 2\sqrt{5}cm$

Example 30.7

A ladder is placed against a wall vertically. If the wall is 4m high and the foot of the ladder is 3m from the wall, how long is the ladder?

Solution



Since the ladder looks slanting, it becomes the hypotenuse side.
i.e. Side AC
By Pythagoras theorem;

$$|AC|^2 = |AB|^2 + |BC|^2 = 16 + 9 = 25$$

$$\Rightarrow |AC| = 5m$$

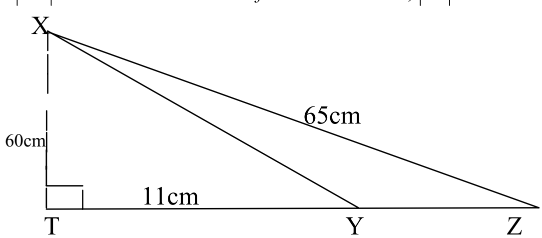
Hence, the ladder is 5m long.

NB

Similarly, the length of the ladder could be given with one other side and asked to find the remaining side. Pythagoras theorem is again applied to find the other side.

Example 30.8

In the diagram below,
$|XT|$ *which is the altitude of* ΔXYZ *is* $60cm$, $|YT| = 11cm$ *and* $|XZ| = 65cm$. *Find* $|YZ|$



Solution

Let $|YZ| = xcm$

Then, from triangle XTZ and applying Pythagoras theorem,

$$|XZ|^2 = |XT|^2 + |TZ|^2$$

$$65^2 = 60^2 + (x+11)^2$$

$$4225 = 3600 + x^2 + 22x + 121$$

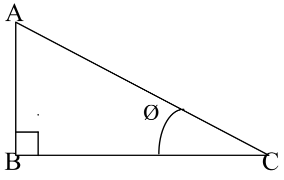
$$\Rightarrow x^2 + 22x - 504 = 0$$

$$\therefore x = -36, 14$$

TRIGONOMETRY

30.2 **Trigonometry Ratios**

There are basically three trigonometric ratios which are: the ***sine*** usually written as "***sin***", the ***tangent*** usually written as "***tan***" and the ***cosine*** usually written as "***cos***".



Considering the right angled triangle above, we define the trigonometric ratios as below and are applied to right angled triangles.

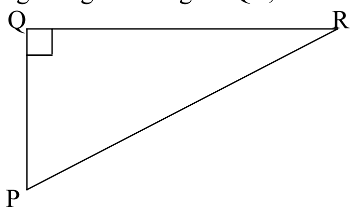
$$Sin\,\theta = \frac{Opposite}{Hypotenuse} = \frac{|AB|}{|AC|}$$

$$Tan\,\theta = \frac{Opposite}{Adjacent} = \frac{|AB|}{|BC|}$$

$$Cos\,\theta = \frac{Adjacent}{Hypotenuse} = \frac{|BC|}{|AC|}$$

Example 30.9

In the right angled triangle PQR, Sin R equals**:**



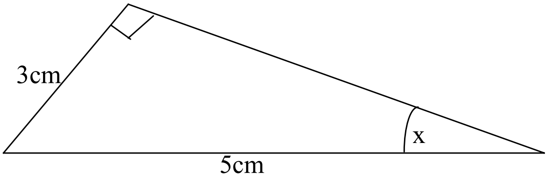
Solution

From diagram,

$$SinR = \frac{Opposite}{Hypotenuse} = \frac{|PQ|}{|PR|} \quad and \quad CosP = \frac{Adjacent}{Hypotenuse} = \frac{|PQ|}{|PR|}$$

$$Hence,\ SinR = CosP$$

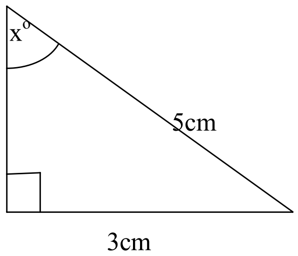
Example 30.10



Calculate correct to the nearest degree, the value of x in the triangle above.

Solution

Redrawing we have;



Since the opposite and hypotenuse are known; we use:

$$Sinx = \frac{Opposite}{Hypothenuse} = \frac{3}{5} = 0.6 \Rightarrow Sinx = 0.6 \Rightarrow x = Sin^{-1}0.6 = 37^{o}$$

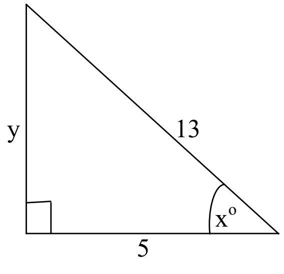
Example 30.11

Given that $Cosx = \frac{5}{13}$, *where x is an acute angle, find the value of* $\tan x$.

Solution

$$Cos\theta = \frac{Adjacent}{Hypotenuse} \Rightarrow Adjacent = 5\,(Since\ corresponds\ to\ the\ numerator)$$

$$and\ \ hypotenuse = 13\,(Sincecorresponds\ to\ the\ deno\min ator)$$



By Pythagoras theorem;

$y^2 + 5^2 = 13^2$

$\Rightarrow y = 12$

$But\ \tan x = \dfrac{Opp}{Adj} = \dfrac{12}{5}$

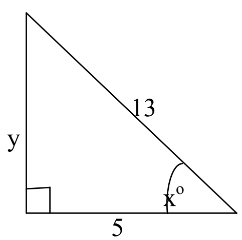
Example 30.12

If x is an acute angle such that

$Cosx = \dfrac{5}{13},\ find\ Sinx.$

Solution

Since $Cosx = \dfrac{Adjacent}{Hypotenuse}, \Rightarrow adjacent = 5\ \ and\ \ hypotenuse = 13$



By Pythagoras theorem;

$y^2 + 5^2 = 13^2$

$\Rightarrow y = 12$

$But\ Sinx = \dfrac{Opposite}{Hypotenuse} = \dfrac{12}{13}$

Example 30.13

Find Q if

$(\theta + 60^o) = 0.0872\ where\ 0^o \leq \theta \leq 90^o$

Solution

$Cos(\theta + 60^o) = 0.0872$

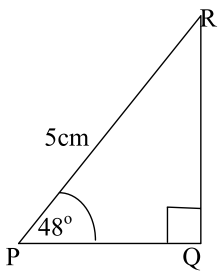
$\Rightarrow \theta + 60 = Cos^{-1}(0.0872)$

$\Rightarrow \theta + 60 = 85$

$\therefore \theta = 25^o$

Example 30.14

In the diagram, PQR is a right angled triangle. Angle RPQ is 48° and $|PR| = 5cm$, *find* $|\overrightarrow{PQ}|$.

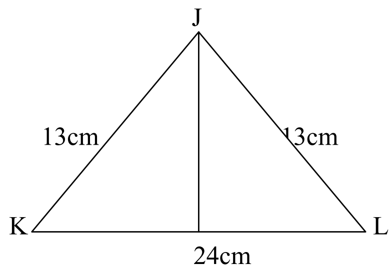


Solution

$Cos48^o = \dfrac{Adjacent}{Hypotenuse} = \dfrac{|PQ|}{5} \Rightarrow 5\cos 48^o = |PQ|$

$\therefore |PQ| = 3.35cm$

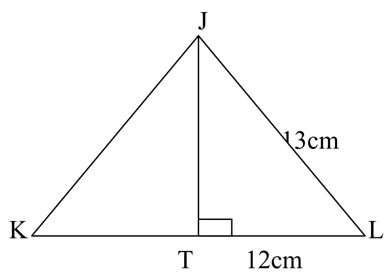
Example 30.15



In the diagram

$|JK| = |JL| = 13cm \ and \ |KL| = 24cm. \ Find \ \ SinJKL$

Solution



From triangle JTL, we have;

$|JL|^2 = |JT|^2 + |TL|^2 \quad But \ |TL| = \tfrac{1}{2}|KL| = \tfrac{1}{2} \times 24 = 12cm$

$13^2 = |JT|^2 + 12^2$

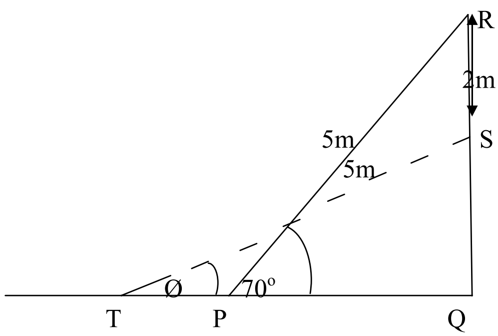
$\Rightarrow |JT| = 5cm$

Hence, $SinJKL = \dfrac{5}{13} \ (i.e. \ considering \ \Delta JKT)$

Example 30.16

A ladder 5m long leans against a vertical wall at an angle of 70° to the ground.
The ladder slips down the wall 2m. Find correct to two significant figures,
(i) the new angle which the ladder makes with the ground
(ii) the distance the ladder has slipped back on the ground from its original position.

Solution



$$Sin70^o = \frac{|QR|}{|PR|} = \frac{|QR|}{5} \Rightarrow |QR| = 4.7m$$

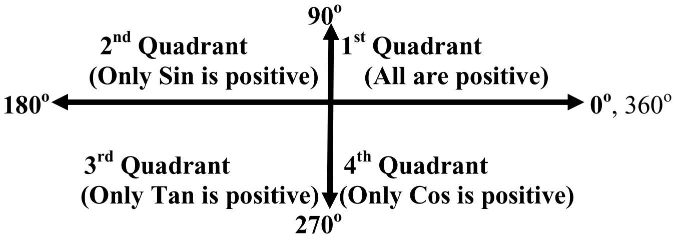
$$\Rightarrow |QS| = |QR| - |RS| = 4.7 - 2 = 2.7m$$

$$Hence,\ Sin\theta = \frac{|QS|}{|TS|} = \frac{2.7}{5} = 0.54 \Rightarrow \theta = 32.68^o$$

Hence, the new angle the ladder makes with the ground is 33° to 2 significant figures.

30.3 **The General Angle And Trigonometric Ratios**

Consider the diagram of the general angle below:



NB

In trigonometry, positive angles are measured in the anti clockwise direction starting from the East pole (the positive x – axis).
In the diagram above, the three trigonometric ratios of angles in the first quadrant (between 0˚ and 90˚) are positive. For that of the second quadrant (between 90˚ and 180˚) only sin is positive. Further, only the tan of angles in the third quadrant (between 180° and 270°) is positive whiles only the cos of angles in the fourth quadrant (between 270° and 360˚) are positive.
Thus, in the second quadrant (between 90˚ and 180˚) we have the following generalizations:

1. Sin (180 – x) = sin x (since only sin is positive)
2. Cos (180 – x) = -cos x
3. Tan (180 – x) = -tan x where x is between 90˚ and 180˚

Again, for the third quadrant (between 180˚ and 270˚)
we have the following generalizations:

1. Sin (x – 180) = -sin x
2. Cos (x – 180) = -cos x
3. Tan (x – 180) = tan x (since only tan is positive)

Further, in the fourth quadrant (between 270 ˚ and 360˚)

we have the generalizations:

1. Sin (360˚ - x) = -sin x

2. Cos (360˚ - x) = cos x (since only cos is positive)
3. Tan (360˚ - x) = -tan x

Example 30.17

If sin x = 0.8192, where 90˚ < x < 180˚, find x

Solution

Sin x = 0.8192
x = $\sin^{-1}$ (0.8192) = 55˚
Hence, from the table chart properties;
Sin (180 – x) = sin x (since 55˚ lies in first quadrant)
Implies, Sin (180 – 55) = sin 55
Sin 125 = Sin 55
Therefore, x = 125˚

Example 30.18

Given that $\tan(x + 25^o) = 5.145$, *where* $0 \le x \le 90$, *fin correct to one decimal place, the value of x.*

Solution

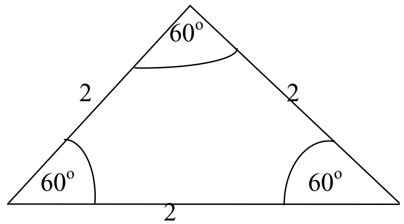
From $\tan(x + 25^o) = 5.145$
Take tan *inverse of both sides*;
$\Rightarrow x + 25^o = \tan^{-1}(5.145)$
$x + 25^o = 79^o \Rightarrow x = 79 - 25 = 54^o$

30.4 **Special Angles**

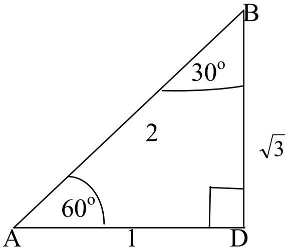
Here, we are going to illustrate how some special angles such as: 30˚, 45˚, 60˚ can be calculated from the trigonometric ratios without using the calculator straightaway.

1. for 30° and 60°

Consider the equilateral triangle below:



Dividing the triangle into two equal parts gives;



Using Pythagoras theorem;

$$|AB|^2 = |AD|^2 + |BD|^2$$

$$2^2 = 1^2 + |BD|^2 \Rightarrow |BD| = \sqrt{3}$$

Hence, Sin 30^o = $\frac{1}{2} = 0.5$

Cos 30^o = $\frac{\sqrt{3}}{2} = 0.866$

Tan 30^o = $\frac{1}{\sqrt{3}} = 0.5774$

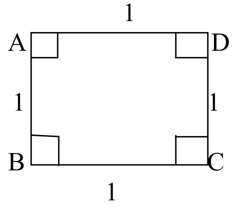
Sin 60^o = $\frac{\sqrt{3}}{2} = 0.866$

Cos 60^o = $\frac{1}{2} = 0.5$

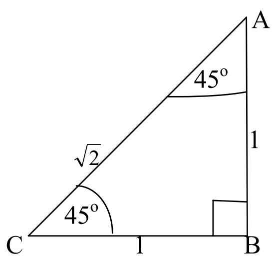
Tan 60^o = $\frac{\sqrt{3}}{1} = 0.732$

2. for 45^o and 90^o

Consider the square below:



Dividing the square diagonally into two gives;



By Pythagoras theorem;

$|AC|^2 = |AB|^2 + |BC|^2$

$|AC|^2 = 1^2 + 1^2 \Rightarrow |AC| = \sqrt{2}$

Hence, $Sin 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.7071(Rationalized)$

$Cos 45 = \frac{1}{\sqrt{2}} = 0.7071$

$Tan 45 = \frac{1}{1} = 1$

NB

Cos 90˚ = 0

Sin 90˚ = 1

Tan 90˚ = infinity (∞)

Cos 0˚ = 1

Sin 0˚ = 0
Tan 0˚ = 0

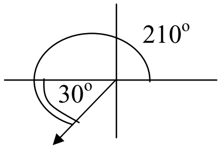
30.5 **Finding The Trigonometric Ratios Of Angles**

We follow the general steps for vector approach distance – bearing problems in chapter 33 to sketch the angle in the Cartesian plane and taking the trigonometric ratio of the angle it makes with the x – axis. We then apply the sign convention of respective quadrant to the result.

Example 30.19

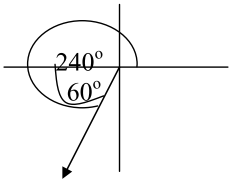
Find the value of the following angles:
(i) cos 210˚ (ii) tan 240˚ (iii) sin 150˚ (iv) tan 315˚

Solution
(i) cos 210˚



Cos 210˚ = -cos 30˚ = -0.866 [Since cosine is negative in 3rd quadrant]

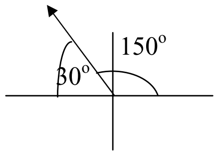
(ii) tan 240˚



Implies, Tan 240˚ = tan 60˚ = 1.732

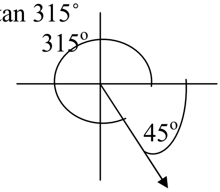
(Since tangent in third quadrant is positive)

(iii) sin 150˚



Implies, Sin 150˚ = sin 30˚ = 0.5
(Since in second quadrant, sine is positive)

(iv) tan 315˚



Tan 315˚ = -tan 45˚ = -1

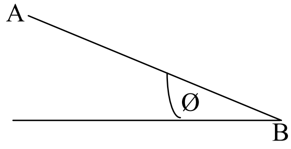
30.6 **Angles Of Elevation And Depression**

Let ***A*** be an object and ***B*** be the position of an observer, then the ***angle of elevation*** of the object A from B is the angle that the ***line AB*** makes with the horizontal.

NB

Angle of elevation is measured at the point below the other point

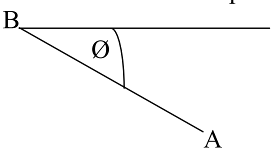
Illustration



Here, Ø is the angle of elevation of A from B.
When angle of elevation is measured in the vice versa, we call it ***angle of depression.*** Thus, the angle of depression of an object, say A from an observer at B is the angle that the line AB makes with the horizontal.

NB

Angle of depression is measured at the point above the other point.

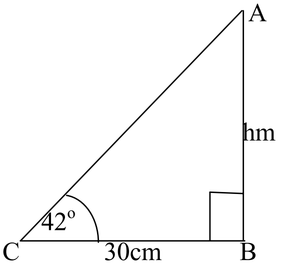


Here, angle of depression of A from B is Ø.

Example 30.20

The angle of elevation of the top of a tree from a point on the ground is 42°. If the point is 30m away from the foot of the tree, find the height of the tree.

Solution



Let height of tree be hm.

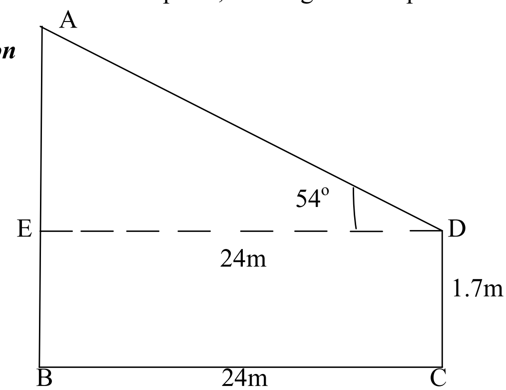
$$\Rightarrow \tan 42^o = \frac{h}{30} \Rightarrow h = 30\tan 42 = 30 \times 0.9004 = 27$$

Hence, the height of the tree is 27m.

Example 30.21

A vertical pole AB is erected on a level ground. A man 1.7m tall stands at C, 24m away from the foot, B of the pole. The angle of elevation of the top A of the pole from the man is 54°. Calculate, correct to one decimal place, the height of the pole.

