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Least-Square Solution of Linear-Quadratic Inverse Eigenvalue Problem (LQIEP) in Hamiltonian Symmetric Matrices

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***Abstract*--This paper investigate the least-square solution of linear-quadratic inverse eigenvalue problem (LQIEP) of Hamiltonian symmetric matrices using the nonsingular value decomposition method where $n \times n$ are real matrix and represents its unique optimal approximation in the least-square solutions**

***Keywords*-- Hamiltonian, inverse, eigenvalue, symmetric, Least-square**

I. Introduction

Many results have been obtained by various researchers from the applications of the least-square problem in linear optimal control, estimate theory, structure design and analysis of various kind of problem. Lovass-Nagy and Powers (1976) studied the least-square solution of an inconsistent singular matrix equation. Sun (1988) gave a general expression of the least-square solution of the symmetric matrix inverse problem. Allwright (1988) proposed a condition for solvability of the least square solution of the simple matrix equation in positive semi definite symmetric matrices set by using the convex analysis method. Xie, Zhang and Hu (2000) obtained least-square solutions of inverse problem for bisymmetric matrices as well as solvability of an IEP by Oduro et al (2012), Oladejo et al (2014) and (2015). In this paper, we investigate and discuss the Least-square solution method of linear-quadratic inverse eigenvalue problem (LQIEP) of Hamiltonian symmetric matrices.

We let

$A \in \Re^{n\times n}, B \in \Re^{n\times m}, Q \in \Re^{n\times n} : Q = Q^T \geq 0$ and

$R \in \Re^{m\times m} : R = R^T > 0$

We consider a linear system of the form:

$$\dot{X} = Ax + Bu \quad X_0 = x_0 \qquad (1)$$

Where u is the admissible control unit and of the form: $u = \phi(t)$ For u to be the admissible control unit, ϕ must be a continuous function, the closed loop system must have a unique solution and its results in $\lim_{t\to\infty} x(t) = 0$.

Then, the control objective is to find a control strategy that minimizes the following cost functional:

$$J(x,\phi) = \int_0^\infty \left[X^T(t)Qx(t) + \phi^T(t)R\phi(t)\right]dt \quad (2)$$

Where: Q is a symmetric positive semi definite matrix i.e $x^T(t)Qx(t) \geq 0$, R is a symmetric positive definite matrix i.e

$\phi^T(t)R\phi(t) > 0$ unless $\phi(t) = 0$

Then, $x^T(t)Qx(t) = \sum_{i=1}^{n} q_i x^2{}_i(t) \qquad (3)$

Where $q_i(s)$ is the magnitude of $x_i(t)$.

For stability, we let (A,B) be stable and introduce K as feedback control such as that the closed loop system:

$$\dot{X} = (A + Bk)x \qquad (4)$$

Where $u = Kx$ and clearly admissible

Which gives $x(t) = e^{(A+Bk)t}x_0$ and satisfy uniqueness condition of the closed loop

Then the cost function yield:

$$J(x_0, K_x) = x_0{}^T \int_0^\infty e^{(A+Bk)^T t}(Q + K^T RK)e^{(A+Bk)t}x_0$$
(5)

Solving equation (5) i.e. the optimal control problem by dynamic programming we let the instantaneous cost for the final state of $x_0 = x$ be:

$$L(x,u) = x^T Qx + u^T Ru \qquad (6)$$

We define the optimal cost or value function as: $V(x) = \inf_{\phi\in u} J(x,\phi) \qquad (7)$

Where $\inf$ the greatest is lower bound and $u(t), 0 \le t \le r$ be the control over $[0, r]$

Then, $J = \int_0^r L(x(t), u(t)dt + V(x(r))$ (8)

$u(t)$ is an arbitrary and the optimal cost satisfies the equation

$$V(x) = \min_{v(t)0 \le t \le r} \left[\int_0^T L(x(t), u(t)dt + V(x(r)) \right] \quad (9)$$

As $\frac{\sigma(r)}{r} \to 0, r \to 0$

$$\Rightarrow \int_0^T L(x(r), u(t)dt - rL(x,u) + 0(r) \quad (10)$$

$$V(x(r) = v(x) + r\frac{\partial v}{rx}(x)(Ax + Bu) + 0(r) \quad (11)$$

Substituting (11) into the equation (9) we get the Hamilton-Jacobi Bellman (HJB) equation for V Satisfies by $V(x)$ .i.e.

$$\min_{u \in R^n} \left\{ \frac{\partial v}{\partial x}(x)(Ax + Bu) + L(x,u) \right\} = 0 \quad (12)$$

We let $R > 0$, minimizing u from equation (12) yields:

$$u^T Ru + 2\alpha^T u + \beta = (u + R^{-1}\alpha) + \beta - \alpha^T R^{-1}\alpha$$
(13)

$$\Rightarrow \min_u (u^T Ru + 2\alpha^T u + \beta) = \beta - \alpha^T R^{-1}\alpha$$

