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Linear-Quadratic Optimal Control Problem (LQOCP) And The Definiteness of an Inverse Eigenvalue Problem (IEP) on A Certain Hermiltian Matrices

Oladejo N K[1], S. K Amponsah[2], F. T Oduro[3]
[1]University for Development Studies, Navrongo, Ghana
[2,3]Kwame Nkrumah University of Science and Technology, Kumasi

***Abstract*-- This paper investigate the definiteness of an inverse eigenvalue problem (IEP) in a certain Hermiltian matrices consists of both singular and non-singular symmetric matrices of rank 1 via Newton's method for solving the inverse eigenvalue problem for non-singular symmetric matrices in the neighborhood of the first type of matrices on linear-quadratic optimal control problem (LQOCP).**

***Keywords*-- Eigenvalue, symmetric, definite, indefinite, inverse, Hermiltian**

I. INTRODUCTION

Based on the recent methods of solving the inverse eigenvalue problem for certain matrices consisting singular symmetric matrices of rank 1 via Newton's method for solving the inverse eigenvalue problem for non-singular symmetric matrices (See Oladejo et.al (2014)) which is in support of some theoretical results on the solvability of the inverse eigenvalue problem for Hermiltian matrices together with numerical examples provided by Oduro et al (2012) and Oduro (2012a, b) as well as Baah Gyamfi (2012).

This paper investigate the definiteness of an inverse eigenvalue problem (IEP) in a certain Hermiltian matrices consists of both singular and non-singular symmetric matrices of rank 1 via Newton's method for solving the inverse eigenvalue problem for non-singular symmetric matrices in the neighborhood of the first type of matrices on linear-quadratic optimal control problem (LQOCP)

Linear Quadratic Optimal Control Problem (LQOCP)

Here we consider a linear system of the form:

$$\dot{x} = Ax + Bu \; ; x_0 = x_0 \qquad (1)$$

Where: u is the admissible control unit and be of the form: $u = \phi(t)$

The control objective is to find a control strategy that minimizes the cost functional.

$$J(x,\phi) = \int_0^{\infty} \left[X^T(t)Qx(t) + \phi^T(t)R\phi(t)\right]dt \quad (2)$$

Where Q is a symmetric positive semi definite matrix. R is a symmetric positive definite matrix. Thus, equation (2) is a control problem called **Linear Quadratic Control Problem**

Since Q is positive semi definite, then, $x^T(t)Qx(t) \geq 0$ and R is positive definite i.e. $\phi^T(t)R\phi(t) > 0$ unless $\phi(t) = 0$

Linear System and the differential Equation

We let

$A \in \Re^{n\times n}, B \in \Re^{n\times m}, Q \in \Re^{n\times n} : Q = Q^T \geq 0$ and $R \in \Re^{m\times m} : R = R^T > 0$

Finding the linear quadratic optimal control for the functional;

$$Ix_i(u) = \int_{t_i}^{t_f} \frac{1}{2}\left[x(t)^T Qx(t) + u(t)^T Ru(t)\right]dt \quad (3)$$

Subject to differential equation

$$\dot{X}(t) = Ax(t) + Bu(t), t \in \left[t_i, t_f\right], x(t_i) = x_i \quad (4)$$

Then the Hamiltonian functional is given by:

$$H(p,x,u,t) = \frac{1}{2}\left[x^T Qx + u^T Ru\right] + p^T\left[Ax + Bu\right] \quad (5)$$

From the above, it then follows that any optimal input $u_\bullet$ and the corresponding state $x_\bullet$ Satisfies:

$$\frac{\partial H}{\partial u}(p_\bullet(t), x_\bullet(t), u_\bullet(t), t) = 0$$

$$\Rightarrow u_\bullet(t)^T R + p_\bullet(t)^T B = 0 \qquad (6)$$

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Thus, $u_\bullet(t) = -R^{-1}B^T p_\bullet(t)$ and the adjoint equation is given as:

$$\left[\frac{\partial H}{\partial x}(p_\bullet(t), k_\bullet(t), u_\bullet(t), t)\right]^T = -\dot{p}_\bullet(t), t \in [t_i, t_f], p_\bullet(t_f) = 0$$

$$\Rightarrow (x_\bullet(t)^T Q + p_\bullet(t)^T A)^T = -\dot{p}_\bullet(t), t \in [t_i, t_f], \quad p_\bullet(t_f) = 0$$

Then,

$$\dot{p}_\bullet(t) = A^T p_\bullet(t) - Qx_\bullet(t), t \in [t_i, t_f], p_\bullet(t_f) = 0 \quad (7)$$