Solution



From figure, $|AB| = |AE| + |EB|$ *i.e. height of pole*

Consider triangle ADE

$$\Rightarrow \tan 54^o = \frac{|AE|}{|DE|} = \frac{|AE|}{24} \Rightarrow |AE| = 33.724$$

But height of pole is

$|AB|$ *and* $|EB| = |CD| = 1.7m$

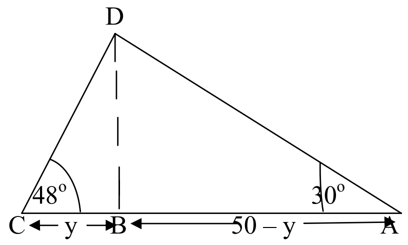
Where

$|AB| = |AE| + |EB| = 33.724 + 1.7 = 34.7$ *to* $1dp$

Example 30.22

Two points A and C, on opposite sides of a vertical pole, are on the same level ground as the foot of the pole, B. the angles of elevation of the top of the pole D from A and C are 30° and 48° respectively. If the distance between A and C is 50m, find BD, the height of the pole.

Solution



Let

$|BC| = y \Rightarrow |BA| = 50 - y$

Consider triangle CBD

$$\tan 48 = \frac{|BD|}{|CB|} = \frac{|BD|}{y} \Rightarrow y = \frac{|BD|}{1.111} \text{---------}(1)$$

Again, from triangle ABD,

$$\tan 30 = \frac{|BD|}{|AB|} = \frac{|BD|}{50-y} \Rightarrow y = 50 - \frac{|BD|}{0.5774} \text{---------}(2)$$

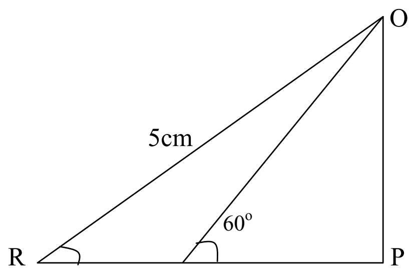
Equate equation (1) and (2)

$$\Rightarrow \frac{|BD|}{1.111} = 50 - \frac{|BD|}{0.5774} \Rightarrow |BD| = 18.99$$

Hence, the height of the pole BD is 19m to 1dp.

Example 30.23

In the diagram, $|OR| = 5m$, $< ORP = 45^o$, $< OQP = 60^o$ *and* $< OPR = 90^o$. Find the distance QP, leaving your answer in surd form.



NOT DRAWN TO SCALE

Solution

From triangle PRO,

$$\sin 45^o = \frac{|OP|}{|OR|} = \frac{|OP|}{5} \Rightarrow |OP| = 5\sin 45^o = \frac{5\sqrt{2}}{2}$$

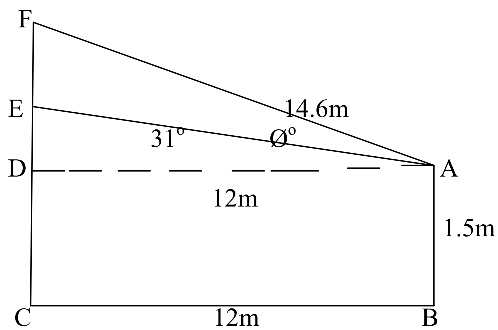
Now from ΔPQO,

$$\tan 60^o = \frac{|OP|}{|QP|} \Rightarrow |QP| = \frac{\frac{5\sqrt{2}}{2}}{\tan 60^o} = \frac{5\sqrt{2}}{2} \times \frac{1}{\sqrt{3}} = \frac{5\sqrt{6}}{6}\, m\,(ie.\ Rationalize\ \frac{1}{\sqrt{3}})$$

Example 30.24

A boy 1.5m tall is standing 12m away from a church building which has a tower on top of its roof. The top of the cross on the tower is 14.6m away from the boy's head
(eyes). If the boy has to raise his eyes through an angle of 31° in order to see the top of the roof, calculate
(a) Correct to the nearest degree, the angle through which the boy must raise his eyes to see the top of the cross on the tower;
(b) Correct to one decimal place, the height of the top of the cross from the ground
(c) Correct to one decimal place the height of the Church building.

Solution



(a) Let the required angle be Ø°

From triangle ADF,

$$Cos(31+\theta) = \frac{|AD|}{|AF|} = \frac{12}{14.6} = 0.8219$$

$$\Rightarrow 31+\theta = Cos^{-1}(0.8219) = 35$$

$$\Rightarrow \theta = 4^o$$

Therefore, the angle the boy must raise his eyes to see the top of the cross on the tower is 4°

(b) From triangle ADF

$$|AF|^2 = |AD|^2 + |DF|^2$$

$$14.6^2 = 12^2 + |DF|^2$$

$$213.16 = 144 + |DF|^2$$

$$\Rightarrow |DF| = 8.3m$$

But height, $|CF| = |CD| + |DF| = 1.5 + 8.3 = 9.8m$

(c) Height of church building is $|CE|$

Consider triangle ADE,

$$\Rightarrow Tan31^o = \frac{|DE|}{12} \Rightarrow |DE| = 12\tan 31^o = 7.2m$$

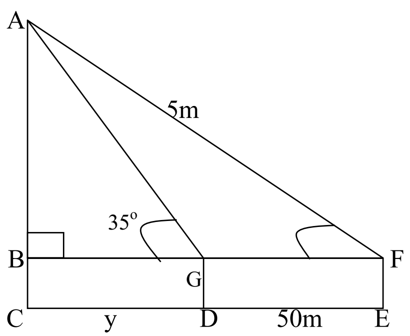
But height

$|CE| = |CD| + |DE| = 1.5 + 1.7 = 8.7m$

Example 30.25

A man 1.7m tall observes the angle of elevation of the tip of a tower to be 35°. He mobes 50m away from the tower and now observes the angle of elevation to be 28°. How far above the ground is the tip of the tower to **three** significant figures.

Solution



Considering ΔABG

$$\Rightarrow \tan 35^o = \frac{|AB|}{|BG|} = \frac{|AB|}{y} \Rightarrow y\tan 35^o = |AB| ------(1)$$

Now, consider ΔABF

$$\Rightarrow \tan 28^o = \frac{|AB|}{|CE|} = \frac{|AB|}{y+50} \Rightarrow (y+50)\tan 28^o = |AB| ---(2)$$

Equate (1) *and* (2)

$$\Rightarrow y\tan 35^o = (y+50)\tan 28^o \Rightarrow \frac{y\tan 35^o}{\tan 28^o} = y+50$$

$$1.3169y = y + 50 \Rightarrow 1.3169y - y = 50$$

$$\frac{0.3169y}{0.3169} = \frac{50}{0.1369} \Rightarrow y = 157.8m$$

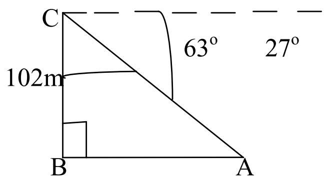
$|AB| = 157.8\tan 35^o = 110.49m$

$\therefore \textit{Tip of the tower from the ground} = |AB| + |BC| = 110.49 + 1.7 = 112m$

Example 30.26

A cliff is 102m high. From the top of the cliff the angle of depression of a boat is 27°. Find the distance of the boat from the foot of the cliff.

Solution



From triangle ABC,

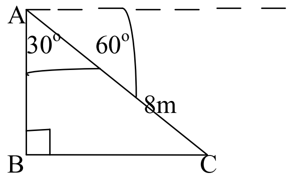
$$\tan 63^o = \frac{|AB|}{BC} = \frac{|AB|}{102} \Rightarrow |AB| = 200m$$

Hence, the distance of the boat from the foot of the cliff is 200m.

Example 30.27

A ladder 8m long leans against a vertical wall.
The angle of depression from the top of the ladder is 60°.
How far is the foot of the ladder from the wall?

Solution

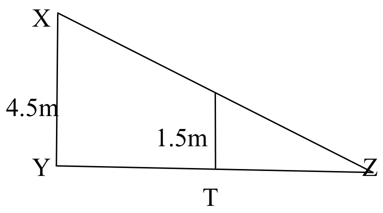


From triangle ABC,

$$Sin30^o = \frac{|BC|}{|AC|} = \frac{|BC|}{8} \Rightarrow |BC| = 4m$$

Hence, the foot of the ladder is 4m from the wall

Example 30.28



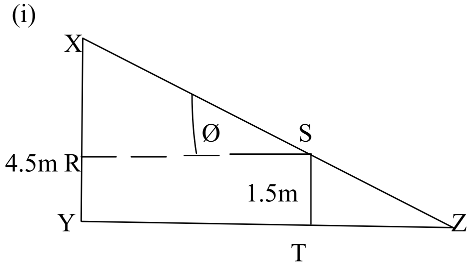
In the diagram above, XY represents a street lamp. ST represents a boy standing 6m away from the street lamp. TZ is the shadow of the boy cast by the street lamp. The height of the boy is 1.5m and the street lamp is 4.5m high.

(i) Find $|TZ|$

(ii) Find the angle of elevation of the top of the street lamp from the boy, correct to the nearest degree.

Solution

(i)



By similarities;

$$\frac{|XZ|}{|ST|} = \frac{|YZ|}{|TZ|} \ (\Delta YXZ \ is \ an \ enlargement \ of \ \Delta TSZ)$$

$$\Rightarrow \frac{4.5}{1.5} = \frac{6+|TZ|}{|TZ|}$$

$$\Rightarrow |TZ| = 3m$$

(ii) Let the angle of elevation be Ø°

From triangle XRS,

$$\tan\theta = \frac{|XR|}{|RS|} = \frac{4.5-1.5}{6} = \frac{3}{6} = 0.5$$

$$\Rightarrow \theta = 26.57^o$$

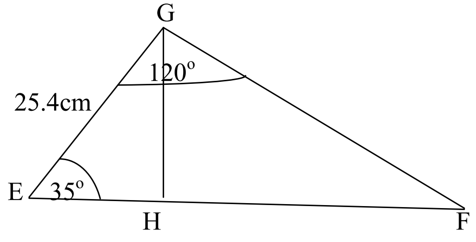
Hence, angle of elevation is to 27° 1dp

EXERCISE

QUE. A

If $Sin\theta = \frac{3}{5}$, find the value of tan Ø.

QUE. B



In the figure, angle GEF = 35˚, angle EGF = 120˚, $|EG| = 25.4cm$ and GH is perpendicular to EF. Calculate: $|EF|$ correct to three significant figures.

QUE. C

If $Sinx = \frac{1}{q}$, which of the following is equal to tan x?

QUE. D

If sin x $^\circ$ = 0 6239 and cos y$^\circ$ = 0.6239, find (x + y)

QUE. E

If $\tan x = \frac{15}{8}$, and x is acute, find the value of sin x.

QUE. F

If $Sinx = \frac{8}{17}$, find the value of $\frac{\tan x}{1 + 2\tan x}$

QUE. G

Given that 5sin x = 4.33 where $0^o \leq x \leq 90^o$, find x correct to the nearest degree.

QUE. H

If $Sin\theta = \frac{3}{5}$, find the value of 2SinØ + 3SinØ

QUE. I

If Sin5x = Cos20^o, find the value of x.

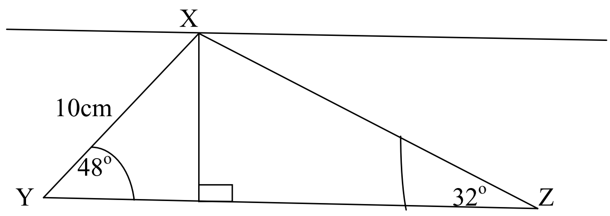
QUE. J

If tan Ø = 1, evaluate sin Ø + cos Ø, leaving your answer in surd form.

QUE. K

A fishing traveler travels 200km on a bearing of 077° and then 150^o on a bearing of 167^o. Find how far the traveler is from the starting point?

QUE. L



In the diagram XYZ is a triangle. If <XYZ = 48˚, <YZX = 32˚ and XY = 10cm,
which of the following expressions equal to the magnitude of XZ?

QUE. M

A man standing 40m from a vertical pole observes that the angle of elevation of the top on the pole is 18˚. Assuming that his eye is 1.8m above the base of the pole, calculate correct to the nearest meter the height of the pole.

QUE. N

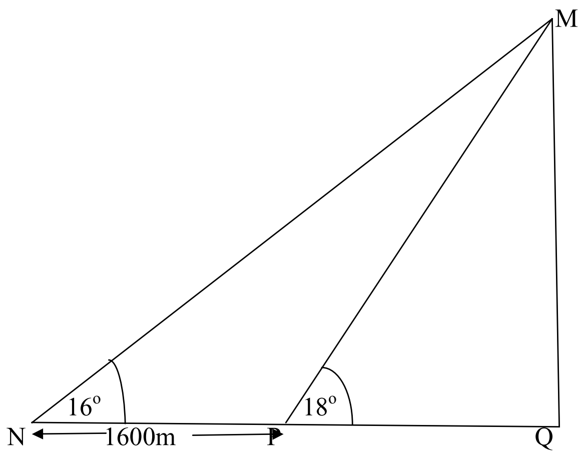
The point M is on the same level as the foot of a vertical pole. The angle of elevation of the top of the pole from M is 55˚. If M is 10m from the foot of the pole, calculate correct to the nearest metre, the height of the pole.

QUE. O

Two given points S and T which are at opposite sides of a tower are 20m apart. S, T and the foot of the tower are in the same straight line. The angle of elevation of the top of the tower from S and T are 32˚ and 45˚ respectively. Find correct to one decimal place, the height of the tower.

QUE. P

A surveyor at sea level observed that the angle of elevation of the top of a mountain, M, from two points N and P due west of it are 16° and 18° respectively, as shown in the diagram. If NP = 1600m and the base of the mountain, Q is vertically below M, calculate the height of the mountain. Let G be the length of P to M.



QUE. Q

If $\sin 5x = \cos 20^{o}$, *find the value of* x.

QUE. R

If $\sin\theta = \frac{3}{5}$, *find the value of* $2\sin\theta + 3\cos\theta$

QUE. S

If $\tan\theta = 1$, *Evaluate* $\sin\theta + \cos\theta$, *leaving your answer in surd form*.

QUE. T

A ladder 8.5metres long leans against a vertical wall. The top (T) of the ladder makes an angle of 58^{o} with the wall. How far, correct to one decimal place, is the foot (F) of the ladder from the wall?

QUE. U

In a right-angled triangle, one of the acute angles is 30^{o}. if the length of the side adjacent to this angle is 10.05cm, find the length of the corresponding opposite side.

QUE. V

The foot of a ladder is 2.5m from a vertical wall. If the ladder makes an angle of 16^{o} with the wall, find how high up the wall it reaches.

QUE. W

The side road to a house on top of a hill off the main street is inclined at an angle of 12.5^{o} to the horizontal. If the top of the hill is 18m above the level of the main street, how far along the side road is the house from the street?

CHAPTER 31

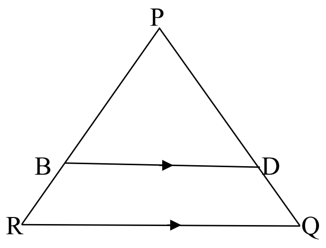
ENLARGEMENT, REDUCTION AND SIMILARITY

31.1 **Enlargement And Reduction**

Enlarging a plane figure increases the size of the figure but does not alter the shape of the figure.
Likewise, reducing a plane figure decreases its size but does not change the shape.
We enlarge or reduce plane figures by a certain scale factor, say ***k*** calculated from the corresponding lengths or sides of the figure. Thus, we find the ratio of the corresponding sides of the enlarged figure to that of the original figure. We equate the results to find one side unknown as illustrated in the examples below.

Example 31.1

In the diagram below, PQR is an enlargement of PBD. If $|PD| = 12cm \text{ and } |PQ| = 36cm,$ find the scale factor of the enlargement.



Solution

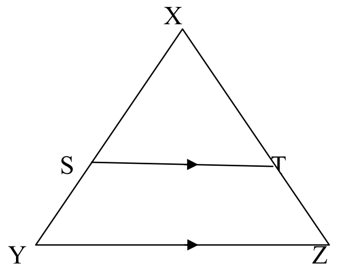
The corresponding sides are $|PD| \text{ and } |PQ|$. Likewise, $|RQ| \text{ and } |BD|$ are also corresponding sides.
Hence,

Scale factor $= \frac{|PQ|}{|PD|} = \frac{36}{12} = 3$

Therefore, Scale factor = 3

Example 31.2

In the figure below triangle XYZ is an enlargement of triangle XTS. Given that $|XS| = 30cm \ \ and \ \ |SY| = 45cm,$ what is the scale factor for the enlargement?



Solution

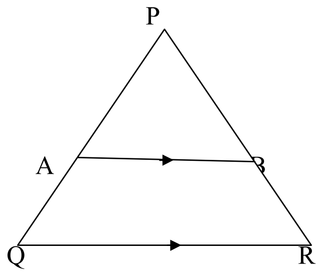
Corresponding sides are $|XY| \ and \ |XS|$

But $|XY| = |XS| + |SY| = 30 + 45 = 75cm$

Implies, Scale factor $= \dfrac{|XY|}{|XS|} = \dfrac{75}{30} = 2.5$

Example 31.3

If triangle PQR is the enlargement of triangle PAB where $|PB| = 10cm, |PR| = 30cm \ \ and \ \ |AB| = 18cm,$ find the length of the side $|QR|$



Solution

Form diagram;

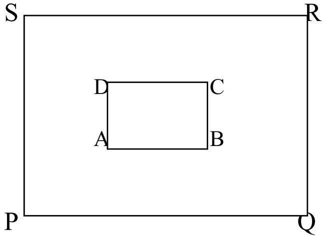
Scale factor $=\frac{|PR|}{|PB|}=\frac{|QR|}{|AB|}$

$\Rightarrow\ \frac{|PR|}{|PB|}=\frac{|QR|}{|AB|}$

$\frac{30}{10}=\frac{|QR|}{18}\Rightarrow|QR|=54cm$

Example 31.4

In the diagram below, square PQRS is an enlargement of square ABCD where $|PQ|=8cm\ and\ |AB|=4cm.$ Find the scale factor and the area of square PQRS.



Solution

Scale factor $= \frac{|PQ|}{|AB|} = \frac{8}{4} = 2$

Since ABCD is a square, it implies

That; $|AB| = |BC| = |CD| = |AD|$

And

Since PQRS is also a square, it means

$|PQ| = |QR| = |RS| = |PS|$

But Area = Length x height

Length $= |PQ| = 8cm \ \ and \ \ height = |QR| = 8cm$ Implies,

Area = 8cm x 8cm = 64cm^2

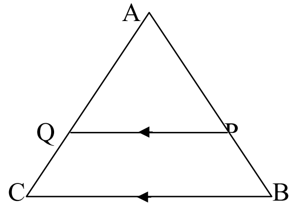
Example 31.5

In triangle ABC, $|AB| = 5cm, \ |BC| = 8cm, \ |AC| = 6cm$. P is a point on AB such that $|AP| = 2cm$. The line through P parallel to BC meets AC at Q, calculate

$(i) \ |PQ| \quad (ii) \ |QC|$

Solution

(i) Given $|AB| = 5cm, \ |BC| = 8cm, \ |AC| = 6cm \ \ and \ \ |AP| = 2cm$.



