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Romberg-Type Integration Methods

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Abstract

In this paper we derived and studied two new numerical methods for solving definite integrals. The approach is based on the Newton-Cotes formulas and Romberg integration. Numerical examples are given to demonstrate the efficiency and accuracy of the methods. The results obtained by these methods are comparable to those of Romberg integration but with lesser computational cost involved.

Keywords: Newton-Cotes formula, definite integrals, Romberg integration, Richardson extrapolation.
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Introduction

Consider the definite integral

$$I = \int_a^b f(x)dx \qquad a \le x \le b \tag{1}$$

This equation is a mathematical model of many problems in science and engineering. Often the functions involved have no explicit anti-derivatives or are not easily obtained. Hence we use numerical techniques to approximate equation (1). Among numerous numerical methods for approximating equation (1) are Newton-Cotes formulas, Adaptive Quadrature methods, Hermite's Quadrature methods, Romberg integration and Gaussian Quadrature [1], [2], [5], and [6]. Romberg integration uses the extended trapezoidal rule (i.e. degree one Newton-Cote formula), which is not very accurate since its truncation error is only of $O(h^2)$, to obtain the preliminary estimates to (1) before extrapolation process is applied to improve the accuracy of the estimates. Romberg [4] first described extrapolation procedure in connection with the trapezoidal rule. The use of mid-point rule was also mentioned by Romberg and later by Haive [3].

In this paper, we introduce two numerical methods based on the Newton-Cotes formulas of degrees 2 and 4.

Method 1

The extended Newton-Cotes formula of degree 2 i.e. Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx = \frac{h}{3}\{f(x_0)+f(x_n)+4[f(x_0+h)+f(x_0+3h)+\cdots+f(x_n-h)] + 2[f(x_0+2h)+f(x_0+4h)+\cdots+f(x_n-2h)]\}+O(h^4) \tag{2}$$

With $h={}^{(b-a)}\!/_{2m}$ and $a=x_0<x_1<\cdots<x_{2m}=b$, where $x_l = x_0 + lh$ for each $l = 0,1,\cdots,2m$

The extended Simpson's rule is applied with 2^k (k=1,2,3,$\cdots$) subintervals to obtain preliminary estimates of (1). But as k increases the number of subintervals increases and the number of function evaluation grows exponentially. Each time we increase k by 1 we double the work. Therefore more computational effort is required to obtain the original Simpson's 1/3 estimates. However, the number of function evaluations is considerably reduced if $S_{k,1}$ is computed from a known value of $S_{k-1,1}$ using

$$S_{k,1} = \frac{1}{2}S_{k-1,1} + \frac{h_{k-1}}{6}\{4[f(x_0+\tfrac{1}{2}h_{k-1})+f(x_0+\tfrac{3}{2}h_{k-1})+f(x_0+\tfrac{5}{2}h_{k-1})+\cdots+f(x_0+(2^{k-1}-\tfrac{1}{2})h_{k-1})] - 2[f(x_0+h_{k-1})+f(x_0+3h_{k-1})+f(x_0+5h_{k-1})+\cdots+f(x_0+(2^{K-1}-1)h_{k-1})]\} \tag{3}$$

Where $h_k = {}^{(b-a)}\!/_{2^k}$

Now with (3), $S_{k,1}$ can be calculated using only 2^{k-1} additional function evaluations instead of 2^k+1 function evaluations which could have been required if $S_{k,1}$ were to be calculated directly from Simpson's 1/3 rule. We now prove the formula (3) above.

Proof

Let $S_{k,1}$ $(k=1,2,\cdots)$ denote the Simpson's 1/3 rule estimates of I using 2^k subintervals of width $h_k = {}^{(b-a)}\!/_{2^k}$ .Therefore

$$S_{k,1} = \frac{h_k}{3}\{f(x_0)+f(x_n)+4[f(x_0+h_k)+f(x_0+3h_k)+f(x_0+5h_k)+\cdots+f(x_0+(2^k-1)h_k)] + 2[f(x_0+2h_k)+f(x_0+4h_k)+\cdots+f(x_0+(2^k-2)h_k)]\} \tag{4}$$

Letting *k=k-1,* then we obtain from (4)

$$S_{k-1,1} = \frac{h_{k-1}}{3}\{f(x_0)+f(x_n)+4[f(x_0+h_{k-1})+f(x_0+3h_{k-1})+f(x_0+5h_{k-1})+\cdots + f(x_0+(2^{k-1}-1)h_{k-1})]+2[f(x_0+2h_{k-1})+f(x_0+4h_{k-1})+\cdots+f(x_0+(2^{k-1}-2)h_{k-1})]\} \tag{5}$$

