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Existence and Uniqueness Solution of an Optimal Control Problem Via Stochastic Differential Equation

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***Abstract*--This paper deal with the existence and uniqueness solutions of an optimal control problem using stochastic differential equations, the important properties and the solutions of such equations. A particular consequence is the connection with the classic partial differential equation (PDE) methods for studying diffusions, the Kolmogorov forward (Fokker-Planck) and backward equations. Where the Stochastic Differential Equations (SDE) is considered as an ordinary differential equations (ODE) driven by white noise we justified the connection between the Ito's integral and white noise in the case of non-random integrands (interpreted as test functions). The sequence of ODEs, driven by approximations to white noise limiting to an SDE which is very important in the stochastic modelling of physical systems and simulation of SDE on a computer was also considered.**

***Keywords*-- Existence, uniqueness, stochastic, optimal, white noise**

I. INTRODUCTION

We let R^n be the real $n-$dimensional linear space of column vector x with components $x_1 \dots x_n$ and C^n be the corresponding space of complex column vectors.

For $X \in R^n, X^t$ will denote the transpose of X. $X \in C^n, X^H$ be the conjugate transpose.

A real $M \times N$ matrix $A = (a, i, j)$ defines a linear mapping from R^n to R^m, written as $A \in R^{mxn}$ or $A = L(R^n, R^m)$ to denote either the matrix or the linear operator respectively.

Similarly, for a complex $M \times N$ matrix A will be written as $A \in R^{mxn}$ or $A \in L(C^n, C^m)$

II. STOCHASTIC PROCESSES

Let (Ω, F, P) be a probability space. Suppose that I is a set of non-specified instants time occasional by $[0, \infty]$.

Let (E, ε) be another measurable space. If the set E is endowed with a metric, then, the Borel $\sigma-$algebra that is generated by this metric will be denoted by $\beta(E)$.

Mostly E will be $\Re$ or $\Re^d$ and ε the ordinary Borel $\sigma-$algebra on it. Then a random element of E is a map from Ω into E that is F/ε measurable while a stochastic process X with time set I is a collection $\{X_t, t \in I\}$ of random elements of E. For each ω the map $t \to X_t(\omega)$ is called a path, trajectory or realization of X.

We consider the stochastic differential equation of the form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$
$$X_0 = x \qquad (1)$$

Which is a suggestive notation for the Ito' process of the form:

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \qquad (2)$$

If there exists a stochastic process X_t that satisfies equation (2), we say that it solves the stochastic differential equation (1).

Here we find conditions on the coefficients b and σ that guarantee the existence and uniqueness solutions.

We let W_t be an m-dimensional Wiener process, A be a $n \times n$ matrix and B be a $n \times m$ matrix.

Then, the n-dimensional equation:

$$dX_t = AX_t dt + BdW_t \qquad X_0 = x \qquad (3)$$

is called a linear stochastic differential equation which can be solve explicitly by

$$X_t = e^{At}x + \int_0^t e^{A(t-s)}BdW_s \qquad (4).$$

Applying Ito's rule, the solution is unique.

We let Y_t be another solution with the same initial condition.

Then, $X_t - Y_t = \int_0^t A(X_s - Y_s)ds \qquad (5)$

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$$\frac{d}{ds}(X_t - Y_t) = A(X_t - Y_t),\ X_0 - Y_0 = 0 \quad (6)$$

This yields a standard unique solution of: $X_t - Y_t = 0$

For nonlinear b and σ we write down the solution in explicit form, which resorts to a less explicit proof of existence where it is often resolved by imposing Lipchitz conditions and proved by Picard iteration.

Let a function $f : \Re^n \to \Re^m$ be Lipchitz continuous of the form:

$$\exists K < \infty : \|f(x) - f(y)\| \le K\|x - y\|, \forall, x.y \in \Re^n.$$
(7)

A function $g : S \times \Re^n \to \Re^m$ is Lipchitz uniformly in S if $\|g(s,x) - g(s,y)\| \le K\|x - y\|$ for a constant $K < \infty$ which does not depend on S.

For simplicity, we concentrate on the finite time horizon $[0,T]$, where we consider filtered probability space $(\Omega, F, \{F_t\}_{t\in[0,T]}, \wp)$ on which we defined an m-dimensional F_t-Wiener process W_t.

We let X_0 be an F_0-measurable n-dimensional random variable, we then seek a solution to the equation:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \quad (8)$$

Where $b : [0,T] \times \Re^n \to \Re^m$ and $\sigma : [0,T] \times \Re^n \to \Re^{n\times m}$ are at least measurable

III. EXISTENCE THEOREM

(i). We let $X_0 \in \mathcal{L}^2(\wp)$

(ii). b, σ are Lipchitz continuous uniformly on $[0,T]$

(iii). $\|b(t,0)\| and \|\sigma(t,0)\|$ are bounded on $t \in [0,T]$.