Form figure, ABC is an enlargement of triangle APQ.

Hence, by enlargement;

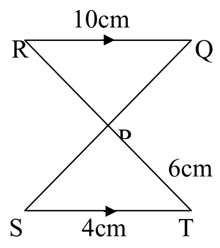
$$\frac{|BC|}{|PQ|} = \frac{|AB|}{|AP|} \Rightarrow \frac{8}{|PQ|} = \frac{5}{2} \Rightarrow |PQ| = 3.2cm$$

(ii) Similarly;

$$\frac{|AC|}{|AQ|} = \frac{|AB|}{|AP|} \Rightarrow \frac{6}{|AQ|} = \frac{5}{2} \Rightarrow |AQ| = 2.4cm$$

But $|QC| = |AC| - |AQ| = 6 - 2.4 = 3.6cm$

Example 31.6



In the diagram above, PQ is parallel to ST, $|ST| = 4cm, |RQ| = 10cm \ \ and \ \ |PT| = 6cm$.

(i) Find $|RP|$

(ii) If the area of triangle PRQ is 90cm^2, find the area of triangle PST.

Solution

QUE. 11

From diagram, PRQ is an enlargement of SPT.

Implies, RQ corresponds to ST and RP corresponds to PT

Hence, Scale factor $= \frac{|RQ|}{|ST|} = \frac{|RP|}{|PT|}$

$$\Rightarrow \frac{|RQ|}{|ST|} = \frac{|RP|}{|PT|} \Rightarrow \frac{10}{4} = \frac{|RP|}{6} \Rightarrow |RP| = 15cm$$

QUE. 12

NB

Generally, under every enlargement, the relationships between the area of the enlarged figure and original figure are given by:

[Area of enlarged figure RQP] =

[Area of original figure PST] x [scale factor]^2

[Volume of enlarged figure] =

[volume of original figure] x [scale factor]^3

[Length of enlarged figure] =

[Length of original figure] x [scale factor]

Hence, here;

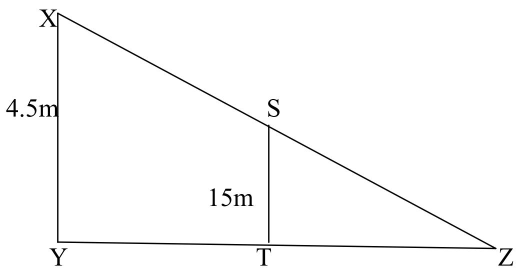
Area of enlarged figure RPQ = 90cm^2

Scale factor $= \frac{|RQ|}{|ST|} = \frac{10}{4} = 2.5$

Hence, from relation;

$$90 = area\ of\ \ PST \times 2.5^2 \ \Rightarrow Area\ of\ \ PST = \frac{90}{6.25} = 14.44cm^2$$

Example 31.7



In the diagram above, XY represents a street lamp. ST represents a boy standing 6m away from the street lamp. TZ is the shadow of the boy cast by the street lamp. The height of the boy is 1.5m and the street lamp is 4.5m high. Find $|TZ|$

Solution

By similarities;

$$Scale\ \ factor = \frac{|XZ|}{|ST|} = \frac{|YZ|}{|TZ|}\ (\Delta YXZ\ is\ an\ enlargement\ of\ \Delta TSZ)$$

$$\Rightarrow \frac{4.5}{1.5} = \frac{6+|TZ|}{|TZ|}$$

$$\Rightarrow |TZ| = 3m$$

31.2 **Similarity**

We say that two plane figures are ***similar*** if they have the same shape. E.g. A figure and its corresponding enlargement are said to be similar.

NB

Two similar figures have equal corresponding angles and the ratio of the lengths of their corresponding sides are also equal. For two figures P and Q, we have;

$$\frac{Area\ of\ P}{Area\ of\ Q} = \left[\frac{length\ of\ side\ of\ P}{length\ of\ side\ of\ Q}\right]^2$$

And

$$\frac{Volume\ of\ P}{Volume\ of\ Q} = \left[\frac{length\ of\ side\ of\ P}{length\ of\ side\ of\ Q}\right]^3$$

Example 31.8

Circle P is an enlargement of circle Q of Scale factor 3. if the area of circle Q is 126cm^2 find the area of circle P and state its radius.

(Take $\pi = \frac{22}{7}$)

Solution

Area of circle = $\pi r^2\ \ where\ r = radius$

Implies, area of circle Q $= \pi r^2 \Rightarrow 126 = \frac{22}{7} r^2 \Rightarrow r = 6.324cm$

From, [Area of enlarged figure RQP] = [Area of original figure PST] x [scale factor]^2

i.e. area of P = [area of Q] x [scale factor]^2

= 126 x 3^2 = 126 x 9 = 1134cm^2

Example 31.9

Triangle ABC is an enlargement of that of PQR with scale factor –2. If the area of triangle ABC is 60cm^2, find the area of triangle PQR.

Solution

Area of ABC = [Area of PQR] x [scale factor]^2

Implies, 60 = [Area of PQR] x $(-2)^2$

60 = Area of PQR x 4

$\Rightarrow 60 = [Area\ of\ PQR] \times 4 \ \Rightarrow Area\ of\ PQR = 15cm$

Example 31.10

A cone, P of height 30cm has a volume of 320cm^3. Find the volume of a cone Q of height 90cm.

Solution

Using

$$\frac{Volume\ of\ P}{Volume\ of\ Q} = \left[\frac{length\ of\ one\ side\ of\ P}{length\ of\ corresponding\ side\ of\ Q}\right]^3$$

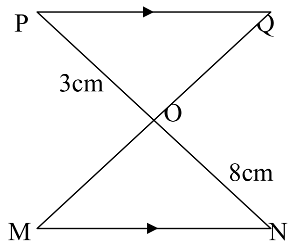
$$\frac{320}{Volume\ of\ Q} = \left[\frac{30}{90}\right]^3 \Rightarrow Volume\ of\ Q = 2880cm^3$$

NB

We say that two triangle are similar if they have three pairs of corresponding angles equal and also the three ratios of corresponding sides are equal. ***Similar triangles*** are illustrated in Examples: 31 .5, 31.6 and 31.7

EXERCISE

QUE. A



In the diagram, PQ is parallel to MN,
$|MN| = 7cm, |NO| = 8cm \ and \ |PO| = 3cm. \ If \ MOP \ and \ PON$ are straight lines, find $|PQ|$

QUE. B

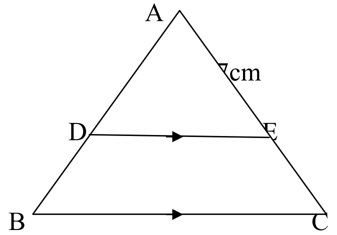
In the figure below, PQRS is an enlargement of ABCD such that the area of PQRS IS $12cm^2$ and that of ABCD is $3cm^2$.
Find the scale factor of the enlargement

QUE. C

Triangle ABC is an enlargement of that of PQR of scale factor 2. Assume the area triangle ABC is $64cm^2$, find the area of triangle PQR.

QUE. D

In the diagram, DE is parallel to BC.
$|DB| = 4cm,\ |AE| = 7cm,\ |EC| = 5cm.\ Find\ |AD|$



QUE. E

Two given cylinders P and Q both have volumes 9 45cm^3 and 280cm^3 respectively. The radius of Q is 6.2cm.
(i) Express the radius of Q as a fraction of that of P
(ii) Find the radius of P
(iii) Determine, correct to two decanal places, the height of P.
[take $\pi = 3.142$]

CHAPTER 32

CONSTRUCTION OF PLANE FIGURES AND THE LOCI

Here, we are required to use ***only*** the pair of compasses and a straight rule.

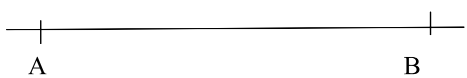
32.1 **Basic Constructions**

32.1.1 **To Construct A Straight Line Of A Given Length**

Steps

1. Using a sharp pencil and a ruler, draw a line of length more than the given required length
2. Using a ruler and the pencil, cut the line towards the left end and label that point A.
3. With your pair of compasses, measure the given length on your ruler
4. With the point of the compasses at A, draw an arc to cut the line towards the right end
5. Label the intersection of the arc with the line as B

Illustration



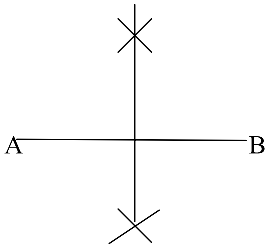
32.1.2 **The Bisection Of A Straight Line**

Bisection means dividing into ***two equal*** parts. The straight line that divides a straight line into two equal parts is called a ***perpendicular bisector*** or a ***mediator.***

Steps

1. Using a straight rule, draw a line with the given dimension and mark the ends of the line A and B
2. With the pair of compasses, step at A and open to any radius more than half of line AB and draw two arcs above and below the line. Similarly, step at B with the same radii and draw two arcs to intersect the first two arcs
3. Join the points of intersection with a straight line

Illustration

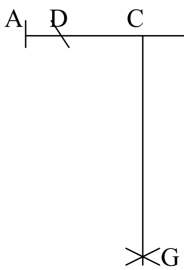


32.1.3 **Constructing A Perpendicular At A Point On A Straight Line**

Steps

1. Draw the given line says AB using a rule
2. Mark the given point on the line say C
3. Step at C, with the pair of compasses to any radius and draw two arcs, one on the
Left and the other on the right of point C and name the arcs D and E.
4. Step at D with any radius more than half of DE and draw an arc either above or below line AB. With the same radius step at E and draw a similar are to intersect each other at say G.
5. Join G, the point of intersection of the arcs to C on the line AB.

Illustration

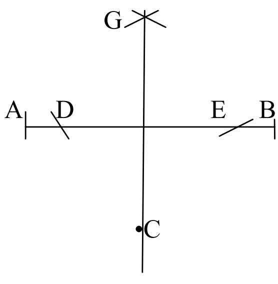


32.1.4 **Constructing A Perpendicular To A Given Line From A Given Point Outside The Given Line**

Steps

1. Draw the given line, say AB
2. Mark the given point say C outside line AB
3. Step at C and draw two arcs on the line AB and name the arcs D and E.
4. Step at D and E with the same radius and draw two arcs to intersect each other on the opposite side of point C.
5. Join this intersection to point C with a straight rule.

Illustration

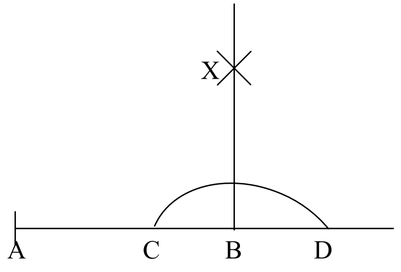


32.1.5 **Constructing A Perpendicular At The End Of A Given Line**

Steps

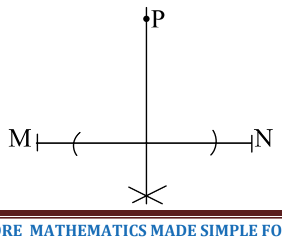
1. Draw the given line, say AB.
2. Extend the line at the given end point.
3. Step at the given end, say B with any radius and draw a semi-circle above the
   Line AB to touch AB at C and D
4. With a radius of more than half the distance of CD, draw two arcs stepping at C
   and D to intersect each other at X.
5. Join point X to B to form the perpendicular.

Illustration



Example 32.1

Which of the following describes accurately the construction below?



Solution

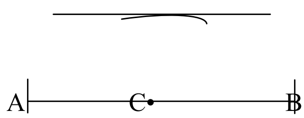
The perpendicular through P to MN

32.1.6 **Constructing A Line Parallel To A Given Line At A Given Point Of A Given Distance**

Steps

1. Draw the given line say AB
2. Mark a point say C, on the line AB
3. Step at C with the compasses to any radius and draw an arc above or below the line AB
4. Drag the two set squares from line AB to the top of the arc and draw a straight line parallel to AB

Illustration



32.2 **Construction Of Angles**

32.2.1 **The Bisection Of Angles**

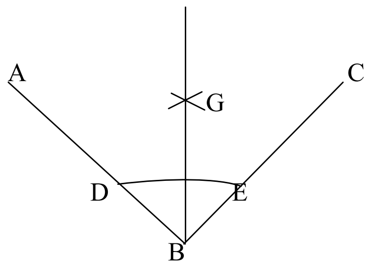
When an angle is bisected, we say it has been divided into two equal parts.
Thus, bisecting angle 120˚ will given two angles each of 60˚.

Steps

1. Draw the angle $A\hat{B}C$. i.e. Angle is at B.
2. Step at B and open the compasses to any radius and draw an arc to cut AB and BC at D and E.
3. Step at D with a radius more than half of DE and draw an arc and with the same radius step at E and draw a similar arc to intersect each other at G

4. Join the point of intersection to point B.

Illustration



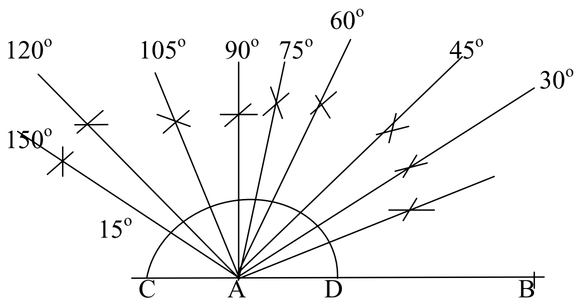
32.2.2 **General Angle Constructions**

Constructing 15˚, 30˚, 45˚, 60˚, 75˚, 90˚, 105˚, 120˚, 150˚

Steps

1. Draw line AB and extend line at A
2. Draw a semi-circle above A to touch AB at C and D
3. Step at D with the same radius of semi-circle and draw an arc on the semi-circle and
name it E. Step E and draw another arc with the same radius on the semi-circle away from E and name it F.
4. Join A through E to obtain 60˚ and A through F to obtain 120˚
5. Bisect angle 60˚ to obtain 30˚ and angle 30˚ to obtain 15˚.
6. Again bisect angle between 30˚ and 60˚ to obtain 45˚ and angle between 60˚ and
120˚ to obtain 90˚.
7. Bisect angle between 60˚ and 90˚ to get angle 75˚ and angle between 90˚ and 120˚ to get 105˚.

Illustration

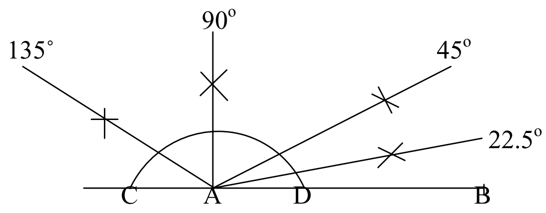


Constructing 90˚, 45˚, 135˚, 22.5˚

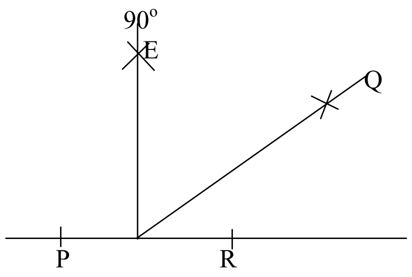
Steps

1. Draw line AB and extend line from A.
2. Step at A and open to any radius and draw a semi-circle above A to touch AB at C and D
3. Step at C and open to more than half of CD and draw an arc above the semi-circle. Step at D with the same radius and draw a similar arc to intersect each other at E.
4. Join E to A to obtain 90˚.
5. Bisect the 90˚ angle to obtain 45˚
6. Similarly, bisect 45˚ to obtain 22.5˚
7. Bisect the angle between 90˚ and the horizontal line to obtain 135˚.

Illustration



Example 32.2

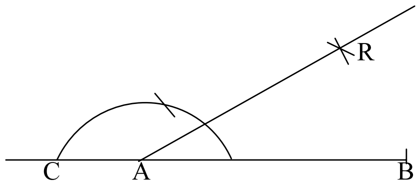


The construction above is required in constructing angle POQ. What is the size of angle POQ?

Solution

Angle POQ = $90^\circ + 45^\circ = 135^\circ$
(Since Q is a bisection of 90°)

Example 32.3



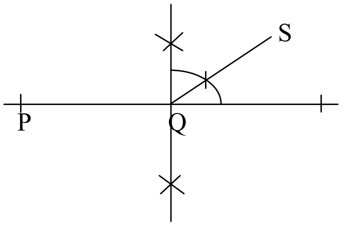
What is the size of the angle XYR in the diagram above?

Solution

Angle XYR = 30˚ (since R is a bisection of 60˚).

Example 32.4

What is <PQS in the construction below?



Solution

Angle PQS = 90˚ + 30˚ = 120˚ (since S marks 60˚ form horizontal line)

32.3 **Construction Of Triangles**

In the construction of triangles, a question must contain one of the following values:
1. All the three sides of the triangle
2. Two of its sides and an angle
3. One side and two angles.

Example 32.5

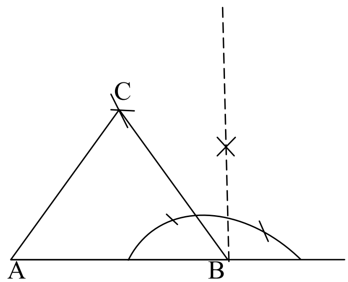
(a) Using a smaller and a pair of compasses only, construct triangle ABC in which

$|AB| = 8cm, |BC| = 9cm \ and \ angle \ ABC = 75^o$

(b) Measure angle ABC

Solution

(a)



(b) Angle BAC = 47 $^\circ$

Steps

1. Draw line AB = 8cm
2. Draw a semi-circle above line AB to touch AB.
3. Step at one of the intersections and draw two arcs on the semi-circle to get 60°
  and 120° respectively.
4. Bisect the angle between 60° and 120^o arcs to obtain 90° and join it to B with a dotted line.

NB

We use dotted lines because 90˚ is not stated any where in the question)

5. To get 75˚, bisect the angle between the 60˚ and 90˚ angles.
6. Join the 75˚ to B with a straight line
7. Measure 9cm on a ruler using the compasses and stepping at B, mark on 75˚ line to get C.
8. Place the protractor on line AB and measure angle BAC.

NB

In constructions, we do not construct ***two sides*** or ***two angles*** consecutively. Instead, after first constructing a side, the second thing we construct is an angle. Thus, we move from a side to construct an angle and never two sides before an angle.

Example 32.6

(a) Using a ruler and a pair of compasses only, construct.

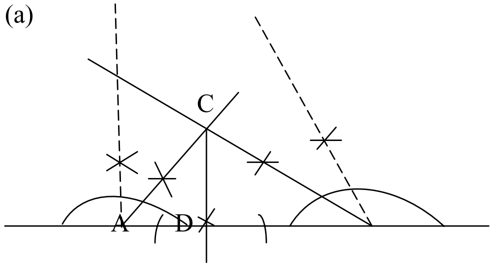
(i)Triangle ABC with $|AB| = 10cm,\ angle\ ABC = 30^o\ and\ angle\ BAC = 45^o$

(ii) A perpendicular from C to meet AB at D.