Now substituting $h_k = \tfrac{1}{2}h_{k-1}$ in equation (4)

$$S_{k,1} = \frac{1}{3}*\frac{h_{k-1}}{2}\{f(x_0)+f(x_n)+4[f(x_0+\tfrac{1}{2}h_{k-1})+f(x_0+\tfrac{3}{2}h_{k-1})+f(x_0+\tfrac{5}{2}h_{k-1})+\cdots + f(x_0+(2^{k-1}-\tfrac{1}{2})h_{k-1})]+2[f(x_0+h_{k-1})+f(x_0+2h_{k-1})+\cdots+f(x_0+(2^{k-1}-1)h_{k-1})]\}$$

Then, the above equation can be written as

$$S_{k,1} = \frac{1}{2}*\frac{h_{k-1}}{3}\{f(x_0)+f(x_n)+4[f(x_0+\tfrac{1}{2}h_{k-1})+f(x_0+\tfrac{3}{2}h_{k-1})+f(x_0+\tfrac{5}{2}h_{k-1})+\cdots + f(x_0+(2^{k-1}-\tfrac{1}{2})h_{k-1})]+4[f(x_0+h_{k-1})+f(x_0+3h_{k-1})+\cdots+f(x_0+(2^{k-1}-1)h_{k-1})] + 2[f(x_0+2h_{k-1})+f(x_0+4h_{k-1})+\cdots+f(x_0+(2^{k-1}-2)h_{k-1})] - 2[f(x_0+h_{k-1})+f(x_0+3h_{k-1})+\cdots+f(x_0+(2^{k-1}-1)h_{k-1})]\}$$

Rearranging the above equation, we obtain

$$S_{k,1} = \frac{1}{2} * \frac{h_{k-1}}{3} \{ f(x_0) + f(x_n) + 4[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + f(x_0 + 5h_{k-1}) + \cdots$$
$$+ f(x_0 + (2^{k-1} - 1)h_{k-1})] + 2[f(x_0 + 2h_{k-1}) + f(x_0 + 4h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2)h_{k-1})]\}$$
$$+ \frac{1}{2} * \frac{h_{k-1}}{3} \{4[f(x_0 + \tfrac{1}{2}h_{k-1}) + f(x_0 + \tfrac{3}{2}h_{k-1}) + f(x_0 + \tfrac{5}{2}h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - \tfrac{1}{2})h_{k-1})]$$
$$-2[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1)h_{k-1})]\}$$

Ss substituting (5) in the above equation, we have

$$S_{k,1} = \frac{1}{2} S_{k-1,1} + \frac{h_{k-1}}{6} \{4[f(x_0 + \tfrac{1}{2}h_{k-1}) + f(x_0 + \tfrac{3}{2}h_{k-1}) + f(x_0 + \tfrac{5}{2}h_{k-1}) + \cdots$$
$$+ f(x_0 + (2^{k-1} - \tfrac{1}{2})h_{k-1})] - 2[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1)h_{k-1})]\} \quad (proved)$$

Hence the prove of equation (3).

In order to produce approximations of high order accuracy, we use linear combination of initial estimates by employing Richardson extrapolation. Now

$$I = S_{k,1} + \alpha_1 h_k^{\ 4} + \alpha_2 h_k^{\ 6} + \alpha_3 h_k^{\ 8} + \cdots \tag{6}$$

With halve the value of h and double the number of subintervals, then

$$I = S_{k+1,1} + \alpha_1 (\tfrac{h_k}{2})^4 + \alpha_2 (\tfrac{h_k}{2})^6 + \alpha_3 (\tfrac{h_k}{2})^8 + \cdots$$
$$= S_{k+1,1} + \alpha_1 \tfrac{h_k^4}{16} + \alpha_2 \tfrac{h_k^6}{64} + \alpha_3 \tfrac{h_k 8}{256} + \cdots \tag{7}$$

We shall now eliminate terms involving h_k^4 from (6) and (7) by multiplying (7) by 16 and subtracting (6), so that

$$I = \frac{1}{15}(16S_{k+1,1} - S_{k,1}) - \tfrac{1}{20}\alpha_2 h_k^{\ 6} - \tfrac{1}{16}\alpha_3 h_k^{\ 8} + \cdots$$
$$I = S_{k+1,2} - \tfrac{1}{20}\alpha_2 h_k^{\ 6} - \tfrac{1}{16}\alpha_3 h_k^{\ 8} + \cdots \tag{8}$$

Where

$$S_{k+1,2} = \frac{1}{15}(16S_{k+1,1} - S_{k,1}) \tag{9}$$

We see from (8) that $S_{k+1,2}$ is a sixth order approximation to *I* and so should be more accurate than either $S_{k,1}$ or $S_{k+1,1}$ which are fourth order approximations.