Consequently, we get the following equation (8) which is a linear and time variant differential equation in $(x_\bullet, p_\bullet)$ called Hamilton's Equation.

$$\frac{d}{dt}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix}, t \in [t_i, t_f], x_\bullet(t_i) = x_i, p_\bullet(t_f) = 0 \quad (8)$$

From equation (8) above, we consider the case where the Hamiltonian matrix

$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$ is $2n \times 2n$ so that *A*, *Q*, $BR^{-1}B^T$ are all is 2×2 sub-matrices of *H*. Using appropriate row dependence relations, a 4×4 singular Hermiltian matrix representing H above can be constructed as follows

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \\ k_1 & |k_1|^2 & k_1\bar{k}_2 & k_1\bar{k}_2 \\ k_2 & k_2\bar{k}_1 & |k_2|^2 & k_2\bar{k}_3 \\ k_3 & k_3\bar{k}_1 & k_3\bar{k}_2 & |k_3|^2 \end{bmatrix}$$

Here we assume that the singularity of the matrix is due to the row dependence relations specified below:

$$R_{i+1} = k_i R_1$$

$$\Rightarrow a_{22} = k_1(a_{12}) = k_1(a_{21}) = k_1{}^2 a_{11} = |k_1|^2$$

$$a_{23} = k_1(a_{13}) = k_1(a_{31}) = k_1 k_2 a_{11} = \bar{k}_1 k_2$$

$$a_{24} = k_1(a_{14}) = k_1(a_{41}) = k_1 k_3 a_{11} = k_1 k_3$$

$$a_{33} = k_2(a_{13}) = k_2(a_{31}) = k_2{}^2 a_{11} = |k_2|^2$$

$$a_{34} = k_2(a_{14}) = k_2(a_{41}) = k_2 k_3 a_{11} = k_2 \bar{k}_3$$

$$a_{44} = k_3(a_{14}) = k_3(a_{41}) = k_3{}^2 a_{11} = |k_3|^2$$

To solve the inverse eigenvalue problem (IEP) for the singular matrix of rank 1 we use the given nonzero eigenvalue as follows:

$$tr(A) = \lambda = a_{11}(1 + |k_1|^2 + |k_2|^2 + |k_3|^2)$$

So;

$$H = \frac{\lambda}{tr(A)}\begin{bmatrix} 1 & \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \\ k_1 & |k_1|^2 & \bar{k}_1 k_2 & k_1\bar{k}_2 \\ k_2 & k_2\bar{k}_1 & |k_2|^2 & k_2\bar{k}_3 \\ k_3 & k_3\bar{k}_1 & k_3\bar{k}_2 & |k_3|^2 \end{bmatrix}$$

Since we have assumed that the above is also a Hamiltonian matrix of the linear quadratic optimal control problem, we may partition it as follows:

$$\left[\begin{array}{cc|cc} 1 & \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \\ k_1 & |k_1|^2 & \bar{k}_1 k_2 & k_1\bar{k}_2 \\ \hline k_2 & k_2\bar{k}_1 & |k_2|^2 & k_2\bar{k}_3 \\ k_3 & k_3\bar{k}_1 & k_3\bar{k}_2 & |k_3|^2 \end{array}\right] = \begin{bmatrix} A & -BR^{-1}B^t \\ -Q & -A^T \end{bmatrix}$$

Where

R and Q are Hermiltian symmetric matrices and

$A = -A^T$

Thus we have

$$\bar{k}_1 = k_3\bar{k}_2 \Rightarrow k_1 = -k_3\bar{k}_2 \text{ ....}(i)$$

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$$\bar{k}_3 = k_2 k_1 \Rightarrow k_3 = \bar{k}_2 \, k_1$$

$$k_1 = -\left(\bar{k}_2 \, k_1\right) k_2 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$k_1 = \left(\bar{k}_2\right)^2 k_1 \Rightarrow \left(\bar{k}_2\right)^2 = -1$$

$$\bar{k}_2 = \sqrt{-1} \Rightarrow k_2 = -i \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

$$k_3 = ik_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

Substituting k_1, k_2, k_3 into the Hamiltonian matrix gives;

$$a_{11}\begin{bmatrix} 1 & \bar{k}_1 & i & -i\,\bar{k}_1 \\ k_1 & |k_1|^2 & i\bar{k}_1 & -i|k|^2 \\ i & -i\,\bar{k}_1 & 1 & -\bar{k}_1 \\ ik_1 & i|k_1|^2 & -k_1 & |k_1|^2 \end{bmatrix}$$