(b) Measure $|CD|$

Solution

(a)



(b) $|CD| = 3.7cm$

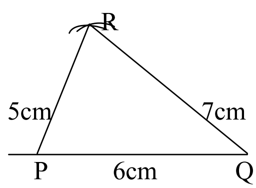
Steps

1. Draw line AB = 10cm
2. At B construct angle 60˚ and bisect it to get 30˚.
3. Join a straight line from B through the 30˚ arc.
4. Construct 90˚ at A and bisect it to set 45˚ arc to meet the line from B at C.
6. Step at C and draw two arcs on AB
7. Step at the arcs draw in step 6 and draw two arcs to intersect
8. Join the intersection to C and name the point on AB where the line touches as D.

Example 32.7

Construct a triangle PQR such that
$|PQ| = 6cm, |QR| = 7cm \ \ and \ |PR| = 5cm$

Solution



Steps

1. Draw line $|PQ| = 6cm$
2. Measure 7cm with the compasses from the ruler and step at Q mark an arc above
3. Measure 5cm from the ruler using the pair of compasses and step at P and mark an arc to intersect that draw in step 2.
4. Join point of intersection to both Q and P.

Example 32.8

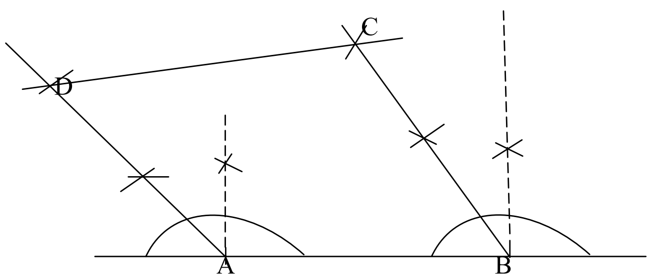
(i) Using a smaller and a pair of compasses only, construct quadrilateral ABCD with
$|AB| = 9cm, |BC| = 10cm, |AD| = 7.5cm, < ABC = 45^o \text{ and } < BAD = 135^o$

(ii) What type of quadrilateral is ABCD?

Solution

Steps

1. Draw $|AB| = 9cm$ and 90˚ at B and then bisect to get 45˚ and draw a line through to C.
2. Draw 90˚ at A and bisect the second 90^o to get 135˚. Measure 10cm on a ruler using the pair of compasses and step at B and mark C.
3. Similarly, measure 7.5cm on the ruler with the compasses and stepping at A, mark D on the 135^o line.



(ii) ABCD is a Rhombus

Example 32.9

(a) Using a ruler and a pair of compasses only,
   (i) construct
   $\Delta PQR \text{ such that } |PQ| = 6.6cm, |QR| = 8.0cm, < PQR = 60^o$

(ii) locate by construction, a point S on PQ such that $|PS| = |SQ|$

(iii) construct ST parallel to QR such that STRQ is parallelogram

(b) Measure

(i) $|PR|$

(ii) $|TR|$

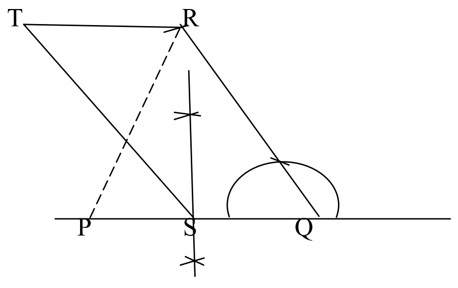
(iii) <STR

Solution

Steps

1. Draw line $|PQ| = 6.6cm$
2. Construct 60^o at Q and draw a straight line through it from Q
3. Measure 8cm with the compasses and step at Q and mark on the 60^o line.
4. Bisect the line $|PQ|$ *and mark S on the* $|PQ|$ *where the bisector meets* $|PQ|$.
5. Place the set square on line QR and support its bottom with a ruler, move along line QR to S and draw a straight line from S parallel to QR
6. Similarly, place the set square on PQ and support its bottom with a ruler and then move along to R and draw a straight line parallel to PQ and mark its intersection with line throught S as T.

(a)



(b) $(i)\ |PR| = 7.2cm \quad (ii)\ |TR| = 3.2cm \quad (iii) < STR = 60^o$

NB

The dotted line is only to show the measurement PR.

32.4 **Types Of Loci**

32.4.1 **Finding The Locus Of A Point Equidistant From A Fixed Point**

Here, the locus is usually a circle with the fixed point as center and the given distance as its radius.

Example 32.10

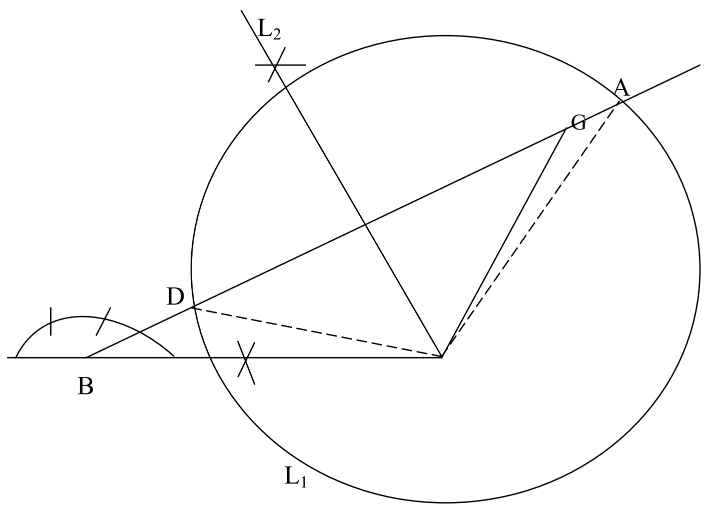
(a) Using a ruler and a pair of compasses only, construct
(i) ΔGBC with angle GBC = 30°, $|BC| = 9.5cm \ and \ |BG| = 12cm$
(ii) L_1, the locus of points 6cm from C
(iii) L_2, the perpendicular from C to BG
(b)(i) Locate A and D, the intersections of L_1 and BG
(ii) Measure $|AD| \ and < ACD$
(iii) Calculate correct to two significant figure the area of minor sector ACD.
(Take $\pi = 3.142$)

Solution

Steps

1. Draw line $|BC| = 9cm$ and construct 60° at B to be bisected to get 30°
2. Measure 12cm and mark G from B and join G to C
3. Measure 6cm and step at C and construct a circle for L_1
4. (i) to draw L_2, step at C with the compasses and mark two arcs on line BG of the same radius.

(ii)Stepping at the two arcs drawn in step 4, draw two arcs above BG to intersect each other
(iii) Join the intersect to C to obtain L_2
5. Mark A and D on BG where the circle L_1 touches the line BG



(b) (ii) $|AD| = 6.7cm \ \ and \ \ < ACD = 70^o$

(iii) Area of sector ACD $= \frac{70}{360} \times 3.142 \times 6^2 = 22cm$

32.4.2 **Finding The Locus Of A Point Equidistant From Two Fixed Points**

This locus is a perpendicular bisector of the two fixed given points. i.e. we bisect the line that joins the two fixed points.

Example 32.11

(a) Using a ruler and a pair of compasses only, construct

(i) Triangle ABC with $|AB| = 8cm, |BC| = 8cm\ and\ angle\ ABC = 90^o$

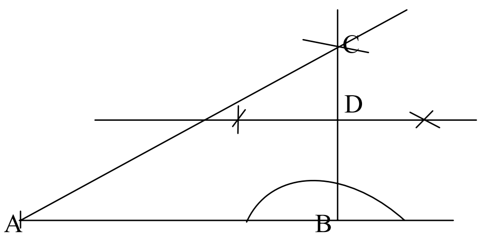
(ii) A point D on AC, which is equidistant from B and C.

(iii) Measure $|BD|$

Solution

Steps

1. Draw line $|AB| = 6cm$ and construct 90˚ at B
2. Draw a line from B through the 90˚ and measure and mark 8cm on it
and that becomes C
3. Join C to A to get a triangle
4. Stepping at B and C with same radius more than half of BC, draw arcs to intersect
5. Join the two intersections with a straight line
6. Mark points of intersection of line drawn in step 5 and line AC as D



(iii) $|AB| = 5cm$

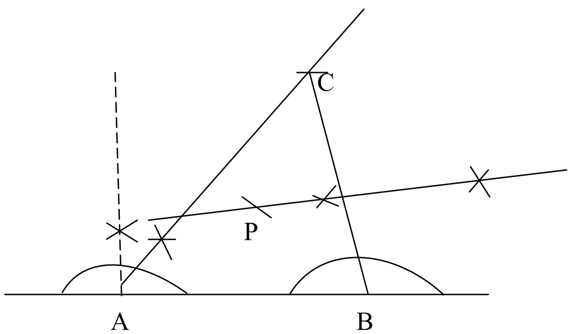
Example 32.12

Using a ruler and a pair of compasses only, construct triangle ABC in which angle A = 45°, $|AB| = 7cm\ and\ |AC| = 6cm$. Locate a point P, inside the triangle ABC, 5cm from A and equidistant from B and C. Find $|PB|$

Solution

Steps

1. Draw $|AB| = 7cm$ and construct 45° at A
2. Measure 9cm and stepping at A mark C
3. Join C to A and to B
4. Bisect B and C
5. Measure 5cm and step at A and mark P on the bisector of BC drawn in step



$|PB| = 4.7cm$

Example 32.13

(a) Using a ruler and a pair of compasses only, construct the quadrilateral ABCD such that

$|AB| = 7cm, |BC| = 5cm, angle\ ABC = 120^o\ \ and\ |AD| = |AC| = |DC|$

(b) Construct the locus

(i) L_1, of points equidistant from B and C

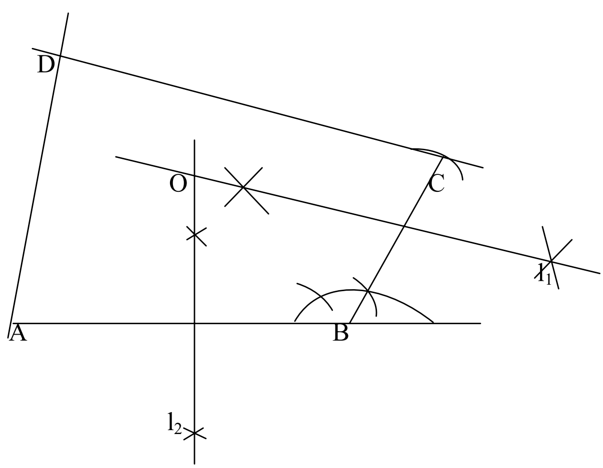
(ii) L_2, of points equidistant B and A

(c) (i) Locate, O, the point of intersection of L_1, and L_2

(ii) With O as center construct a circle to pass through A, B and C
(d) Measure
(i) Angle BCD
(ii) The radius of the circle

Solution

Steps
1. Draw $|AB| = 7cm \ \ and \ \ < ABC = 120^o$
2. Measure 5cm and mark C from B
3. Step at A and open your compasses to C and draw an arc and with that same radius mark another arc to intersect at D
4. Join D, the point of intersection to C and A.
5. Bisect BC and name it l_1, and AB and name it l_2 to meet at O.
6. Step at O and Draw a circle to touch A, B and C.



(d) (i) <BCD = 95˚ and (ii) Radius = 6.3cm

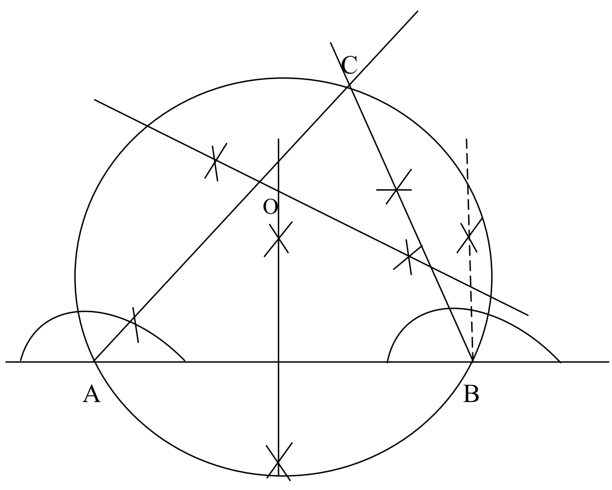
32.4.3 **Finding The Locus Of A Point Equidistant From Three Fixed Points**

The locus of a point equidistant from three fixed points is a circle drawn to touch all the three points (or vertices).
To get this circle, we bisect any two of the sides to meet each other at a point called the ***center*** of the circle to be drawn.

Example 32.14

(a) Using a ruler and a pair of compasses only (i) Construct triangle ABC such that $|AB| = 8cm,\ angle\ BAC = 60^o\ \ and\ \ angle\ ABC = 75^o$ (ii) Locate the point O inside triangle ABC equidistant from A, B and C (iii) Construct a circle with center O which passes through A.
(b) Measure (i) $|OA|$ (ii) Angle ACB

Solution



(b) $(i)\ |OA| = 5.4cm\ \ (ii) < ACB = 45^o$

Example 32.14

(a) Using a ruler and a pair of compasses only, construct:

(i) Triangle PQR such that
$|PQ| = 8.5cm, < QPR = 60^o$ *and* $|PR| = 7.5cm$

(ii) the locus l_1 of points equidistant from P and R

(iii) the locus l_2 of point equidistant from Q and R

(iv) locate the point of intersection I of the loci l_1 and l_2

(b) (i) Construct a circle passing through the three vertices of the triangle PQR

(ii) Find the radius of the circle

(iii) Measure $|QR|$

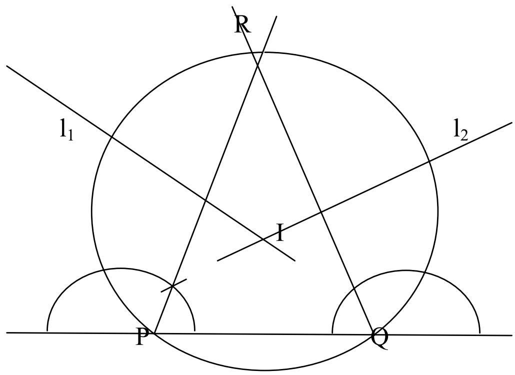
Solution

Steps

1. Draw line PQ 8.5cm and construct angle 60^o at P
2. Draw a straight line through the 60^o from P
3. With the compasses, measure 7.5cm on the ruler and step at P and mark an arc on the line drawn in step 2 as R
4. Bisect line PR and name the bisector l_1
5. Again, bisect line QR and name the bisector l_2
6. Extend l_1 and l_2 to meet each other at I
7. With the compasses, step at I and open to P and draw a circle to touch Q and R as well
8. Place the measurement of the compasses used to draw the circle on the ruler and take its measurement as the radius of the circle drawn

(b)

(i)



(ii) Radius = 4.6cm (ii) line QR = 8.3cm

32.4.4 **Finding The Locus Of A Point Equidistant From Two Intersecting Straight Lines**

This locus is the bisector of the angle between the two straight lines which is usually

located at the vertex where the ***letter*** is ***repeated*** in both lines. Thus, between AB and BC, we bisect at B since it is repeated letter in AB and BC.

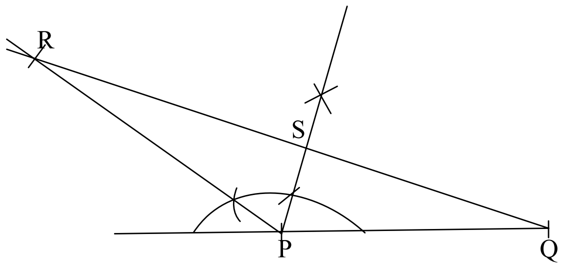
Example 32.15

Using a ruler and a pair of compasses only, construct
(a) (i) triangle PQR, with $|PQ| = 5.5cm, |PR| = 6.5cm \ and < QPR = 120^o$

(ii) A point S on QR such that it is equidistant from $|PQ| \ and \ |PR|$

(b) Measure $|PS|$

Solution



(b) $|PS| = 3cm$

Steps

1. Draw $|PQ| = 5.5cm$ and draw 120° at P
2. From P mark R, 6.5cm away and join it to P and Q
3. Bisect angle at P to touch QR at S

32.4.5 **Finding The Locus Of A Point Equidistant From Three Intersecting Straight Lines**

This locus is a circle drawn to touch all the three sides of the triangle. To get the center of the circle, we bisect any two of the angles to intersect. The intersection becomes the ***center*** of the circle and radius is the distance from the centre to any side

Example 32.16

Using a ruler and a pair of compasses only,
(a) Construct triangle PQR such that
$|PQ| = 9cm, < PQR = 75^o, < QPR = 60^o$
(b) Locate a point T inside triangle PQR such that it is equidistant from RQ, RP and PQ.
(c) Construct the circle which touches the three sides of triangle PQR and measures its radius

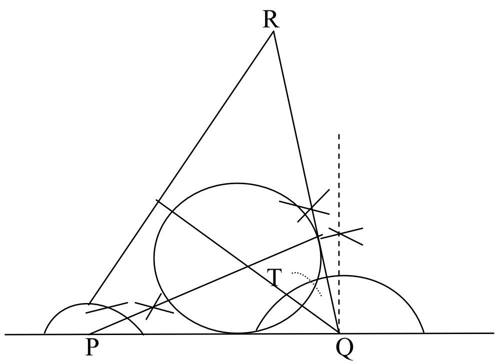
Solution

Steps

1. Draw line $|PQ| = 9cm,\ 75^o\ \ at\ Q\ and\ 60^o\ \ at\ P$

2. Bisect angle PQR and angle QPR to intersect at T

3. With PT as radius, draw a circle to touch all three sides of the triangle



(c) Radius = 3cm

EXERCISE

QUE. A

Use a ruler and a pair of compasses only for the following constructions

(a) Construct triangle ABC in which $|BC| = 6cm,\ angle\ \ ABC = 45^o,\ |AB| = 10cm$

(b) Locate the point D inside ABC such that D is equidistant from AB and AC and 5cm from B

(c) Construct a straight line through D to cut AB at X and AC at Y such that AX = AY (d) Measure $|AY|$

QUE. B

(a) Using a ruler and a pair of compasses only construct
(i) Triangle ABC, where $|AB| = 7cm, |AC| = 8cm, < BAC = 105^o$
(ii) X, the locus of points 6cm from C
(iii) Y, the locus of points equidistant from AB and BC to cut X at P and R.
(b) Measure (i) $|BC|$ (ii) $|PR|$