We repeat the extrapolation process to produce eighth order estimate of *I*. From (8), we have

$$I = S_{k,2} + \beta_1 h_k^{\ 6} + \beta_2 h_k^{\ 8} + \beta_3 h_k^{\ 10} + \cdots \tag{10}$$

We have h again, so that

$$I = S_{k+1,2} + \beta_1 (\tfrac{h_k}{2})^6 + \beta_2 (\tfrac{h_k}{2})^8 + \beta_3 (\tfrac{h_k}{2})^{10} + \cdots$$
$$I = S_{k+1,2} + \beta_1 \tfrac{h_k^6}{64} + \beta_2 \tfrac{h_k 8}{256} + \beta_3 \tfrac{h_k 10}{1024} + \cdots \tag{11}$$

We multiply (11) by 64 and subtract (10), in order to eliminate h_k^4. So, we have

$$I = \frac{1}{63}(64S_{k+1,2} - S_{k,2}) - \tfrac{1}{84}\beta_2 h_k^{\ 8} - \tfrac{5}{336}\beta_3 h_k^{\ 10} + \cdots$$
$$I = S_{k+1,3} - \tfrac{1}{84}\beta_2 h_k^{\ 8} - \tfrac{5}{336}\beta_3 h_k^{\ 10} + \cdots \tag{12}$$

Where

$$S_{k+1,3} = \frac{1}{63}(64S_{k+1,2} - S_{k,2}) \tag{13}$$

$S_{k+1,3}$ is an eighth order approximation to *I*.

This process can be continued, and generally if

$$I = S_{k,j} + \varepsilon_1 h_k^{\,2j+2} + \varepsilon_2 h_k^{\,2j+4} + \cdots \tag{14}$$

Then

$$I = S_{k+1,j} + \varepsilon_1 (\tfrac{h_k}{2})^{2j+2} + \varepsilon_2 (\tfrac{h_k}{2})^{2j+4} + \cdots \tag{15}$$

The terms involving $h_k^{\,2j+2}$ can be eliminated by multiplying (15) by $2^{2j+2} = 4^{j+1}$ and subtracting (14) to obtain

$$(4^{j+1} - 1)I = 4^{j+1} S_{k+1,j} - S_{k,j} - \tfrac{3}{4}\varepsilon_2 (h_k)^{2j+4} - \cdots$$

Or equivalently

$$I = S_{k+1,j+1} + O(h^{2j+4}) \tag{16}$$

Where

$$S_{k+1,j+1} = \frac{(4^{j+1} S_{k+1,j} - S_{k,j})}{4^{j+1} - 1} \;;\; j = 1,2,3,\cdots \quad k = 1,2,3,\cdots \tag{17}$$

Method II

For extended Newton-Cotes of degree 4 (i.e. Boole's rule)

$$\begin{aligned}\int_a^b f(x)dx = \frac{2h}{45}\{&7[f(x_0) + f(x_n)] + 32[f(x_0+h) + f(x_0+3h) + \cdots + f(x_n - h)] \\ &+ 12[f(x_0+2h) + f(x_0+6h) + \cdots + f(x_n - 2h)] + 14[f(x_0+4h) + f(x_0+8h) \\ &+ \cdots + f(x_n - 4h)]\} + O(h^6)\end{aligned} \tag{18}$$

With $h = {}^{(b-a)}\!/_{4m}$ and $a = x_0 < x_1 < \cdots < x_{4m} = b$, where $x_l = x_0 + lh$ for each $l = 0,1,\cdots,4m$ is used to obtain the preliminary estimates of (1). Similarly to reduce the number of function evaluations from 2^k+1 to additional 2^{k-1}, we calculate $P_{k,2}$ from $P_{k-1,2}$ with 2^k $(k = 2,3,4,\cdots)$ subintervals using the formula

$$\begin{aligned}P_{k,2} = \frac{1}{2}P_{k-1,2} + \frac{h_{k-1}}{45}\{&32[f(x_0 + \tfrac{1}{2}h_{k-1}) + f(x_0 + \tfrac{3}{2}h_{k-1}) + f(x_0 + \tfrac{5}{2}h_{k-1}) + \cdots \\ &+ f(x_0 + (2^{k-1} - \tfrac{1}{2})h_{k-1})] - 20[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + f(x_0 + 5h_{k-1}) + \cdots \\ &+ f(x_0 + (2^{k-1} - 1)h_{k-1})] + 2[f(x_0 + 2h_{k-1}) + f(x_0 + 6h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2)h_{k-1})]\}\end{aligned} \tag{19}$$