Thus; $tr(A) = \lambda = 2a_{11}\left(1 + |k_1|^2\right)$

Then the solution of the IEP is given by

$$H = \frac{\lambda}{2\left(1 + |k_1|^2\right)}\begin{bmatrix} 1 & \bar{k}_1 & i & -i\,\bar{k}_1 \\ k_1 & |k_1|^2 & i\bar{k}_1 & -i|k|^2 \\ i & -i\,\bar{k}_1 & 1 & -\bar{k}_1 \\ ik_1 & i|k_1|^2 & -k_1 & |k_1|^2 \end{bmatrix}$$

II. ILLUSTRATION

Given that $a_{11} = 1, k_1 = 2 \Rightarrow \lambda = 10$

Hence;

$$H = \begin{bmatrix} 1 & \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \\ k_1 & |k_1|^2 & k_1\,\bar{k}_2 & k_1\,\bar{k}_2 \\ k_2 & k_2\,\bar{k}_1 & |k_2|^2 & k_2\,\bar{k}_3 \\ k_3 & k_3\,\bar{k}_1 & k_3\,\bar{k}_2 & |k_3|^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & i & 2i \\ 2 & 4 & 2i & -4i \\ i & -2i & 1 & 2 \\ 2i & 4i & -2 & 4 \end{bmatrix}$$

Algorithms to determine the definiteness of an inverse eigenvalue problem (IEP) for nonsingular Hamiltonian symmetric matrix

To solve the inverse eigenvalue problem (IEP) for the Hamiltonian equation associated with the LQOC problem:

$$Ix_i(u) = \int_{t_i}^{t_f} \frac{1}{2}\left[x(t)^T Qx(t) + u(t)^T Ru(t)\right]dt$$

Subject to differential equation

$$\dot{X}(t) = Ax(t) + Bu(t), t \in \left[t_i, t_f\right], x(t_i) = x_i$$

Given the Hamiltonian equation of the form:

$$\frac{d}{dt}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix}, t \in \left[t_i, t_f\right], x_\bullet(t_i)$$

$$= x_i, \, p_\bullet(t_f) = 0$$

Solving the IEP for the matrix;

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

Since there are repeating diagonal elements we solve the IEP by Newton's method for two distinct target eigenvalues λ_1, λ_2 which therefore give rise to two (2) functions with independent variables being the diagonal elements of matrix A which is a sub-matrix of H:

Given two distinct target eigenvalues λ_1, λ_2 (repeated for each)

Step 1: Determine the characteristic functions i.e.

$$f_1(a_{11}, a_{22}) = \lambda_1^{\ 2} - 2(trA)\lambda_1 + \det H$$

$$f_2(a_{11}, a_{22}) = \lambda_2^{\ 2} - 2(trA)\lambda_2 + \det H$$

Step 2: Find the Jacobian from the function thus:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial a_{11}} & \frac{\partial f_1}{\partial a_{22}} \\ \frac{\partial f_2}{\partial a_{11}} & \frac{\partial f_2}{\partial a_{22}} \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda_1 + 2a_{22} & -\lambda_1 + 2a_{11} \\ -\lambda_2 + 2a_{22} & -\lambda_2 + 2a_{11} \end{bmatrix}$$

*Step 3.*Establish the Determinant

$$Det = 2(\lambda_1 - \lambda_2)(a_{22} - a_{11})$$

Step 4 Find the inverse of 2×2 Jacobian matrix is given as:

$$J^{-1} = \frac{1}{2(\lambda_1 - \lambda_2)(a_{22} - a_{11})} \begin{bmatrix} 2a_{22} - \lambda_1 & 2a_{11} - \lambda_1 \\ 2a_{22} - \lambda_2 & 2a_{11} - \lambda_2 \end{bmatrix}$$

Step 5: Apply the Newton's method in H.i.e.

$$X^{(n+1)} = X^{(n)} - J^{-1}(X^{(0)}) \underline{f}(X^{(n)})$$

Step 6: Substitute

$X^{(0)} = \begin{bmatrix} a_{11}^{\ (0)} \\ a_{22}^{\ (0)} \end{bmatrix}$ into H replacing the original diagonal element.