QUE. C

(a) Using a ruler and a pair of compasses only, construct
(i) A quadrilateral ABCD,
$|AB| = 8cm, |AD| = 6cm, |BC| = 10cm, < BAD = 60^o \text{ and } < ADC = 135^o$
(ii) The locus L_1, of points equidistant from BC and CD
(iii) The line l_2, from B perpendicular to L_1
(b) (i) Locate E, the point of intersection of L_1 and L_2
(ii) Measure $|DE|$

QUE. D

Using a ruler and a pair of compasses only, (a) construct (i) triangle ABC such that
$|AB| = 8cm, < BAC = 105^o \text{ and } < ABC = 30^o$
(ii) The locus, L_1 of points equidistant from A and B
(iii) The locus L_2 of points equidistant from B and C
(b) Locate P, the point of intersection of L_1 and L_2
(c) Using PC as radius, draw a circle
(d) Measure (i) $|BC|$ (ii) the radius of the circle

QUE. E

(a) Using a ruler and a pair of compasses only,
(i) Construct triangle PQR such that
$|PQ| = 6.6cm, |QR| = 8cm \text{ and } < PQR = 60^o$
(ii) Locate by construction, a point S on PQ such that $|PS| = |SQ|$

(iii) Construct ST parallel to QR such that STRQ is a parallelogram
(b) Measure (i) $|PR|$ (ii) $|TR|$ (iii) <STR

QUE. F

(a) Using a ruler and a pair of compasses **only**, construct
(i) $\Delta PQR \text{ in which } |QR| = 7cm, |PQ| = 8cm \text{ and } < PQR = 75^o.$
(ii) the locus l_1 of points equidistant from PR and QR.
(iii) the locus l_2 of points equidistant from Q and R
(b) Locate the point S, the intersection of l_1 and l_2
(c) Measure
(i) $|SR|$
(ii) <PRQ

QUE. G

Using a ruler and a pair of compasses only,
(a) Construct a quadrilateral PQRS in which
$|PQ| = 6.5cm, |PS| = 7.0cm,$
$Q\hat{P}S = 105^o,\ R\hat{Q}P = 120^o \text{ and } \overline{PQ} \text{ is parallel to } \overline{SR}.$
(b) Measure $(i)\ |QR|$ $(ii)\ |SR|$

QUE. H

Using a ruler and a pair of compasses only, construct:
(i) A trapezium ABCD such that
$|AB| = 6.8cm, < ABC = 120^o, BC // AD,$
$|AD| = 10.6cm \text{ and } |AC| = 9.3cm$
(ii) locus l_1 of points equidistant from A and C
(iii) locus l_2 of points equidistant from B and C
(iv) If S is the point of intersection of l_1 and l_2, measure $|AS|$

QUE. I

Construct a triangle ABC, given that
$|AB| = 10cm, |BC| = 11cm \ \ and \ |AC| = 12cm.$
Measure <ABC

QUE. J

Construct $\Delta RPQ \ with < RPQ = 60^o, |PQ| = 9cm \ and \ |PR| = 8cm.$
Measure $|RQ|$

QUE. K

Construct $\Delta DEF \ \ with < FDE = 75^o, |DE| = 11cm \ \ and < DEF = 45^o$
Measure $|DF| \ and < DFE$

QUE. L

Construct a parallelogram ABCD having
$|AB| = 10cm, |BC| = 7cm \ and \ < ABC = 105^o. \ Measure \ |AC|.$

QUE. M

Construct a quadrilateral PQRS, with $|PQ| = 8cm, < SPQ = 60^o,$
$< PQR = 75^o, |PS| = 6cm \ and \ |QR| = 7cm$
$Measure \, |RS| \ and < QRS.$

CHAPTER 33

BEARINGS

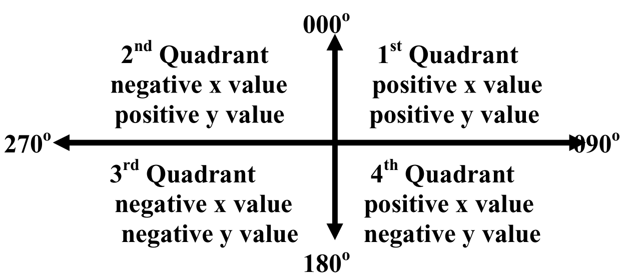
When vectors are represented in ***magnitude*** and ***direction*** form, we call it ***bearings*** usually measured from the geographical north in the clockwise direction.

33.1 **Changing Bearings Into Column Vectors**

Given that (V, Ø) is a magnitude bearing vector with '***v***' as its magnitude and Ø the acute angle the vector makes with the x-axis, then we can change it into a column vector form by using the relation $\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$. Thus, the x – component corresponds to (vcos Ø) and y - component corresponds to (vsin Ø)

NB

In using the relation above, we take note of the ***quadrant*** in which the vector lies after plotting it and apply the appropriate ***sign convention*** to it by using the chart below.



Thus, we multiply the resulting x and y- values by the signs (i.e. multiply by +1 when the sign is positive and multiply by -1 when the sign is negative) of that quadrant.

General Steps (Bearing To Column Vector Form)

1. Represent the vector on the Cartesian plane considering the given angle, taking measurements from north
2. Measure the angle the vector makes with the x axis as Ø in the relation.
NB: Usually, when the vector lies in the first quadrant, we subtract angle given from 90˚ to get Ø. Again, when vector lies in the second quadrant, we subtract 270˚
from the angle to get Ø. Further, when vector lies in third quadrant, we subtract bearing from 270^o and finally when the vector lies in fourth quadrant, we subtract 90˚
From the bearing.
3. Taking V in relation as the given magnitude in the given vector and Ø as the angle the vector makes with x – axis, find $\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$
4. After computing step 3, we then multiply the result by the corresponding sign convention of its quadrant.

Example 33.1

Express the following as column vector

(a) $\overrightarrow{XY} = (15km,\ 060^o)$

(b) $\overrightarrow{AB} = (2km,\ 130^o)$

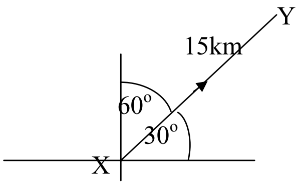
(c) $\overrightarrow{CD} = (30km,\ 220^o)$

(d) $\overrightarrow{EF} = (7km,\ 310^o)$

Solution

(a) $\overrightarrow{XY} = (15km,\ 060^o)$

Plotting



Hence, the angle the vector makes with x-axis = Ø = 90° – 60° = 30°

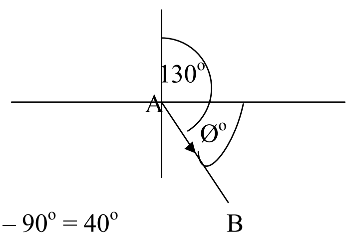
Then, in column form = $\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix} = \begin{pmatrix} 15\cos 30 \\ 15\sin 30 \end{pmatrix} = \begin{pmatrix} 12.99 \\ 0.433 \end{pmatrix}$

Hence, $\vec{XY} = \begin{pmatrix} 13 \\ 0.4 \end{pmatrix}$

NB: Since the vector lies in first quadrant, we multiply x vallue by +1 and y value by +1)

(b) $\vec{AB} = (2km, 130^o)$

Plotting;

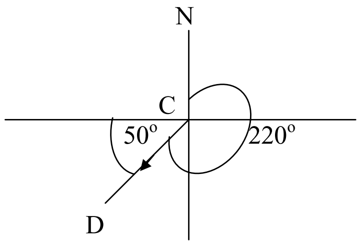


Implies, Ø = 130° – 90° = 40°

Hence, column form = $\begin{pmatrix} v\cos\theta \\ -v\sin\theta \end{pmatrix} = \begin{pmatrix} 2\cos 40 \\ -2\sin 40 \end{pmatrix} = \begin{pmatrix} 1.5321 \\ -1.2856 \end{pmatrix}$

Hence, $\vec{AB} = \begin{pmatrix} 1.5 \\ -1.3 \end{pmatrix}$

(c) $\overrightarrow{CD} = (30km,\ 220^o)$

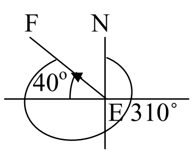


Implies, Ø = 270° - 220° = 50°

Hence, column form = $\begin{pmatrix} -v\cos\theta \\ -v\sin\theta \end{pmatrix} = \begin{pmatrix} -30\cos 50 \\ -30\sin 50 \end{pmatrix} = \begin{pmatrix} -19.28 \\ -22.98 \end{pmatrix}$

Hence, $\overrightarrow{CD} = \begin{pmatrix} -19.28 \\ -22.98 \end{pmatrix}$

(d) $\overrightarrow{EF} = (7km,\ 310^o)$



$\theta^o = 310 - 270 = 40^o$

Implies, column form $\begin{pmatrix} -v\cos\theta \\ v\sin\theta \end{pmatrix} = \begin{pmatrix} -7\cos 40 \\ 7\sin 40 \end{pmatrix} = \begin{pmatrix} -5.3623 \\ 4.4995 \end{pmatrix}$

Hence, $\overrightarrow{EF} = \begin{pmatrix} -5.4 \\ 4.5 \end{pmatrix}$

33.2 **Changing Column Vectors Into Bearing Form**

Steps

1. We first find the magnitude of the given vector

2. Represent (plot) the vector on the Cartesian plane
3. Join the tip of the vector drawn to the x - axis to form a triangle
4. Using $\tan\theta = \frac{Opposite}{Adjacent}$ to find the angle Ø that the vector makes with the x – axis
5. Find the bearing. **NB:** To get the bearing when the vector lies in the ***first quadrant***, we subtract Ø˚ in step 4 from 90˚.
Again, when in the ***second quadrant***, we add Ø˚ to 270˚, whiles when in the ***third quadrant***, we subtract Ø˚ from 270^o and finally, when vector lies in the ***fourth quadrant***, we add Ø˚ to 90˚ for the bearing.

Example 33.2

Express the following into magnitude bearing form:

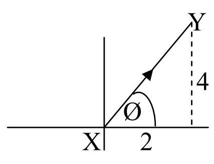
(i) $(i)\ \overrightarrow{XY} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $(ii)\ \overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $(iii)\ \overrightarrow{PQ} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ $(iv)\ \overrightarrow{ST} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

Solution

(i) $\overrightarrow{XY} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

First, $|\overrightarrow{XY}| = \sqrt{2^2 + 4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

Plotting;



Now, $\tan\theta = \frac{Opp}{Adj} = \frac{4}{2} = 2 \ \Rightarrow \theta = \tan^{-1}(2) = 63.4349$

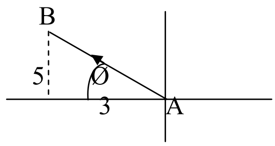
Hence, bearing = 90^o - 63.4^o = 26.6^o

Therefore, magnitude bearing = ($2\sqrt{5}$, 26.6^o)

(ii) $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{(-3)^2 + 5^2} = \sqrt{9+25} = \sqrt{34}$

Plotting;



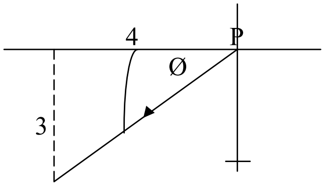
$\Rightarrow \tan\theta = \frac{5}{3} = 1.67 \qquad \Rightarrow \theta = 59^o$

Bearing = $270^o + 59^o = 329^o$

Therefore, magnitude bearing = $(\sqrt{34}, 59^o)$

(iii) $\vec{PQ} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \Rightarrow |\vec{PQ}| = \sqrt{(-4)^2 + (-3)^2} = 5$

Next,

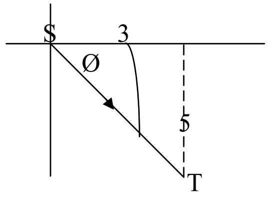


$\tan\theta = \frac{3}{4} = 0.75 \qquad \Rightarrow \theta = 36.9^o \qquad \Rightarrow bearing = 270 - 36.9 = 233.1^o$

Therefore, $(5, 233^o)$

(iv)

$\vec{ST} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \Rightarrow |\vec{ST}| = \sqrt{34}$



$\tan\theta = \frac{5}{3} = 1.67 \quad \Rightarrow \theta = 59^o \quad Hence,\ bearing = 90 + 59 = 149^o$

Implies, $(\sqrt{34}, 149^o)$

NB

In computing $\tan\theta$, we do not attach the negative sign to negative values. For instance, in the example above, we do not use $\tan\theta = \frac{-5}{3}$.

Instead, we use $\tan\theta = \frac{5}{3}$.

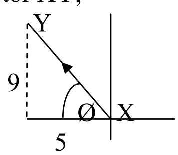
Example 33.3

Find the bearing of Y from X if $\vec{XY} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$

Solution

$|\vec{XY}| = \sqrt{106}$

Plotting vector XY;



$\tan\theta = \frac{9}{5} = 1.8 \quad \Rightarrow \theta = 60.9^o \Rightarrow Bearing = 270^o + 60.9^o = 330.9^o$

33.3 **Back – Bearings**

When the direction of a particular bearing is reversed, we call it the ***back – bearing***.

Thus, for the bearing of B from A, the back bearing will be the bearing of A from B.

Generally, if the bearing of B from A is Ø ˚, then the bearing of A from B (called the back bearing) will be **180 + Ø** if Ø<180˚ and **Ø – 180** if Ø>180˚

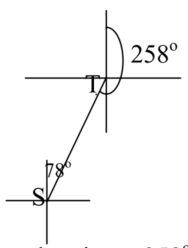
Example 33.4

The bearing of a point S from another point T is 258˚.
Find the bearing of T from S.

Solution

Method 1
Since Ø˚ = 258^o and Ø>180^o, we find Ø – 180 = 258 – 180 = 78˚ as the bearing

Method 2



Thus, bearing = 258^o – 180^o = 78^o

NB

Here, we measure the given bearing in the question at the point that ***comes after*** the word "***from***" in drawing the vector whiles the required bearing is measured from the north to the vector drawn at the other point that ***comes before*** the word "***from***".

Example 33.5

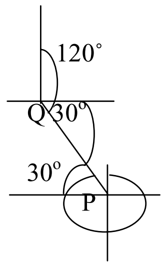
If the bearing of P from Q is 120˚, find the bearing of Q from P.

Solution

Method 1

Since $Ø = 120^\circ$ and $Ø<180^\circ$, we find $180 + Ø = 180 + 120 = 300^\circ$
Therefore, the bearing of Q from P is 300°

Method 2



Hence, bearing of Q from P is $270^\circ + 30^\circ = 300^\circ$.

33.4 **Distance – Bearing Problems**

Usually, bearings give directions in the form of angles.
Sometimes, bearing are given with the geographical directions.
E.g. **N30E** will lie between the north and east and **S50W** will lie between the south and the West poles.
Again, bearings can also be written in three significant figures such as "005˚ ", 280˚, 330˚ without the geographical directions indicated.
In solving distance - bearing problems, we use either of the following:
1. **The vector approach**
(i.e. changing magnitude bearing into column form)
2. **The graphical method**.

Example 33.6

Three villages P, Q and R are situated in such a way that:

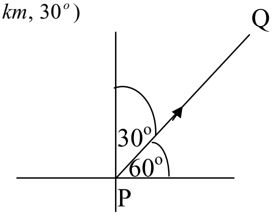
$\vec{PQ} = (15km, 030^o)$ *and* $\vec{QR} = (20km, 120^o)$ Find (i) $|\vec{PR}|$

(ii) the bearing of R from P

Solution

Solution By Vector Approach

$\vec{PQ} = (15km, 30^o)$

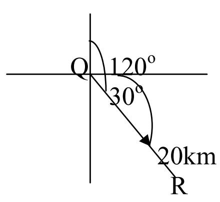


Implies, Ø = 60°

Therefore, $\vec{PQ} = \begin{pmatrix} 15\cos 60 \\ 15\sin 60 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 12.99 \end{pmatrix}$

For

$\vec{QR} = (20km, 120^o)$



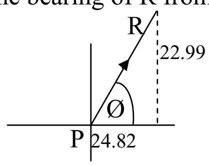
Implies, Ø = 30°

$\vec{QR} = \begin{pmatrix} 20\cos 30 \\ 20\sin 30 \end{pmatrix} = \begin{pmatrix} 17.32 \\ -10 \end{pmatrix}$

(i) But $\vec{PR} = \vec{PQ} + \vec{QR} = \begin{pmatrix} 7.5 \\ 12.99 \end{pmatrix} + \begin{pmatrix} 17.32 \\ -10 \end{pmatrix} = \begin{pmatrix} 24.82 \\ 2\ .99 \end{pmatrix}$

$$|\vec{PR}| = \sqrt{24.82^2 + 22.99^2} = \sqrt{1144.5725} = 33.8315 = 34km$$

(ii) For the bearing of R from P, we use $\vec{PR}$



$$\tan\theta = \frac{22.99}{24.82} = 0.926 \quad \Rightarrow \theta = 42.7997^o \Rightarrow Bearing = 90 - 42.7997 = 47^o$$

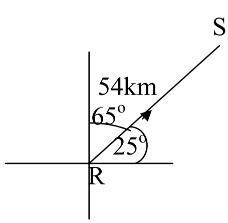
Example 33.7

A ship sails from port R on a bearing of 065° to port S a distance of 54km. It then sails on a bearing of 155° from S to Q a distance of 80km.
Find correct to 1 decimal place: (i) the distance between R and Q
(ii) the bearing of Q from R

Solution

Vector Approach

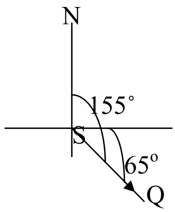
For $\vec{RS} = (54km,\ 065^o)$



Implies, Ø = 25°

$\vec{RS} = \begin{pmatrix} 54\cos 25 \\ 54\sin 25 \end{pmatrix} = \begin{pmatrix} 48.94 \\ 22.82 \end{pmatrix}$

For $\vec{SQ} = (80km, 155^o)$



Ø = 65°

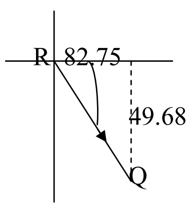
Implies, $\vec{SQ} = \begin{pmatrix} 80\cos 65 \\ -80\sin 65 \end{pmatrix} = \begin{pmatrix} 33.81 \\ -72.50 \end{pmatrix}$

(i) Distance between R and Q is $|\vec{RQ}|$

But $\vec{RQ} = \vec{RS} + \vec{SR} = \begin{pmatrix} 48.94 \\ 22.82 \end{pmatrix} + \begin{pmatrix} 33.81 \\ -72.50 \end{pmatrix} = \begin{pmatrix} 82.75 \\ -49.68 \end{pmatrix}$

$\Rightarrow |\vec{RQ}| = \sqrt{82.75^2 + (-49.68)^2} = 96.52km$

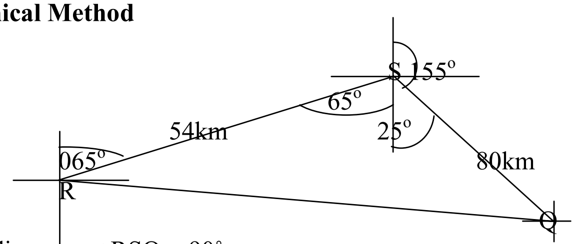
(ii) bearing of Q from R is bearing of $\vec{RQ}$



Implies, $\tan\theta = \dfrac{49.68}{82.75} = 0.6 \quad \Rightarrow \theta = 31^o \ \ and \ \ hence, bearing = 90 + 31 = 121^o$

Graphical Method

(a)



From diagram, $< RSQ = 90^\circ$

NB

We always draw the diagram to form an angle of 90˚ at one of the vertices and then apply the Pythagoras theorem in solving for the side asked of.