Where $h_k = {}^{(b-a)}\!/_{2^k}$

Proof

Let $P_{k,2}$ $(k = 2,3,\cdots)$ denote the Boole's rule estimates of I using 2^k subintervals of width $h_k = {}^{(b-a)}\!/_{2^k}$ Therefore

$$\begin{aligned}P_{k,2} = \frac{2}{45}h_k\{&7[f(x_0) + f(x_n)] + 32[f(x_0 + h_k) + f(x_0 + 3h_k) + f(x_0 + 5h_k) + \cdots \\ &+ f(x_0 + (2^k - 1)h_k)] + 12[f(x_0 + 2h_k) + f(x_0 + 6h_k) + \cdots + f(x_0 + (2^k - 2)h_k)] \\ &+ 14[f(x_0 + 4h_k) + f(x_0 + 8h_k) + \cdots + f(x_0 + (2^k - 4)h_k)]\}\end{aligned} \tag{20}$$

Letting *k=k-1,* then we obtain from (20)

$$\begin{aligned}P_{k-1,2} = \frac{2}{45}h_{k-1}\{&7[f(x_0) + f(x_n)] + 32[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + f(x_0 + 5h_{k-1}) + \cdots \\ &+ f(x_0 + (2^{k-1} - 1)h_{k-1})] + 12[f(x_0 + 2h_{k-1}) + f(x_0 + 6h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2)h_{k-1})] \\ &+ 14[f(x_0 + 4h_{k-1}) + f(x_0 + 8h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 4)h_{k-1})]\ \}\end{aligned} \tag{21}$$

Now substituting $h_k = \tfrac{1}{2}h_{k-1}$ in equation (20)

$$P_{k,2} = \frac{2}{45} * \frac{1}{2} h_{k-1} \{7[f(x_0) + f(x_n)] + 32[f(x_0 + \tfrac{1}{2} h_{k-1}) + f(x_0 + \tfrac{3}{2} h_{k-1}) + f(x_0 + \tfrac{5}{2} h_{k-1})$$
$$+ \cdots + f(x_0 + (2^{k-1} - \tfrac{1}{2}) h_{k-1})] + 12[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1) h_{k-1})]$$
$$+ 14[f(x_0 + 2h_{k-1}) + f(x_0 + 4h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2) h_{k-1})] \}$$

This above equation can be rewritten as

$$P_{k,2} = \frac{1}{2} * \frac{2}{45} h_{k-1} \{7[f(x_0) + f(x_n)] + 32[f(x_0 + \tfrac{1}{2} h_{k-1}) + f(x_0 + \tfrac{3}{2} h_{k-1}) + f(x_0 + \tfrac{5}{2} h_{k-1})$$
$$+ \cdots + f(x_0 + (2^{k-1} - \tfrac{1}{2}) h_{k-1})] + 32[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots$$
$$+ f(x_0 + (2^{k-1} - 1) h_{k-1})] - 20[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1) h_{k-1})]$$
$$+ 12[f(x_0 + 2h_{k-1}) + f(x_0 + 6h_{k-1}) + f(x_0 + 10h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2) h_{k-1})]$$
$$+ 14[f(x_0 + 4h_{k-1}) + f(x_0 + 8h_{k-1}) + f(x_0 + 12h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 4) h_{k-1})]$$
$$+ 2[f(x_0 + 2h_{k-1}) + f(x_0 + 6h_{k-1}) + f(x_0 + 10h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2) h_{k-1})]\}$$

Rearranging, this equation we have

$$P_{k,2} = \frac{1}{2} * \frac{2}{45} h_{k-1} \{7[f(x_0) + f(x_n)] + 32[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + f(x_0 + 5h_{k-1}) + \cdots$$
$$+ f(x_0 + (2^{k-1} - 1) h_{k-1})] + 32[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1) h_{k-1})]$$
$$+ 12[f(x_0 + 2h_{k-1}) + f(x_0 + 6h_{k-1}) + f(x_0 + 10h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2) h_{k-1})]$$
$$+ 14[f(x_0 + 4h_{k-1}) + f(x_0 + 8h_{k-1}) + f(x_0 + 12h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 4) h_{k-1})]\}$$
$$+ \frac{1}{2} * \frac{2}{45} h_{k-1} \{32[f(x_0 + \tfrac{1}{2} h_{k-1}) + f(x_0 + \tfrac{3}{2} h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - \tfrac{1}{2}) h_{k-1})]$$
$$- 20[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1) h_{k-1}) + 2[f(x_0 + 2h_{k-1})$$
$$+ f(x_0 + 6h_{k-1}) + f(x_0 + 10h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2) h_{k-1})]\}$$