*Step 7.*Determine the definiteness of the matrix

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Numerical examples on the definiteness of an inverse eigenvalue problem (IEP)

$(\lambda_1 \quad \lambda_2)$	$(1 \quad 2)$	$(-1 \quad -2)$	$(0 \quad 1)$	$(0 \quad -1)$	$(1 \quad -2)$
$f\left(x^{(0)}\right)$	$\begin{bmatrix}-9\\-16\end{bmatrix}$	$\begin{bmatrix}11\\24\end{bmatrix}$	$\begin{bmatrix}0\\-9\end{bmatrix}$	$\begin{bmatrix}0\\11\end{bmatrix}$	$\begin{bmatrix}-9\\24\end{bmatrix}$
$\begin{bmatrix}\frac{\partial f_1}{\partial a_{11}} & \frac{\partial f_1}{\partial a_{22}}\\ \frac{\partial f_2}{\partial a_{11}} & \frac{\partial f_2}{\partial a_{22}}\end{bmatrix}$	$\begin{bmatrix}7 & 1\\6 & 0\end{bmatrix}$	$\begin{bmatrix}9 & 3\\10 & 4\end{bmatrix}$	$\begin{bmatrix}8 & 2\\7 & 1\end{bmatrix}$	$\begin{bmatrix}8 & 2\\9 & 3\end{bmatrix}$	$\begin{bmatrix}7 & 1\\10 & 6\end{bmatrix}$
$Det = 2(\lambda_1 - \lambda_2)(a_{22} - a_{11})$	-6	+6	-6	+6	18
$\frac{1}{Det}\begin{bmatrix}2a_{22}-\lambda_1 & 2a_{11}-\lambda_1\\2a_{22}-\lambda_2 & 2a_{11}-\lambda_2\end{bmatrix}$	$-\frac{1}{6}\begin{bmatrix}7 & 1\\6 & 0\end{bmatrix}$	$\frac{1}{6}\begin{bmatrix}9 & 3\\10 & 4\end{bmatrix}$	$-\frac{1}{6}\begin{bmatrix}8 & 2\\7 & 1\end{bmatrix}$	$\frac{1}{6}\begin{bmatrix}8 & 2\\9 & 3\end{bmatrix}$	$\frac{1}{18}\begin{bmatrix}7 & 1\\10 & 6\end{bmatrix}$
$X^{(n)} - J^{-1}(X^{(0)})\underline{f}(X^{(n)})$	$(X^{(1)}) = \begin{bmatrix}3.167\\3.667\end{bmatrix}$	$(X^{(1)}) = \begin{bmatrix}-0.5\\-3.223\end{bmatrix}$	$\left(X^{(1)}\right) = \begin{bmatrix}-2\\2.5\end{bmatrix}$	$\left(X^{(1)}\right) = \begin{bmatrix}-2.67\\-1.5\end{bmatrix}$	$\left(X^{(1)}\right) = \begin{bmatrix}3.17\\1\end{bmatrix}$
$\begin{bmatrix}1 & 2 & i & 2i\\-2 & 4 & -2i & 4i\\i & -2i & 1 & 2\\2i & 4i & -2 & 4\end{bmatrix}$	$\begin{bmatrix}3.2 & 2 & i & 2i\\-2 & 3.7 & -2i & 4i\\i & -2i & 3.2 & 2\\2i & 4i & -2 & 3.7\end{bmatrix}$	$\begin{bmatrix}-0.5 & 2 & i & 2i\\-2 & -3.2 & -2i & 4i\\i & -2i & -0.5 & 2\\2i & 4i & -2 & -3.2\end{bmatrix}$	$\begin{bmatrix}-2 & 2 & i & 2i\\-2 & 2.5 & -2i & 4i\\i & -2i & -2 & 2\\2i & 4i & -2 & 2.5\end{bmatrix}$	$\begin{bmatrix}-2.7 & 2 & i & 2i\\-2 & -1.5 & -2i & 4i\\i & -2i & -2.7 & 2\\2i & 4i & -2 & -1.5\end{bmatrix}$	$\begin{bmatrix}3.2 & 2 & i & 2i\\-2 & 1 & -2i & 4i\\i & -2i & 3.2 & 2\\2i & 4i & -2 & 1\end{bmatrix}$
Interpretation	+ve definite	-ve definite	indefinite	-ve semi definite	+ve semi definite

III. CONCLUSION

Based on the recent methods of solving the inverse eigenvalue problem for certain matrices consisting singular symmetric matrices of rank 1 via Newton's method for solving the inverse eigenvalue problem for non-singular symmetric matrices which is in support of some theoretical results on the solvability of the inverse eigenvalue problem for Hermitian matrices, we have successfully investigate and established the definiteness of an inverse eigenvalue problem (IEP) in a certain Hermiltian matrices consisting positive definite, negative definite, indefinite, negative semi definite and positive semi definite through numerical examples of a non-singular symmetric matrices of rank 1 via Newton's method in the neighborhood of the first type of matrices on linear-quadratic optimal control problem (LQOCP).

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