Thus,

By Pythagoras;

$$\left|\overrightarrow{RQ}\right|^2 = \left|\overrightarrow{RS}\right|^2 + \left|\overrightarrow{SR}\right|^2 = 54^2 + 80^2 = 9316 \Rightarrow \left|\overrightarrow{RQ}\right| = 96.52km$$

(b) Let <QRS = Ø

$$\tan\theta = \frac{80}{54} = 1.481815 \Rightarrow \theta = 56^o \;\; Hence,\, bearing = 65 + 56 = 121^o\,(From\ north\ to\ vector)$$

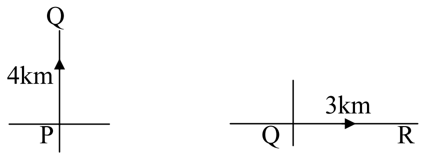
Example 33.8

A ship sails due north from a point P to a point Q, 4km away. It then sails on a bearing of 90˚ to a point R, 3km from Q. Find the distance between P and R

Solution

Vector Approach

$$\overrightarrow{PQ} = (4km,\ 000^o),\ \overrightarrow{QR} = (3km,\ 090^o)$$



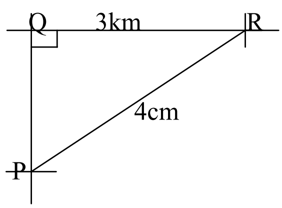
$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} 4\cos 90 \\ 4\sin 90 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ *and* $\overrightarrow{QR} = \begin{pmatrix} 3\cos 0 \\ 3\sin 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

But $|\overrightarrow{PR}| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5km$

Hence, the distance between P and R is 5 km.

Graphical Method



By Pythagoras theorem;

$|\overrightarrow{PR}|^2 = |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 = 4^2 + 3^2 \Rightarrow |\overrightarrow{PR}| = 5km$

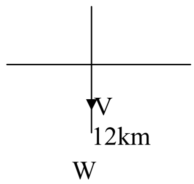
Example 33.9

W is 12km on a bearing of 180˚ from V and X is 9km on a bearing of 090˚ from W. What is the bearing of V from X?

Solution

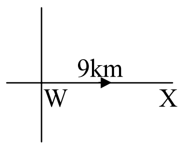
$\overrightarrow{VW} = (12km, 180^o)$, $\overrightarrow{WX} = (9km, 090^o)$

For $\overrightarrow{VW} = (12km, 180^o)$



$$\vec{VW} = \begin{pmatrix} 12\cos 90 \\ -12\sin 90 \end{pmatrix} = \begin{pmatrix} 0 \\ -12 \end{pmatrix}$$

$\vec{WX} = (9km, 090^o)$

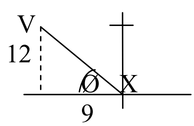


$$\vec{WX} = \begin{pmatrix} 9\cos 0 \\ 9\sin 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \Rightarrow \vec{XV} = -\vec{VX}$$

$$But\ \vec{VX} = \vec{VW} + \vec{WX} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -12 \end{pmatrix}$$

$$\vec{XV} = -\vec{VX} = -\begin{pmatrix} 9 \\ -12 \end{pmatrix} = \begin{pmatrix} -9 \\ 12 \end{pmatrix}$$

Plotting;



$\tan\theta = \dfrac{12}{9} = 1.33 \Rightarrow \theta = 53^o$ *Hence, bearing of V from X is* $270 + 53 = 323^o$

Example 33.10

A, B, X and Y are four points in a horizontal plane. B is on a bearing of 090˚.

from A. X is 7.5m due north of B and on a bearing of 052° form A. Y is due north of A and on a bearing of 340° from B. Calculate, correct to three significant figures:

(a) $\vec{AB}$ (b) $\vec{AY}$ (c) The components of XY (d) the distance and bearing of Y from X.

TRY: (a) 9.6m (b) 26.4m (c) 18.9m (d) 21.2m and 333°

Example 33.11

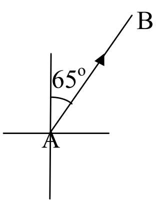
A ship sails from a point A in a direction 065° to a point B, 24km away. From B, the ship sails 18km due east to a point C. from C, the ship then sails 30km due north to a point D. Calculate the bearing of D from A.

Solution

Vector Approach

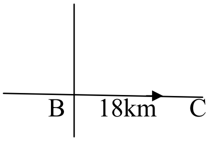
$\vec{AB} = (24km, 065^{o}),\ \vec{BC} = (18km, 90^{o})\ \ and\ \ \vec{CD} = (30km, 000^{o})$

For $\vec{AB} = (24km, 065^{o})$



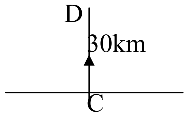
$$\vec{AB} = \begin{pmatrix} 24\cos 25 \\ 24\sin 25 \end{pmatrix} = \begin{pmatrix} 21.75 \\ 10.14 \end{pmatrix}$$

For $\vec{BC} = (18km, 090^{o})$



$\vec{BC} = \begin{pmatrix} 18\cos 0 \\ 18\sin 0 \end{pmatrix} = \begin{pmatrix} 18 \\ 0 \end{pmatrix}$

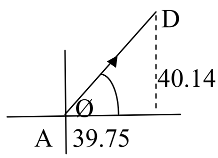
For $\vec{CD} = (30km, 000^o)$



$\vec{CD} = \begin{pmatrix} 30\cos 90 \\ 30\sin 90 \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$

But $\vec{AD} = \vec{AB} + \vec{BC} + \vec{CD} = \begin{pmatrix} 21.75 \\ 10.14 \end{pmatrix} + \begin{pmatrix} 18 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 30 \end{pmatrix} = \begin{pmatrix} 39.75 \\ 40.14 \end{pmatrix}$

Now,



$\Rightarrow \tan\theta = \dfrac{40.14}{39.75} = 1.0098113 \Rightarrow \theta = 45.3^o$

Hence, the bearing of D from A = 90+45.3=44.7^o

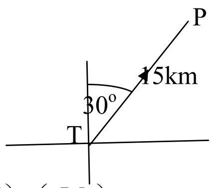
Example 33.12

A ship leaves a point T and sails on a bearing of 030^o to a point P, 15km away. It then sails on a bearing of 120^o to a point Q, 20km away from P. Calculate $|\overrightarrow{TQ}|$

Solution

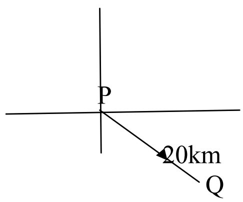
From question, $\vec{TP} = (15km, 030^o)$, $\vec{PQ} = (20km, 120^o)$

For $\vec{TP} = (15km, 030^o)$



$$\vec{TP} = \begin{pmatrix} 15\cos 60 \\ 15\sin 60 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 12.99 \end{pmatrix}$$

For $\vec{PQ} = (20km, 120^o)$



$$\Rightarrow \vec{PQ} = \begin{pmatrix} 20\cos 30 \\ 20\sin 30 \end{pmatrix} = \begin{pmatrix} 17.32 \\ -10 \end{pmatrix} \Rightarrow \vec{TQ} = \vec{TP} + \vec{PQ} = \begin{pmatrix} 7.5 \\ 12.99 \end{pmatrix} + \begin{pmatrix} 17.32 \\ -10 \end{pmatrix} = \begin{pmatrix} 24.82 \\ 2.99 \end{pmatrix}$$

$$\Rightarrow |\vec{TQ}| = \sqrt{24.82^2 + 2.99^2} = 25km$$

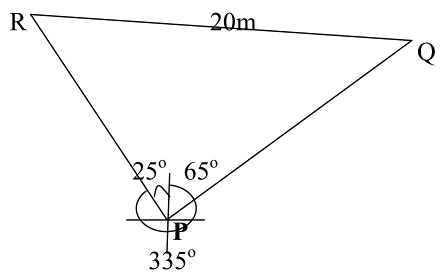
Example 33.13

P, Q, R are three points on a horizontal plane. Q is on a bearing of 065^o from P and R is on a bearing of 335^o from P. If

$If\ |PQ| = 12m\ and\ |RQ| = 20m,\ find\ |RP|$

Solution

Illustration



From diagram above, since <RPQ = 90°, we apply Pythagoras dtheorem.

$\Rightarrow |RP|^2 = |RQ|^2 - |PQ|^2 = 20^2 - 12^2 = 400 - 144 = 256$

$\Rightarrow |RP|^2 = 256 \Rightarrow |RP| = \sqrt{256} = 16m$

$\therefore |RP| = 16m$

EXERCISE

QUE. A

Express (30km, 240°) in the form $\begin{pmatrix} a \\ b \end{pmatrix}$

QUE. B

Express the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ in the form (r, Ø),

where r is in km and Ø in degrees.

QUE. C

Express the vector (3.4km, 043°) in the component form

QUE. D

P is equivalent from Q and R. the bearing of Q from P is 70^o and the bearing of R from P is 130^o. What is the bearing of R from Q?

QUE. E

A boy walks a distance of 5km on a bearing of 240^o. Which of the following diagrams shows his direction?

QUE. F

A man starts from a point S and walks 1000m on a bearing of 025^o. How far north is he from his starting point?

QUE. G

P, Q and R are three villages on level ground. Q is 4km on the bearing of 040^o from P, while R is 3km on the bearing of 130^o from Q. Calculate the distance and bearing of P from R and state PR in the distance bearing form.

QUE. H

A ship sails from port X to port Y, 500km away on a bearing of 145^o and then sails 1200km from port Y, on a bearing of 235^o to port Z.

QUE. I

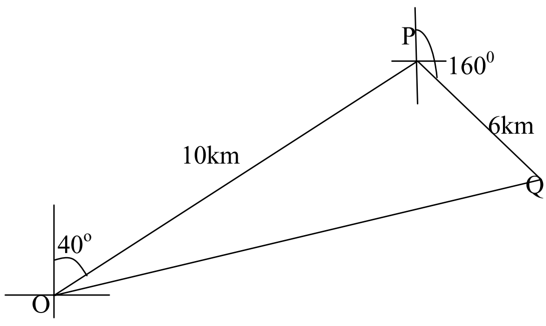
A village P is 10km from a lorry station, Q on a bearing of 065^o. village R is 8km from Q on a bearing of 155^o. calculate (i) the distance of R from P to the nearest kilometers (ii) the bearing of R from P to the nearest degree

M is village of $\vec{PR}$ such that $\vec{QM}$ is perpendicular to $\vec{PR}$, find the distance of M from P to the nearest kilometer

QUE. J

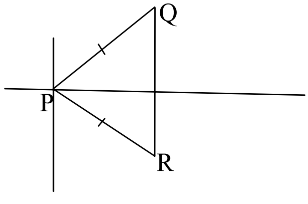
In the diagram, $|OP| = 10km, |PQ| = 6km,$ the bearing of P from O is 040^o while the bearing of Q from P is 160^o. (a) Calculate, correct to three significant figures (i) $|OQ|$

(b) How far south of P is Q?, correct to three significant figures.



QUE. K

The diagram shows the positions of three points P, Q, R. The bearing of Q from is 040°. R is directly under Q and $|PQ| = |PR|$, find the bearing of R from P.

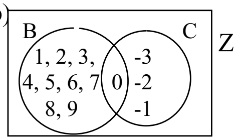


ANSWERS TO EXERCISES

CHAPTER 1

A. (a) (i) $\{---,-5,-3,-1,1,3,5,---\}$ (ii) $\{1,3,5,7,9\}$

(b)



B. $\{x : x > 2\} = \{3,4,5,6,7,8,9\}$

C. **(a)** P = {18, 21, 24, 27, 30, 33, 36}, Q = $\{18,24,36\}$,
R = {18, 20, 22, 24, 26, 28, 30, 32, 34, 36}
(b) (i) {18, 24, 36} (ii) {18, 24, 36} (iii) {18, 24, 30, 36}
(c) $P \cap Q = Q \cap R$

D. $\{1,2,3,4,5,6\}$

E. {-2, -1, 0, 1} Or {x:-3<x<2}

F. 4

G. (a) 2 (b) 14 (c) 12

H. (ii) 11

I. (ii) 6 (iii) 66

J. (i) 30 (ii) 10 (iii) 45

K. (i) 20% (ii) 45% (iii) 20% (iv) 35% (v) 10%

L. (i) 54 (ii) 40

M. (i) 4 (ii) 3

N. (i) 22 (ii) 31

O. 31

P. (i) 6 (ii) 76

Q. (i) {2, 3, 4, 5, 6, 7, 8} (ii) {3, 5} (iii) {1, 2, 3, 4, 5, 6, 8}
(iv) {2, 3, 4, 5} (v) {2, 3, 4, 5, 7} (vi) {2, 3, 4, 5}
(vii) {2, 3, 4, 5, 7}

R. (i) {2, 4, 6, …, 18} (ii) {7, 8, 9, …} (iii) {13, 15, 17, …}
(iv) {1, 3, 5, 7, 9, 11, 13} (v) {6, 8, 10, …, 78}
(vi) {13, 15, 17, …, 55}

S. (i) {x:x is a natural number between 2 and 7}
(ii) {x:x is an odd natural number between 0 and 8}

(iii) {x:x is an odd natural number greater than 4}
(iv) {x:x is a natural number greater than 3}
(v) {x:x is an even natural number between 5 and 83}
(vi) {x:x is an odd natural number between 8 and 52}

T. 1. {1, 5, 7, 8, 10,11} 2. {a, b, c, d, 1, 2, 3, 4}
3. {a, b, c, d, x, y, z}
4. {1, 3, 4, 5, 8} 5. {5, 10, 15, 20, 25}

U. 1. AUB =
{Monday, Tuesday, Thursday, Friday, Saturday, Sunday}
AnB = {Sunday}
2. AUB = {1, 2, 3, 6, 7, 9, 18, 27, 54} and AnB = {2, 3}
3. AUB = {2, 3, 4, 5, 6, 9, 12} and AnB = {3, 6}
4. AUB = {1, 2, 3, 4, 5, 6, 7, 11, 13, 17, 19} and AnB = { }

W. 40

X. 1. 100 2. 50 3. 90

Y. (a) 40 (b) 10 (c) 5 (d) 65

CHAPTER 2

A. (b) (i) Invalid (ii) Valid (iii) Invalid

B. 1. Not Valid 2. Valid 3. Valid 4. Not Valid 5. Valid
6. Valid 7. Valid
8. Valid 9. Not Valid 10. Not Valid

C. (b) (i) Invalid (ii) Invalid (iii) Invalid

D. (ii) (a) Not Valid (b) Valid

CHAPTER 3

A. 1. 10 2. $\frac{25}{4}$ *OR* $6\frac{1}{4}$ 3. $\frac{16}{3}$ *OR* $5\frac{1}{3}$ 4. 4 5. $\frac{61}{45}$ *OR* $1\frac{16}{45}$
6. $1\frac{31}{57}$
7. $3\frac{1}{5}$ 8. $\frac{3a+2b}{a^2-4b^2}$ 9. $1\frac{31}{57}$ 10. $\frac{4x+2}{(x-2)(x+3)}$

B. $5.75, \frac{28}{5}, 4\frac{2}{3}$ *and* 4.03

C. 140,400.00

D. $N\,2400.00$

E. 1. 0.5 2. 0.25 3. 0.125 4. 0.0625 5. 0.12
6. 0.375
7. 0.3125 8. 0.175 9. 0.22 10. 0.46
11. 0.3375
12. 0.24375 13. 0.333333 … 14. 0.166666 …
15. 0.142857142857 … 16. 0.111111 …
17. 0.090909090 … 18. 0.666666 …
19. 0.444444 …

F. 1. $\frac{225}{1000}$ 2. $\frac{74}{100}$ 3. $\frac{3875}{10000}$ 4. $\frac{575}{1000}$ 5. $\frac{4875}{10000}$ 6. $\frac{35625}{100000}$
7. $\frac{368}{1000}$ 8. $\frac{59375}{100000}$ 9. $\frac{796875}{1000000}$ 10. $\frac{2}{3}$ 11. $\frac{7}{11}$ 12. $\frac{54}{111}$
13. $\frac{108}{111}$ 14. $\frac{8}{33}$ 15. $\frac{90}{111}$ 16. $\frac{57}{90}$ 17. $\frac{561}{1980}$ 18. $\frac{14}{15}$
19. $\frac{102}{111}$ 20. $\frac{6}{11}$

G. 1. $3\frac{5}{21}$ 2. $4\frac{7}{12}$ 3. $-12\frac{4}{15}$ 4. $-17\frac{11}{12}$ 5. $-11\frac{2}{35}$
6. $19\frac{5}{6}$ 7. $9\frac{13}{20}$ 8. $17\frac{5}{12}$ 9. $20\frac{27}{35}$ 10. $-2\frac{5}{6}$ 11. $10\frac{7}{8}$
12. $13\frac{1}{6}$ 13. $-1\frac{2}{15}$ 14. $10\frac{7}{8}$ 15. $13\frac{1}{6}$ 16. $-1\frac{2}{15}$ 17. $2\frac{9}{10}$
18. $3\frac{31}{35}$ 19. $2\frac{4}{15}$ 20. $3\frac{1}{4}$

H. 1. $-20\frac{1}{4}$ 2. $-33\frac{3}{5}$ 3. $-37\frac{33}{35}$ 4. $47\frac{7}{10}$ 5. -100.45
6. $1\frac{1}{10}$ 7. $\frac{3}{7}$ 8. 2 9. $1\frac{11}{27}$ 10. $\frac{3}{4}$ 11. $\frac{6}{7}$ 12. $\frac{2}{3}$
13. $17\frac{1}{3}$ 14. -30 15. -119 16. $17\frac{1}{3}$ 17. $16\frac{4}{11}$
18. 45 19. $-86\frac{4}{11}$ 20. -75