Minimizing u by $u = -R^{-1}\alpha$, substituting into equation (13) yield;

$$\Rightarrow u = -\frac{1}{2}R^{-1}B^T \frac{\partial v}{\partial x}(x) \quad (14)$$

$$\frac{\partial v^T}{\partial x}(x)(Ax) + x^T Qx - \frac{1}{4}\frac{\partial v}{\partial x}(x)BR^{-1}B^T \frac{\partial v^T}{\partial x}(x) = 0$$
(15)

Solving equation (15), by apply trial

Solution. $V(x) = x^T px$ for $p \ge 0$,

$$\frac{\partial}{\partial x_k}(x^T px) = \frac{\partial}{\partial x_k}\sum_{ij=1}^n x_i p_{ij} x_j = \sum_j p_{kj} x_j + \sum_i x_i p_i k$$

$$= (Px)_k + (p^T x)_k \ = 2(Px)_k \quad (16)$$

Substituting into equation (15) yields:

$$x^T (A^T P + PA - PBR^{-1}B^T P + Q)x = 0 \quad (17)$$

Since this is true for all x . P Satisfy the matrix quadratic equation of the form:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (18)$$

This is called the Algebraic Riccati Equation

In term of P_i, the minimizing u would then be given as:

$$u = -R^{-1}B^T Px \quad (19)$$

Since $x_0 Px_o$ is constant and $u = -R^{-1}B^T Px$ is admissible with $R > 0$, then, the optimal control will be:

$$u(t) = -R^{-1}B^T Px(t)$$

While the optimal cost is: $V(x) = x^T Px$

Theorem 1

Let $p_\bullet$ be a solution of the Riccati equations:

$$\dot{P}_\bullet(t) = -P_\bullet(t)A - A^T P_\bullet(t) + P_\bullet(t)BR^{-1}B^T P_\bullet(t) - Q, t \in [t_i, t_f], p_\bullet(t_f) = 0 \quad (20)$$

We let $x_\bullet$ be the solution of the equation

$$\dot{X}_\bullet(t) = [A - BR^{-1}B^T P_\bullet(t)]x_\bullet(t), t \in [t_i, t_f], \quad x_\bullet(t_i) = x_i \quad (21)$$

Given that $P_\bullet(t) = P_\bullet(t)x_\bullet(t)$

Then $(x_\bullet, p_\bullet)$ is the unique solution of equation (20)

Proof

From the equations (20) and (21)

$$\frac{d}{dt}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} = \begin{bmatrix} CA - BR^{-1}B^T P_\bullet(t)x_\bullet(t) \\ P_\bullet(t)x_\bullet(t) + P_\bullet(t)\dot{X}_\bullet(t) \end{bmatrix} \quad (22)$$

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$$\begin{bmatrix} \dfrac{Ax_\bullet(t) - BR^{-1}B^T P_\bullet(t)}{\left(-P_\bullet(t)Ax_\bullet(t) - A^T P_\bullet(t)x_\bullet(t)\right)} + \dfrac{1}{\left(P_\bullet(t)BR^{-1}B^T P_\bullet(t)x_\bullet(t)\right)} \\ \dfrac{1}{-(Qx_\bullet(t) + P_\bullet(t)Ax_\bullet(t))} - \dfrac{1}{\left(P_\bullet(t)BR^{-1}B^T P_\bullet(t)x_\bullet(t)\right)} \end{bmatrix} \quad (23)$$

$$\Rightarrow \begin{bmatrix} Ax_\bullet(t) - BR^{-1}B^T P_\bullet(t) \\ -Qx_\bullet(t) - A^T P_\bullet(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A - BR^{-1}B^T \\ -Q - A^T \end{bmatrix} \begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} \quad (24)$$

Then $x_\bullet$ and $p_\bullet$ satisfies $x(t_i) = x_i$ and $P_\bullet(t_f) = P(t_f)x_\bullet(t_f) = 0x_\bullet(t_f) = 0$,

Then, $(x_\bullet, p_\bullet)$ satisfies equation (20) (ii). For uniqueness, if (x_1, p_1) and (x_2, p_2) satisfies equation (20) and

$\bar{x} = x_1 - x_2, \bar{p} = p_1 - p_2$ satisfies

$$\frac{d}{dt}\begin{bmatrix} \bar{x}(t) \\ \bar{p}(t) \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^T \\ Q - A^T \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{p}(t) \end{bmatrix}, t \in [t_i, t_f], \bar{x}(t_i)$$

$$\bar{p}(t_f) = 0 \quad (25)$$

$$\Rightarrow Q = \bar{p}(t_f)^T \bar{x}(t_f) - \bar{p}(t_i)^T \bar{x}(t_i)$$

$$= \int_{t_i}^{t_f} \frac{d}{dt}\left(\bar{p}(t)^T \bar{x}(t)\right) dt$$

$$= \int_{t_i}^{t_f} \begin{bmatrix} \left(-Q\bar{x}(t) - A^T \bar{p}(t)\bar{x}(t)\right) + \\ \left(\bar{p}(t)^T A\bar{x}(t) - BR^{-1}B^T \bar{p}(t)\right) \end{bmatrix} dt \quad (26)$$

$$= \int_{t_i}^{t_f} \left[\bar{x}(t)^T Q\bar{x}(t) + \bar{p}(t)^T BR^{-1}B^T \bar{p}(t)\right] dt \quad (27)$$