Then there exists a solution X_t associated with stochastic differential equation, moreover $X_t, b(t, X_t)$ and $\sigma(t, X_t)$ are in $\mathcal{L}^2(\mu_T \times \wp)$.

Proof

For any F_t-adapted $Y \in \mathcal{L}^2(\mu_T \times \wp)$

We introduce a nonlinear equation of the form:

$$(\beta(Y))_t = X_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dW_s \quad (9)$$

We assume that $\beta(Y)$ is an F_t-adapted process in $\mathcal{L}^2(\mu_T \times \wp)$.

We find a fixed point of the operator β. where an F_t-adapted process yields:

$$X \in \mathcal{L}^2(\mu_T \times \wp), \beta(X) = X \quad (10)$$

This is a solution of the stochastic differential equation. We show that β does indeed map to an F_t-adapted process in $\mathcal{L}^2(\mu_T \times \wp)$.

Then,

$$\|b(t,x)\| \le \|b(t,x) - b(t,0)\| + \|b(t,0)\| \le K\|x\| + K' \le C(1 + \|x\|) \quad (11)$$

Where $K, K', C < \infty$ are constants that do not depend on t. We then say that b satisfies a linear growth condition Clearly the same argument holds for σ.

Choosing constant $C : \|\sigma(t,x)\| \le C(1 + \|x\|)$

We estimate:

$$\|\beta(Y)\|_{2.\mu_T\times P} \le \|X_0\|_{2.\mu_T\times P} + \left\|\int_0^t b(s,Y_s)ds\right\|_{2.\mu_T\times P} + \left\|\int_0^t \sigma(s,Y_s)dW_s\right\|_{2.\mu_T\times P} \quad (12)$$

The first term yield:

$$\|X_0\|_{2.\mu_T\times P} = \sqrt{T}\ \|X_0\|_{2.P} < \infty \quad (13).$$

Applying Jensen's inequality, i.e

$(t^{-1}a_s d_s)^2 \le t^{-1}\int_0^t a^2{}_s d_s$ and $Y \in \mathcal{L}^2(\mu_T \times \wp)$

$$\left\|\int_0^t b(s,Y_s)ds\right\|^2{}_{2.\mu T\times p} \le T^2\|b(s,Y)\|^2{}_{2.\mu T\times P}$$

$$\le T^2C^2\|(1 + \|Y\|)\|^2{}_{2.\mu T\times p} \le \infty \quad (14)$$

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Estimating stochastic integral term by Ito isometric:

$$\left\|\int_0 \sigma(s,Y_s)dW_s\right\|^2_{2.\mu T\times p} \le T\|\sigma(,Y)\|^2_{2.\mu T\times P}$$

$$\le TC^2\|(1+\|Y\|)\|^2_{2.\mu T\times p} \le \infty \qquad (15)$$

$\Rightarrow \|\beta(Y.)\|_{2.\mu T\times p} < \infty$ for F_t adapted $Y \in \mathcal{L}^2(\mu_T \times \wp)$

Hence, $\beta(Y)$ is F_t-adapted

(ii).To shows that β is a continuous mapping.

If $\|Y^n. - Y.\|_{2.\mu T\times p} \to 0,$ then,

$$\|\beta(Y^n.) - \beta(Y.)\|_{2.\mu T\times p} \to 0$$

$$\|\beta(Y^n.) - \beta(Y.)\|_{2.\mu T\times P} \le T\|b(.,Y^n) - b(.,Y)\|_{2.\mu T\times p}$$

$$+\sqrt{T}\|\sigma(.,Y^n) - \sigma(.,Y)\|_{2.\mu T\times P} \qquad (16)$$

Apply the Lipchitz condition;

$$\|\beta(Y^n.) - \beta(Y.)\|_{2.\mu T\times P} \le K\sqrt{T}(\sqrt{T}+1)\|Y^n - Y\|_{2.\mu T\times P} \qquad (17)$$

Where K is a Lipchitz constant for both b and σ.

(iii). To show that $Y^n.$ is a Cauchy sequence in $\mathcal{L}^2(\mu_T \times \wp)$.