CHAPTER 4

A. 1. 22 2. $7\sqrt{2}$ 3. $5\sqrt{6}$ 4. $9\sqrt{7}$ 5. $11\sqrt{5}$ 6. $4\sqrt{17}$

7. $3\sqrt{91}$ 8. $5\sqrt{29}$ 9. $6\sqrt{47}$ 10. $8\sqrt{23}$ 11. $12\sqrt{14}$ 12. $33\sqrt{5}$

B. 1. $\sqrt{18}$ 2. $\sqrt{12}$ 3. $\sqrt{128}$ 4. $\sqrt{\frac{1}{2}}$ 5. $\frac{\sqrt{2}}{\sqrt{12}}$ *or* $\sqrt{\frac{2}{12}}$

6. $\frac{\sqrt{4}}{\sqrt{6}}$ *or* $\sqrt{\frac{4}{6}}$ 7. $\sqrt{175}$

C. 1. $11\sqrt{2}$ 2. $4\sqrt{2}$ 3. $22\sqrt{3}$ 4. $6\sqrt{3}$ 5. $21\sqrt{2}$ 6. $7\sqrt{10}$
7. $10\sqrt{5}$ 8. $5\sqrt{3}$ 9. $-18\sqrt{2}$ 10. $19\sqrt{15}+30$ 11. $1\frac{1}{3}$
12. $\sqrt{5}$ 13. $\frac{70+18\sqrt{5}}{5}$ 14. $22\sqrt{3}$ 15. $14\sqrt{5}$

D. 1. $\sqrt{6}$ 2. $\sqrt{15}$ 3. $\sqrt{91}$ 4. $\frac{1}{2}\sqrt{35}$ 5. $\frac{1}{2}\sqrt{110}$ 6. $\frac{1}{9}\sqrt{30}$

7. $\frac{6}{5}\sqrt{5}$ 8. $\frac{11}{65}\sqrt{429}$ 9. 3 10. $\frac{10}{9}\sqrt{5}$ 11. $\dfrac{2\sqrt{5}+5\sqrt{2}}{10}$

12. $\dfrac{6+\sqrt{6}}{6}$

13. $15-\sqrt{21}$ 14. $8+\sqrt{35}$ 15. $\dfrac{5+3\sqrt{3}}{4}$ 16. $\dfrac{-13+5\sqrt{7}}{3}$

17. $-1+2\sqrt{2}$ 18. $\dfrac{3+\sqrt{2}}{7}$ 19. $\dfrac{3\sqrt{35}}{25}$ 20. $\dfrac{\sqrt{10}}{4}$

E. 1. $\sqrt{2}$ 2. $\dfrac{15-5\sqrt{3}}{3}$ 3. $12\sqrt{5}-20$ 4. $21\sqrt{7}-42$

5. 0.02646

F. 1. 11 2. $24-18\sqrt{6}$ 3. 23 4. $84-60\sqrt{2}$ 5. 92 6. $130\sqrt{2}-\sqrt{2}-65\sqrt{3}$

7. $15\sqrt{2}-9\sqrt{5}$ 8. $168\sqrt{3}+21\sqrt{7}$ 9. 1 10. $-30-2\sqrt{15}$

11. 172 12. 516 13. $24-6\sqrt{3}-8\sqrt{6}-6\sqrt{2}$

14. $168+56\sqrt{5}-18\sqrt{35}-30\sqrt{7}$ 15. $105\sqrt{5}-120\sqrt{2}-105\sqrt{3}-24\sqrt{30}$

16. $1\frac{1}{3}$

CHAPTER 5

A. 1. 2 2. $\frac{3}{2}$ 3. $\frac{3}{4}$ 4. $\frac{1}{2}$ 5. $\frac{5}{3}$ 6. 2 7. $\frac{1}{3}$ 8. $\frac{1}{4}$ 9. 0

10. $\frac{4}{3}$ 11. $\frac{3}{2}$ 12. $-\frac{1}{4}$ 13. -4 14. $-\frac{1}{2}$ 15. $\frac{1}{3}$ 16. $\frac{2}{3}$ 17. 25

18. 4 19. $\frac{2}{3}$ 20. 3 21. $\frac{1}{5}$ 22. $\frac{1}{2}$ 23. $\frac{1}{4}$ 24. 2 25. 4

26. $\frac{1}{27}$ 27. $3^{\frac{4}{3}}$ 28. 100 29. $\frac{25}{3}$ 30. $\frac{1}{3}$ 31. $\frac{2}{3}^{\frac{2}{3}}$ 32. $\frac{5}{8}$

33. -2 34. $\frac{1}{2}$ 35. $\frac{1}{36}$ 36. $\frac{11}{10}$ 37. -1 38. 9 39. 2, 3

40. $\frac{1}{2}$, 1 41. $-\frac{2}{3}$, $-\frac{1}{2}$ 42. -1 43. $-\frac{3}{4}$ 44. 1 45. 5

B. 1. 1 2. $\frac{27}{64}$ 3. 125 4. $x^{-\frac{7}{12}}$ 5. 1 6. $\frac{16}{81}$ 7. $\frac{x+1}{x^2}$ 8. $\frac{1}{2}$

9. $3^{\frac{5}{2}}$ 10. $\frac{1}{10}$ 11. 3 12. 9 13. $\frac{1}{5}$ 14. 8 15. $\frac{1}{6}$ 16. 4

17. 216 18. $\frac{1}{7}$ 19. 16 20. $\frac{1}{125}$

C. 1. $x=4,\ y=2$ 2. $m=4,\ n=1$ 3. $x=2,\ y=-1$ 4. $x=1,\ y=2$

D. 3

CHAPTER 6

A. 1. $q^p = r$ 2. $5^2 = 25$ 3. $3 = 27^{\frac{1}{3}}$ 4. $27 = (\frac{1}{3})^{-3}$ 5. $0.001 = 10^{-3}$
6. $2^7 = 128$ 7. $2 = 2^1$ 8. $8 = 2^3$ 9. $\frac{1}{16} = 2^{-4}$ 10. $64 = 2^6$
11. $\frac{1}{128} = 2^{-7}$ 12. $256 = 2^8$

B. 1. $4 = \log_2 16$ 2. $\log_{\frac{1}{3}} 81 = -4$ 3. $\frac{3}{2} = \log_9 27$ 4. $\log_{16} 64 = \frac{3}{2}$
5. $\log_{10} 0.01 = -2$ 6. $0 = \log_{\frac{1}{8}} 1$ 7. $\log_2 16 = 4$ 8. $\log_{10} 1000 = 3$
9. $\log_9 3 = \frac{1}{2}$ 10. $\log_3 n = 2$ 11. $\log_{25} n = -1$ 12. $\log_{16} n = 3$
13. $\log_a m = 2$ 14. $\log_2 m = a$ 15. $\log_2 x = r$ 16. $\log_a n = 0.3$
17. $\log_{\frac{1}{2}} n = -3$ 18. $\log_a n = x$ 19. $\log_5 625 = 4$ 20. $\log_2 \frac{1}{8} = -3$
21. $\log_8 64 = 2$ 22. $\log_5 n = x$

C. 1. 4 2. 6 3. -3 4. $\frac{1}{4}$ 5. 100 6. $\frac{1}{125}$
7. 2 8. 10 9. 4 10. 9 11. -3 12. $\frac{1}{81}$ 13. 125
14. 4 15. 10 16. 1 17. 3 18. -2 19. 4 20. 3

D. 1. 47.26 2. 85.90 3. 32.98 4. 48.09 5. 13.20
6. 84.18 7. 2.096 8. 48.25 9. 1.160 10. 1.459
11. 0.8383 12. 0.8935 13. 38.59 14. 1.510 15. 3157
16. 5.903 17. 10.69 18. 291.7 19. 19.10 20. 3.370
21. 37.01 22. 1.808 23. 1.714 24. 78.54

E. 1. 0.9309 2. 0.9566 3. 0.6906 4. 0.0249 5. 0.9422
6. 0.5711 7. 0.5523

F. 1. 0 2. 1 3. $\bar{2}$ 4. $\bar{3}$ 5. $\bar{5}$ 6. 4 7. 0 8. 6

G. 1. 1.0792 2. 1.3802 3. 1.8574 4. 0.4772 5. 0.1762
6. 1.2554 7. 1.4772 8. 1.7782 9. 1.1762 10. 1.6812
11. 1.5564 12. 2.1584 13. 2.0792 14. 0.9542
15. -0.4772
16. 0.4771 17. 0.1760 18. 0.2594 19. 0.2386
20. -0.3891
21. 0.8891 22. 0.9771 23. 3.4648 24. 2.3520
25. -0.9542 26. 1.8485 27. 3.8485
28. $\bar{1}.8485$ 29. $\bar{4}.8485$ 30. 4.8485
31. 6.8485 32. $\bar{2}.8485$ 33. 7.8485 34. 2.0970 35. 0.3495
36. 2.7960 37. 0.4660

H. 1. 3 2. 3 3. 2 4. 1 5. $\log_{10}\frac{25}{2}$ 6. $\log_{10}\frac{3}{8}$

CHAPTER 7

A. $(a)\ T=\dfrac{5.4MT}{6T-37}$ $(b)\ V=-1.7\times10^{8}$

B. (i) $m=\dfrac{n(q-p)}{3q-5p}$ (ii) $r=\sqrt{\dfrac{1}{3}\left(\dfrac{6y}{\pi h}-h^{2}\right)}$ $(iii)\ r=\dfrac{mp}{eq^{2}}$

C. (i) 0 (ii) -38.1 (iii) 11

D. (i) -11 (ii) $r=4p(p-q)$ $(iii)\ -7.7$

E. $(i)\ n=10$ $(ii)\ (a)\ d=u+\dfrac{fu}{u-f}$ $(b)\ 84.5$

F. 1. $x=\frac{a}{3}$ 2. $x=\frac{b-7}{5}$ 3. $x=2c-10$ *or* $x=2(c-5)$ 4. $x=\frac{d}{u+v}$

5. $x=\frac{e}{m}+c$ 6. $x=\frac{df-c}{m}$ 7. $x=\frac{1}{2}\left(\frac{ag}{m}-d\right)$ 8. $x=d\left(b-\frac{2h}{a}\right)$

9. $x=\frac{1}{d}\left(\frac{1}{b}-3c\right)-2$ 10. $x=d+\frac{1}{e}\left(by-\frac{m}{a}\right)$ *or* $x=d-\frac{1}{e}\left(\frac{m}{a}-by\right)$

11. $x=d(a-m+1)$ 12. $x=\frac{1}{m}(2pq-c)$ 13. $x=\frac{b}{mn}(y-k)$

14. $x=\frac{z(a+b)-c}{m}$ 15. $x=\frac{b}{a}\{w(m-n)-c\}$

G. 1. $y=\frac{3a^2}{b^2}$ 2. $y=\frac{2c^2}{b^2}$ 3. $y=\sqrt[3]{\frac{c-a}{b}}$ 4. $y=\sqrt{(d-3e)}$ 5. $y=\frac{a}{(e^4+b)}$

6. $y=\sqrt[3]{\frac{am}{f+b}}$ 7. $y=\frac{bx}{gx-a}$ 8. $y=\frac{a^2b}{ch^2}$ 9. $y=\frac{b}{ab-1}$ 10. $y=\sqrt[3]{\frac{f-am}{b}}$

11. $y=\frac{a}{b+h^2}$ 12. $y=\sqrt{b-\frac{a}{k}}$ 13. $y=\sqrt[3]{\frac{1}{a}\left(n-\frac{m}{l}\right)}$ 14. $y=\sqrt{\frac{2(3m-bx)}{3a}}$

15. $y=\sqrt{\frac{n-a}{3b}}$ 16. $y=\sqrt[3]{\frac{1}{c}\left(\frac{p^2}{a^2}-b\right)}$ 17. $y=\frac{a^3}{q^3-a^2c}$ 18. $y=(a-br)^2$

19. $y=\sqrt{\frac{3cs+b}{a}}$ 20. $y=\frac{1}{5}\left(\frac{a}{u^2}+3\right)$ 21. $y=\sqrt{\frac{5bc}{d(bv-2a)}}$ 22. $y=\sqrt{\left(3-\frac{c}{w-b}\right)}$

23. $y=\left(\frac{5b}{a-5z}\right)^2$ 24. $y=\frac{1}{d}\sqrt[5]{\frac{x^2kl}{h}}$

H. 1. $v=\frac{B-35}{B-7}$ 2. $v=\frac{3}{2}(k-5)$ 3. $v=\frac{e+7d}{d+5}$ 4. $v=\frac{25-3f}{7}$ 5. $v=\frac{6-k}{5}$

6. $v=\frac{1}{a}\left(3d-\frac{3b}{c}-5\right)$ 7. $v=\frac{1}{4}(15-5a-b)$ 8. $v=\sqrt{(a^2+ab+c^2)}$

9. $v=\sqrt{(p^2+pr+q^2)}$ 10. $v=\frac{1}{12a}(9a^2-r^2)$

I. 1. $u=\frac{3+2k}{k-5}$ 2. $u=\frac{3x-ac}{2a-1}$ 3. $u=\frac{my}{c-1}$ 4. $u=\frac{e-bc}{bd-a}$ 5. $u=\frac{ax^2+kd+c}{md-b}$

6. $u=\frac{a-be^2}{1-e^2}$ 7. $u=\frac{1-f}{fb-a}$ 8. $u=\frac{4\pi l-h^2g}{4\pi^2-h^2}$ 9. $u=\sqrt{\frac{pm^2}{a^2-m^2}}$ 10. $u=\frac{a^3-n^3}{3n^3-a^3}$

J. 1. $a=10m/s^2$ 2. $k=125$ 3. $p=4.808\times10^7$ 4. $n=31$

5. $t = 5.738 \times 10^{-1}$ 6. $n = 31$ 7. $S_{10} = 79.92$ 8. $d = 6.1$

9. $x = -2$ or -2.5 10. (a) $h = \frac{v}{\pi r^2} - \frac{2r}{3}$ (b) $h = 7.545cm$ 11. $r = 41\%$

CHAPTER 8

A. 1. $5\frac{5}{9}$ 2. 12 3. 7 4. -10 5. 2

B. (i) $\frac{5}{2}$ (ii) $\frac{3}{4}$

C. 15 ***Hint:*** Use $48 + x = 3(36 - x)$

D. 27, 28, 29

E. 63, 65

F. ¢ 50.00, ¢ 100.00, ¢ 300.00

G. 8

H. 82, 84, 86

I. 15, 30, 36

J. 10 years, 50 years

K. 20, 34

L. ¢ 80.00, ¢ 60.00

M. 1. -87 2. -24 3. -3 4. 6 5. $\frac{5}{2}$

6. 2 7. 18 8. -7 9. 18 10. 0 11. $-\frac{28}{5}$

12. $\frac{10}{29}$ 13. 12 14. 12 15. 6 16. $\frac{5}{18}$

17. 1 18. -4 19. $\frac{3}{2}$ 20. $\frac{121}{48}$ 21. $\frac{19}{13}$

CHAPTER 9

A. (i) $\{p, q : p = 5,\ q = 2\}$ (ii) $\{x, y : x = 5,\ y = -3\}$

(iii) $\{x, y : x = 5, y = 2\}$ (iv) $\{p, q : p = 2, q = -3\}$

B. (i) ¢ 500.00 (ii) ¢ 2,000.00 (ii) ¢ 6000.00

C. ¢ 21,600.00

D. (i) -2 (ii) (α) $x = 2$ *when* $y = 3$ *and* $x = 3$ *when* $y = 2$

(β) $x = -3$ *and* $y = 1$

E. 1. $x = \frac{8}{3},\ y = -\frac{7}{9}$ 2. $x = 2,\ y = 1$ 3. $x = 1,\ y = 1$ 4. $x = 20,\ y = -6$

5. $x=-10,\ y=8$ 6. $a=\frac{1}{7},\ b=-\frac{32}{35}$ 7. $x=7,\ y=-2$ 8. $m=14,\ n=14$
9. $x=1,\ y=7$ 10. $p=10,\ q=9$ 11. $s=3,\ t=5$ 12. $m=-3,\ n=-5$
13. $x=\frac{1}{2},\ y=\frac{1}{3}$ 14. $y=6,\ z=\frac{3}{4}$ 15. $a=-6,\ b=2$ 16. $m=\frac{5}{2},\ n=-\frac{13}{4}$
17. $p=2,\ q=0$ 18. $x=4,\ y=1$ 19. $a=-1,\ b=-2$ 20. $m=-1,\ n=5$

F. 1. $x=1, y=3$ 2. $a=3, b=2$ 3. $s=1, t=1.5$ 4. $m=3, n=5$
5. $x=2, y=6$ 6. *no solution* 7. $a=4, b=3$ 8. $x=-5,\ y=-1$
9. $a=\frac{5}{8}, b=4$ 10. $x=0, y=-3$ 11. $p=2, q=1$ 12. $s=1,\ t=-2$
13. $a=-1,\ b=-1$ 14. $s=20, t=21$ 15. $x=1, y=-\frac{1}{3}$ 16. $m=4, n=\frac{2}{5}$
17. $x=4, y=6$ 18. $k=8,\ l=9$ 19. $x=-\frac{13}{15}, y=\frac{27}{10}$ 20. $p=3,\ q=3$
21. $x=-1,\ y=-3$ 22. $a=4, b=7$ 23. $m=6,\ n=8$ 24. $x=-\frac{1}{2},\ y=0$
25. $p=8, q=9$ 26. $x=1,\ y=-\frac{2}{3}$

G. 1. 45years, 15years 2. 53 3. 57 and 72 4. p=12, q=18
5. 20, 35 6. 15m, 25m 7. 100kg, 90kg 8. Kodwo ¢5000, Esi ¢4000
9. Speed of bird is 112kph, Speed of wind are 128kph and 64kph
10. Orange ¢20.00, Pineapple ¢150.00 11. 23, 15
12. 50kph, 60kph 13. 5kph, 18kph 14. 15girls, 20boys
15. $400cm^3$, $600cm^3$; 1.2kg, 2. 16. 40years, 10years 17. 17, 20 18. $\frac{9}{17}$

CHAPTER 10

A. (i) 7 (ii) 26 (iii) 16
B. (a) 2340^o (b) 2880^o (c) 3240^o (d) 4140^o
C. (a) 8 (b) 5 (c) 18 (d) 24 (e) 6 (f) 16
D. (g) 20
E. 144^o
F. (a) 37.5 (b) 24^o (c) 37.5^o

CHAPTER 11

A. $|AB| = 3\sqrt{10}$ *and* $m = -2$

B. $(i)\ \frac{1}{6}$ $(ii)\ 2$

C. $(i)\ y + x - 4 = 0$ $(ii)\ y + 2x - 4 = 0$ $(iii)\ y - 5x - 4 = 0$

D. 1. $3, (0,2), (-\frac{2}{3}, 0)$ 3. $\frac{5}{3}, (0, \frac{3}{5}), (\frac{-9}{25}, 0)$ 5. $-1, (0, \frac{4}{3}), (\frac{4}{3}, 0)$
7. $-2, (0, 4), (4, 0)$