Then, from equation (27), we obtain

$$\dot{X}(t) = A\bar{x}(t), t \in [t_i, t_f], \bar{x}(t) = 0 \quad (28)$$

$$\dot{P}(t) = -A^T \bar{p}(t), t \in [t_i, t_f], \bar{p}(t) = 0 \quad (29)$$

Then $\bar{x}(t) = 0; \bar{p}(t) = 0, \forall, t \in [t_i, t_f]$

Thus the optimal trajectories $(x_\bullet, u_\bullet)$ are governed by:

$$\dot{x}(t) = \left[A - BR^{-1}B^T P_\bullet(t)\right]x_\bullet(t), t \in [t_i, t_f], \quad x_\bullet(t_i) = x_i \quad (30)$$

$$\dot{u}_\bullet(t) = -R^{-1}B^T P_\bullet(t)x_\bullet(t), t \in [t_i, t_f] \quad (31)$$

Where, $P_\bullet$ is the solution of the Riccati equation

$$\dot{P}_\bullet(t) = -P_\bullet(t)A - A^T P_\bullet(t) + P_\bullet(t)BR^{-1}B^T P_\bullet(t) - Q, t \in [t_i, t_f], P_\bullet(t) = 0 \quad (32)$$

Consequently;

$$\frac{d}{dt}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix}, t \in [t_i, t_f], x_\bullet(t_i) = x_i, p_\bullet(t_f) = 0 \quad (33)$$

Equation (33) is then a linear, time variant differential equation in $(x_\bullet, p_\bullet)$

We then consider the case where the Hamiltonian matrix $H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$ is $2n \times 2n$ so that *A*, *Q*, $BR^{-1}B^T$ are all is 2×2 sub-matrices of *H*.

Using appropriate row dependence relations, a 4×4 singular Hermitian matrix representing H and we assume that the singularity of the matrix is due to the row dependence relations specified above can be constructed as follow: $R_{i+1} = k_i R_1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$

$$a_{11}\begin{bmatrix} 1 & \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \\ k_1 & |k_1|^2 & k_1\bar{k}_2 & k_1\bar{k}_3 \\ k_2 & k_2\bar{k}_1 & |k_2|^2 & k_2\bar{k}_3 \\ k_3 & k_3\bar{k}_1 & k_3\bar{k}_2 & |k_3|^2 \end{bmatrix}$$

$$tr(A) = \lambda = a_{11}(1 + |k_1|^2 + |k_2|^2 + |k_3|^2)$$

To solve the inverse eigenvalue problem (IEP) for the singular matrix of rank 1 for the Hamiltonian equation associated with the LQOC problem we use the given nonzero eigenvalue as follows:

$$Ix_i(u) = \int_{t_i}^{t_f} \frac{1}{2}\left[x(t)^T Qx(t) + u(t)^T Ru(t)\right]dt$$

Subject to differential equation

$$\dot{X}(t) = Ax(t) + Bu(t), t \in [t_i, t_f], x(t_i) = x_i$$

We adopt the following Algorithm:

For the case of Non-Singular Matrix:

Given two distinct target eigenvalues λ_1, λ_2 (repeated for each)

Step 1: Determine the characteristic functions i.e.

$$f_1(a_{11}, a_{22}) = \lambda_1^{\ 2} - 2(trA)\lambda_1 + \det H$$

$$f_2(a_{11}, a_{22}) = \lambda_2^{\ 2} - 2(trA)\lambda_2 + \det H$$

Step 2: Find the Jacobian from the function where:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial a_{11}} & \frac{\partial f_1}{\partial a_{22}} \\ \frac{\partial f_2}{\partial a_{11}} & \frac{\partial f_2}{\partial a_{22}} \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda_1 + 2a_{22} & -\lambda_1 + 2a_{11} \\ -\lambda_2 + 2a_{22} & -\lambda_2 + 2a_{11} \end{bmatrix}$$

Step 3: Apply the Newton's method in H .i.e.

$$X^{(1)} = X^{(0)} - J^{-1}(X^{(0)})\underline{f}(X^{(0)})$$

Step 4: Substitute $X^{(0)} = \begin{bmatrix} a_{11}^{(0)} \\ a_{22}^{(0)} \end{bmatrix}$ into H and replacing the original diagonal element.

II. CONCLUSION

Various results have been obtained by various researchers from the applications of the least-square problem in linear-optimal control, estimate theory, structure design and analysis of various kind of problem. The usual approach to the LQIEP has been reviewed. Based on this results, we have systematically reviewed and investigate the least-square solution of linear-quadratic inverse eigenvalue problem (LQIEP) of Hamiltonian symmetric matrices using the nonsingular value decomposition method solutions

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