We recalled that:

$$\|(\beta(Z))_t - (\beta(Y.))_t\|_{2.p} \le \sqrt{t}\|b(.,Z) - b(.,Y.)\|_{2.\mu t\times p} + \|\sigma(.,Z.) - \sigma(.,Y)\|_{2.\mu t\times p} \qquad (18)$$

By Lipchitz's property, it yields:

$$\|(\beta(Z.))_t - (\beta(Y.))_t\|_{2.p} \le K(\sqrt{T}+1)\|Z - Y.\|_{2.\mu t\times p} \qquad (19)$$

We let $L = K(\sqrt{T}+1)$

By iteration;

$$\|\beta^n(Z) - \beta^n(Y)\|^2_{2.\mu T\times p} = \int_0^T \|(\beta^n(Z))_t - (\beta^n(Y))_t\|^2_{2.p} dt$$

$$\le L^2\int_0^T \|\beta^{n-1}(Z) - \beta^{n-1}(Y)\|^2_{2.\mu t_1\times p} dt \qquad (20)$$

$$\le \ldots\ldots \le L^{2n}\int_0^T \int_0^{t_1} \ldots\int_0^{t_{n-1}} \|Z - Y\|^2_{2.\mu t_n\times p} dt_n \ldots\ldots dt_1 .$$

$$\le \frac{L^{2n}T^n}{n!}\|Z - Y\|^2_{2.\mu T\times p}. \qquad (21)$$

$$\sum_{n=0}^{\infty}\|\beta^{n+1}(Y^0) - \beta^n(Y^0)\|_{2.\mu T\times p} \le \|\beta(Y^0) - Y^0\|_{2.\mu T\times p}\sum_{n=0}^{\infty}\sqrt{\frac{L^{2n}T^n}{n!}} < \infty \qquad (22)$$

Hence, $\beta^n(Y^0)$ is a Cauchy sequence in $\mathcal{L}^2(\mu_T \times \wp)$.

IV. Uniqueness

Suppose that X is the solution of the existence theorem. We let Y be another solution. Then we show that $X = Y.\ \mu_T \times \wp$.a.s since X_t and Y_t have continuous sample paths,

So, $X = Y.\ \mu_T \times \wp$.a.s . Implies that they are $\wp$ a.s. indistinguishable.

Let $Y \in \mathcal{L}^2(\mu_T \times \wp)$, then $\beta^n(Y) = Y.$ and $\beta^n(X) = X$.

$$\|Y. - X.\|^2_{2.\mu T\times p} = \|\beta^n(Y) - \beta^n(X)\|^2_{2.\mu T\times p}$$

$$\le \frac{L^{2n}T^n}{n!}\|Y. - X.\|^2_{2.\mu T\times p} \qquad (23)$$

As $n \to \infty$, $\|Y - X\|_{2.\mu T\times p} = 0 \Rightarrow X = Y.\ \mu_T \times \wp$.a.s

Using Ito' rule:

$$\|Y_t\|^2 = \|X_0\|^2 + \int_0^t \left(2(Y_s)^{\bullet} b(s,Y_s) + \|\sigma(s,Y_s)\|^2\right)ds + \int_0^t 2(Y_s)^{\bullet}\sigma(s,Y_s)dW_s \qquad (24)$$

We let $\tau_n = \inf\{t : \|Y_t\| \ge n\}$.

$$\mathrm{E}\left(\left\|Y_{t\wedge\tau_n}\right\|^2\right) = \mathrm{E}\left(\left\|X_0\right\|^2\right) + \mathrm{E}\left[\int_0^{t\wedge\tau_n}\left(2(Y_s)^{\bullet} b(s,Y_s) + \left\|\sigma(s,Y_s)\right\|^2\right)ds\right] \quad (25)$$

By linear growth condition, we estimate

$$\mathrm{E}\left(\left\|Y_{t\wedge\tau_n}\right\|^2\right) \le \mathrm{E}\left(\left\|X_0\right\|^2\right) + \mathrm{E}\left[\int_0^{t\wedge\tau_n}\left(2C\left\|Y_s\right\|\left(1+\left\|Y_s\right\|\right) + C^2\left(1+\left\|Y_s\right\|\right)^2\right)ds\right]. \quad (26)$$

By applying Fatou's lemma on the left hand and the monotone convergence on the right hand. As $n \to \infty$ using simple estimate

$$(a+b)^2 \le 2(a^2+b^2)$$

$$\Rightarrow \mathrm{E}\left(1+\left\|Y_t\right\|^2\right) \le \mathrm{E}\left(1+\left\|X_0\right\|^2\right) + 2C(2+C)\int_0^t \mathrm{E}\left(1+\left\|Y_s\right\|^2\right)ds \quad (27).$$

By Gromwell's property;

$$\mathrm{E}\left(1+\left\|Y_t\right\|^2\right) \le \mathrm{E}\left(1+\left\|X_0\right\|^2\right)e^{2C(2+C)t} \quad (28)$$

V. CONCLUSION

This paper examined the existence and uniqueness solutions of an optimal control problem using stochastic differential equations and revealed some important properties and the solutions of such equations by considering stochastic differential equations (SDE) as an ordinary differential equations (ODE) driven by white noise and justified the connection between the Ito's integral and white noise in the case of non-random integrands (interpreted as test functions).

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