E. 1. $y = -\frac{3}{4}x + \frac{1}{2}$ 3. $y = \frac{5}{6}x + \frac{7}{6}$ 5. $y = \frac{7}{8}x + \frac{5}{8}$ 7. $y = \frac{3}{25}x + \frac{52}{25}$

F. 1. $y = \frac{2}{3}x + 2$ 3. $y = -\frac{7}{12}x - \frac{5}{4}$ 5. $y = -(x+1)$ 10. $y = -5x + \frac{39}{14}$

CHAPTER 12

A. $(b)\ (i)\ 2, 5$ $(ii)\ 6$

B. $(ii)\ (\alpha)\ 1$ $(\beta)\ 2$ *and* 5

D. (i) 4 (ii) n = 5

E. 1. 6(mod 7) 2. 4(mod 8) 3. 2(mod 3) 4. 3(mod 4)
5. 6(mod 9) 6. 2(mod 4) 7. 2(mod 7) 8. 2(mod 3)
9. 5(mod 8) 10. 8(mod 12) 11. 1(mod 7) 12. 5(mod 7)
13. 6(mod 8) 14. 8(mod 9) 15. 3(mod 4) 16. 3(mod 5)
17. 10(mod 11) 18. 7(mod 13) 19. 5(mod 12)
20. 1(mod 15)

F. 1. 2(mod 6) 2. 3(mod 7) 3. 3(mod 9) 4. 6(mod 9)
5. 2(mod 7) 6. 5(mod 11) 7. 2(mod 13) 8. 6(mod 15)
9. 3(mod 13) 10. 4(mod 7) 11. 0(mod 5) 12. 1(mod 4)
13. 5(mod 9) 14. 2(mod 7) 15. 7(mod 9) 16. 1(mod 3)
17. 11(mod 13) 18. 6(mod 11) 19. 11(mod 12)
20. 5(mod 9) **G.** 1. 3(mod 4) 2. 3(mod 5) 3. 0(mod 6)
4. 1(mod 11) 5. 2(mod 9) 6. 2(mod 7) 7. 0(mod 8)
8. 3(mod 12) 9. 6(mod 13) 10. 5(mod 15) 11. 3(mod 7)
12. 1(mod 8) 13. 7(mod 11) 14. 8(mod 12) 15. 0(mod 12)
16. 0(mod 13) 17. 8(mod 15) 18. 12(mod 14)
19. 0(mod 16) 20. 3(mod 9)

H. 1. {1, 3} 2. {3} 3. {1, 3} 4. {1, 3} 5. {4} 6. {11}
7. {5} 8. {7} 9. {4} 10. {1, 3, 5} 11. {2} 12. {2}
13. {2, 3} 14. { } 15. {2, 3}

CHAPTER 13

A. 1. 5050   2. 7.1   3. 0.002

B. 1. 96×10^{-4} 2. 6.497×10^{3} 3. 6×10^{-7} 4. 1.5×10^{-1} 5. 5×10^{-3}

6. 46.4   7. 2.1×10^{1}

CHAPTER 14

A. 1. $\frac{2}{5}$ 2. 9   3. $-2,5$ 4. $-3,1$ 5. $1\frac{3}{4}$ 6. $1\frac{1}{2}, 2\frac{1}{2}$ 7. $-\frac{3}{2}, 5$ 8. $-4, \frac{3}{2}$

B. (i) 15 (ii) $-\frac{2}{5}$

C. $1 \pm\sqrt{\frac{5}{3}}$ 2 *undefined*   3. $\pm\sqrt{\frac{2}{9}}-\frac{1}{3}$

D. 1. 11 *and* 12   2. 3 *or* $\frac{1}{3}$ 3. -14 *or* 12

E. $(x+3)(x+1+m)$

G. 1. $x=\dfrac{-5+\sqrt{13}}{2}, \dfrac{-5-\sqrt{13}}{2}$ 2. *No real solution*   3. $x=-1,-2$

7. $x=-1, \dfrac{-5}{3}$ 10. $x=-1, \dfrac{-4}{3}$

1. $x=-1, \frac{2}{3}$ 2. $x=\dfrac{-7+\sqrt{37}}{6}, \dfrac{-7-\sqrt{37}}{6}$

4, 7, 8 10 *have no real solutions.*

9. $x=\dfrac{-5+\sqrt{19}}{168}, \dfrac{-5-\sqrt{19}}{168}$

CHAPTER 15

A. 1. 312_6 2. 11000_2 3. 1111221_3 4. 403_{12}

B. 1. $-7, 5$ 2. 6   3. 5   4. 10   5. $-\frac{11}{2}$ *or* 5

C. 1. 8   2. 48   3. 486   4. 109   5. 29

D. 1. 1132_4 2. 123_5

CHAPTER 16

A. a. 14   b. 9   c. -36   d. 24 - 5n   e. 21st term

B. -5865

C. (i) ¢5000.00   (ii) ¢16,400.00

D. -125
E. 60
F. (i) 25 (ii) 31.25 (iii) 61.04 (iv) 11
G. 55880
H. $(a)\ U_n = 2n-1$ $(b)\ U_n = (-1)^{n+1}$ $(c)\ U_n = 2n-2$ $(d)\ U_n = 3n$
$(e)\ U_n = 3n+1$ $(f)\ U_n = (-1)^n 2^{n-1}$ $(g)\ U_n = (n-1)^2$
$(h)\ U_n = 6n-6$ $(i)\ U_n = 5n+2$ $(j)\ U_n = 2n-6$
I. 1. 30 2. 24 3. $\frac{31}{32}$
J. 1. $\sum_{i=1}^{6} i$ 2. $\sum_{i=1}^{6}(-1)^i(2i-1)$ 3. $\sum_{i=1}^{6} i^2$ 4. $\sum_{i=1}^{4}\frac{1}{2+i}$
K. 51
L. 4
M. 26
N. 17
O. (a) 1176 (b) 330 (c) -481 (d) 435 (e) -540
(f) 150 (g) -308
P. $(a)\ U_n = \frac{1}{3}(3)^{n-1}$ $(b)\ U_n = 8\left(-\frac{1}{2}\right)^{n-1}$ $(c)\ U_n = 2(-2)^{n-1}$ $(d)\ U_n = -\frac{1}{3}\left(\frac{3}{4}\right)^{n-1}$

CHAPTER 17

A. (a) (i) 38 (ii) 192 (iii) 2 (b) 3
B. 1. -1 2. -1 3. 9 4. 9
C. 1. -21 2. -5 3. 3.231 4. -1 or 9
D. 1. $\frac{28}{11}$ 2. $\frac{28}{11}$ 3. $\frac{140}{83}$ 4. $\frac{35}{12}$ 5. $\frac{140}{103}$
E. 1. 2 2. 1 3. 1 4. 0
F. 1. $\frac{30}{11}$ 2. $\frac{30}{31}$ *the operation is both Associative and Commutative*
G. 1. Commutative, Associative 2. Commutative, Associative
3. Commutative, Associative 4. Not Commutative, Associative
5. Commutative, Not Associative 6. Not Commutative, Associative
H. 1. Closed 2. (i) Commutative (ii) Associative 3. 0 4. Yes

CHAPTER 18

A. 1. $1-3x$ 2. $\frac{a-1}{a+1}$ 3. xy 4. $\frac{3}{(x-1)(x+2)}$

B. 1. $(2m-7xp)(2m+7xp)$ 2. $y(3x-5z)(3x+5z)$ 3. $(6-5x)(6+7x)$

C. 1. 29 2. ± 7 3. 361 4. 0.96

D. 1. 760 2. 2300 3. 4

E. ***Hint:*** Use $\left(x+\frac{6}{x}\right)^2 = 5^2$

CHAPTER 19

A. (i) defines a function

B. $-\frac{1}{2}$

C. $(i)\ -\frac{2}{3}$ $(ii)\ 0\ or\ -1$ $(iii)\ 1\ or\ 2$

D. $(i)\ Q=\{-2,-1,2\}$

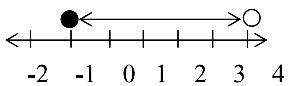
E. -1

F. $(i)\ 1+2x$ $(ii)\ 2x-x^2-2$ $(iii)\ x<-1$

G. $y=3x-1$

CHAPTER 20

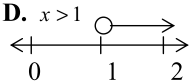
A. $-1 \le x < 4$



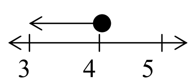
B. $x<4$

C. $\{x : x < 2\}$

D. $x>1$



E. $\{x : x \le 4\}$



F. $x \leq -\frac{41}{13}$

G. $\{x : x < \frac{7}{4}\}$

CHAPTER 21

A. $1\frac{1}{2}hrs$

B. $4.5\ m/s$

C. ¢ 27191

D. $54\ seconds$

E. $3hrs\ 7\frac{1}{2}\ mins.$

F. $2505\ cm$

G. $37.5km/h$

H. $15km/h$

I. $20km/h$

J. $9.6km/h$

CHAPTER 22

A. $(i)\ R = 450 + 0.2V^2 \quad (ii)\ 1730N$

B. $(a)\ F = \frac{2Mm}{d^2} \quad (b)\ 1.41m$

C. $T = \frac{K^2 l}{f^2}$

D. ¢ 4,500.00

E. 1.78

F. $\frac{6}{5}$

G. $\frac{1}{6}$

H. 2, 75, $y = 2x^3$

I. (a) 44.1cm^2 (b) 3cm

J. 600N

K. (a) 1 (b) $0.0625(\frac{1}{16})$

L. (a) 36 (b) 9 (c) 432

M. $(a)\ x = \frac{1}{108}y^2z^3 \quad (b)\ \frac{10}{27}$

N. $(a)\ 17\frac{7}{9}$ $(b)\ 20$
O. (a) 80 (b) 10
P. $14\frac{1}{16}\ or\ 14.0625$
Q. $(a)\ y = mx + c,\ m = -1,\ c = 1$ $(b)\ -1$
R. $(a)\ 10.2$ $(b)\ \frac{1}{\sqrt{2}}\ or\ \frac{\sqrt{2}}{2}$

CHAPTER 23

A. $(c)\begin{pmatrix} -4 \\ -14 \end{pmatrix}$
B. $(c)\begin{pmatrix} 10 \\ -5 \end{pmatrix}$
C. $(c)\ (i)\ 3y = -14x + 63$ $(ii)\ 6\ units$
D. $(c)\ (i)\ reflection\ in\ the\ line\ y = 0$ $(ii)\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
E. $Translation\ by\ the\ vector \begin{pmatrix} -3 \\ 0 \end{pmatrix}$
F. $(9, -4)$

G. $(a)\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $(b)\ A^1(2, -3)$
$(c)\ (i)\ P^1(2,4),\ Q^1(4,3), R^1(3,2)$ $(ii)\ P(4,0),\ Q(6,-1), R(5,2)$
$(iii)\ P^{11}(4,-2),\ Q^{11}(-1,-6),\ R^{11}(2,-5)$
I. (ii) $P_1(3, -2)$, $Q_1(9, -8)$, $R_1(2, -7)$ (iii) $P_2(-2, -3)$, $Q_2(-8, -9)$, $R_2(-7, -2)$ (c) 101.3^o

CHAPTER 24

A. $(d)\ (i)\ x = -1, 3$ $(ii)\ x = -1.7, 3.1$
B. $(c)\ (i)\ x = -1$ $(ii)\ x = -2, 2.5$ $(iii)\ -3 < x < 1$
C. $(c)\ greates\ value\ is\ y = 4\ and\ occurs\ at\ x = 1$
D. $(iii)\ y = -8.6$
E. $(c)\ (i)\ x = -1.5,\ 3.4$ $(ii)\ x = -1.3,\ 2.2$
F. $(c)\ (i)\ x = 1.25$ $(ii)\ 4$ $(iii)\ x = -0.2, 2.7$

G. $(c)\ (i)\ x=-1.1,\ 3.1$ $(ii)\ x=-1.4,\ 3.4$ $(d)\ 1$

H. $(c)\ (i)\ x=-1.5,\ 4.5$ $(ii)\ x=-1.2,\ 4.2$ $(iii)\ y=9.6$

I. (a)

x	-5	-4	-3	-2	-1.5	-1	-0.5	0	1	2	2.5
y	22	**9**	0	**-5**	**-6**	-6	**-5**	-3	1	**15**	22

(c) x = -3.7 or -0.7

J. (a)

x	-4	$-3\frac{1}{2}$	-3	-2	-1	0	1	$1\frac{1}{2}$	2
y	-5	0	4	9	10	7	0	-5	-11

(c) (i) -1.25 (ii) 10.1 (iii) -3.3, 0.7 (iv) -3<x<1

CHAPTER 25

A. $\frac{3}{8}$

B. $\frac{2}{5}$

C. $\frac{2}{3}$

D. $P_r(X\ or\ Y)=P_r(X)+P_r(Y)$

E. $(i)\ \frac{5}{12}$ $(ii)\ \frac{11}{24}$

F. (a) 0 (b) 1 (c) $\frac{1}{2}$

G. $\frac{19}{25}$

$H.\ \frac{19}{36}$

$I.\ \frac{1}{50}$

$J.\ (a)\ \frac{11}{30}$ $(b)\ \frac{4}{75}$ $(c)\ \frac{11}{50}$ $(d)\ \frac{4}{75}$ $(e)\ \frac{23}{75}$

K. $(a)\ \frac{1}{5}$ $(b)\ \frac{4}{5}$

L. $(a)\ \frac{7}{9}$ $(b)\ \frac{2}{9}$

CHAPTER 26

A. $(a)\ (i)\ 26000\ \ (ii)\ 34000\ \ (b)\ (ii)\ 19.2$

B. $(b)\ 5m \qquad (c)\ \frac{7}{15}$

C. $(b)\ (i) 7.25\ \ and\ \ 19.75 \qquad (ii)\ 19\%$

D. $(i)\ 35.5 \qquad (ii)\ 48$

E. $(a)\ 6.4$

G. 2

H. 6

I. 60

J. (i) 11 (ii) 3.0

K. 22

M. (a) 35 (b) 37**:**54

N. (c) (i) 23 (ii) 16%

O. (b) 41years

P. (b) (i) 62% (ii) 44% (iii) 50%

Q. (a) 71kg, 85kg (b) 76kg, 91kg (c) 69% (d) 23%

R. (a) 5.5, 2.9 (b) 15, 2.2 (c) 3, 1.3 (d) 5, 3.0
(e) 187, 1.5 (f) 46, 3.0

CHAPTER 27

A. $\begin{pmatrix} 17 \\ 10 \end{pmatrix}$

B. 13

C. $(a)\ 48^o \qquad (b) \begin{pmatrix} 8.03 \\ 8.92 \end{pmatrix}$

D. (a) (i) B(2, 6) and D(1, -1) (ii) $\overrightarrow{BC} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}\ \ and\ \ \overrightarrow{AD} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

(b) $\overrightarrow{BC} = \overrightarrow{AD}$

E. (10, 2)

F. $(a)\ B(9,10)\ and\ C(8,4)\ \ (b)\ \overrightarrow{AM} = (\frac{9}{2}, 0)$

G. $(i)\ \overrightarrow{QR} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} and\ \overrightarrow{TS} = \begin{pmatrix} 5-x \\ 4-y \end{pmatrix} \Rightarrow x = 4,\ y = 8$

$(ii)\ \sqrt{13} = 3.606 units \qquad (iii)\ (3.606 units,\ 056.31^o)$

(b) x=1, y=2

H. $(i)\ m = 2\ and\ n = 3 \quad (ii)\ |d| = 10.82$

I. $(i)\begin{pmatrix}-3\\5\end{pmatrix} \quad (ii)\ 3\sqrt{5}$

CHAPTER 28

A. (i) p + n (ii) 3.6%
B. px + 40q
C. ¢21,750.00
D. 7:5
E. (i) ¢72,000
F. ¢280,000
H. $33\frac{1}{3}$
I. 4
J. 5%
K. 2.5%
L. (i) ¢38,125
M. (i) (a) 300 (b) 200 (ii) (a) ¢192,000 (b) ¢32,000
N. ¢4,400,000
O. (a) 160 (b) ¢187,500.00
P. ₦798
Q. 6
R. (a) ¢48990.00 (b) ¢3990.00 (c) 30.4%
S. (a) ¢5125.00 (b) ¢3152.72 (c) ¢31392.00 (d) ¢13282.31
(e) ¢33124.00
T. (a) ¢18900.00 (b) ¢22506.12; difference is ¢3606.12
U. ¢146545.50
V. (a) 16.7% (b) $5742.20

CHAPTER 29

A. (ii) 47
B. (i) 60 (ii) 20
C. x = 6, y = 2 and least cost = ¢18.00
F. (b) (i) 6 and 2 (ii) 1 (iii) 18, -10
G. (b) (i) 2 and 7/2 (ii) -1

CHAPTER 30

A. ¾
B. 52cm
C. $\frac{1}{\sqrt{q^2-1}}$
D. 90^o
E. $\frac{15}{17}$
F. $\frac{8}{31}$
G. 60^o
H. $3\frac{3}{5}$
I. 14
J. $\sqrt{2}$
K. 256km
L. $\frac{10Sin48^o}{Sin32^o}$
M. 15m
N. 14m
O. 7.7m
P. 3905m
Q. 14^o
R. $3\frac{3}{5}$
S. $\sqrt{2}$
T. 7.2m
U. 5.8cm
V. 8.7m
W. 83m

CHAPTER 31

A. 2.63cm
B. 2
C. $16cm^2$
D. 5.6cm
E. (i) 2/3 (ii) 18.6cm (3.48cm

CHAPTER 32

A. (d) 6.1cm
B. (b) (i) 11.8cm (ii) 8.6cm
C. (b) (ii) 4cm
D. (d) (i) 11cm (ii) 5.7cm
E. (b) (i) 7.3cm (ii) 3.3cm (iii) 60^{o}
F. (c) (i) 4cm (ii) 58^{o}
G. (b) (i) 7.6cm (ii) 12cm
H. (iv) 5.3cm
I. $<ABC = 70^{o}$
J. $|RQ| = 8.5cm$
K. $|DF| = 9cm \ and < DFE = 59^{o}$
L. $|AC| = 13.5cm$
M. $|RS| = 3.6cm \ and < QRS = 80^{o}$

CHAPTER 33

A. $\begin{pmatrix} -25.98 \\ -15.00 \end{pmatrix}$
B. (5km, 143^{o})
C. $\begin{pmatrix} 2.3 \\ 2.5 \end{pmatrix}$
D. 190^{o}
F. 909m
G. (5km, 77^{o})
I. (a) (i) 13m (ii) 206^{o} (b) 8km
J. (a) (i) 8.72km (ii) 76.6^{o} (iii) 5.64km **K.** 140^{o}

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