Substituting (21) in the equation above, we obtain

$$P_{k,2} = \frac{1}{2} P_{k-1,2} + \frac{1}{45} h_{k-1} \{32[f(x_0 + \tfrac{1}{2} h_{k-1}) + f(x_0 + \tfrac{3}{2} h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - \tfrac{1}{2}) h_{k-1})]$$
$$- 20[f(x_0 + h_{k-1}) + f(x_0 + 3h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 1) h_{k-1})]$$
$$+ 2[f(x_0 + 2h_{k-1}) + f(x_0 + 6h_{k-1}) + f(x_0 + 10h_{k-1}) + \cdots + f(x_0 + (2^{k-1} - 2) h_{k-1})]\}$$

To obtain approximations of high order accuracy we apply extrapolation process to estimates obtained using (19). In a similar manner to that of Method I, it can be shown generally that if

$$I = P_{k,j} + \varepsilon_1 h_k^{\,2j+2} + \varepsilon_2 h_k^{\,2j+4} + \cdots \tag{22}$$

Then

$$I = P_{k+1,j} + \varepsilon_1 (\tfrac{h_k}{2})^{2j+2} + \varepsilon_2 (\tfrac{h_k}{2})^{2j+4} + \cdots \tag{23}$$

The term involving $h_k^{\,2j+2}$ can also be eliminated by multiplying (23) by 2^{2j+2} and subtracting (22) to obtain

$$(4^{j+1} - 1) I = 4^{j+1} P_{k+1,j} - P_{k,j} - \tfrac{3}{4} \varepsilon_2 (h_k)^{2j+4} - \cdots$$

Or equivalently

$$I = P_{k+1,j+1} + O(h^{2j+4}) \tag{24}$$

Where

$$P_{k+1,j+1} = \frac{(4^{j+1} P_{k+1,j} - P_{k,j})}{4^{j+1} - 1} \;;\; j = 2,3,4,\cdots \quad k = 2,3,4,\cdots \tag{25}$$

Numerical Examples

In this section, we use the two methods described to solve some problems. The relative errors $E_{k,j}$ in preliminary estimates and extrapolates are set out in a triangular array.

Example 1: $\int_0^2 \exp(x)\, dx$. The exact solution is $\exp(2)-1$.

Example 2: $\int_1^3 \ln x\, dx$. The exact solution is $3\ln(3)-2$.

Example 3: $\int_0^2 \sqrt{x}\, dx$. The exact solution is $\frac{4\sqrt{2}}{3}$.

Table 1a: Relative Errors for Example 1 using Romberg Integration Method

K	**n**	$E_{k,0}$	$E_{k,1}$	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$
0	1	3.130E1					
1	2	8.198E0	4.957E-1				
2	4	2.075E0	3.372E-2	2.915E-3			
3	8	5.203E-1	2.154E-3	5.002E-5	4.542E-6		
4	16	1.302E-1	1.354E-4	8.007E-7	1.952E-8	1.791E-9	
5	32	3.255E-2	8.473E-6	1.259E-8	7.818E-11	1.918E-12	1.668E-13

Table 1b: Relative Errors for Example 1 using Method II

K	**n**	$E_{k,1}$	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$
0	2	4.957E-1				
1	4	3.372E-2	2.915E-3			
2	8	2.154E-3	5.002E-5	4.542E-6		
3	16	1.354E-4	8.007E-7	1.952E-8	1.791E-9	
4	32	8.473E-6	1.259E-8	7.818E-11	1.918E-12	1.668E-13

Table 1c: Relative Errors for Example 1 using Method II

K	**n**	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$
0	4	2.915E-3			
1	8	5.002E-5	4.542E-6		
2	16	8.007E-7	1.952E-8	1.791E-9	
3	32	1.259E-8	7.818E-11	1.918E-12	1.668E-13

Table 2a: Relative Errors for Example 2 using Romberg Integration Method

K	**n**	$E_{k,0}$	$E_{k,1}$	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$	$E_{k,6}$	$E_{k,7}$
0	1	1.522E1							
1	2	4.120E0	4.195E-1						
2	4	1.060E0	3.976E-2	1.444E-2					
3	8	2.672E-1	2.972E-3	5.199E-4	2.990E-4				
4	16	6.694E-2	1.971E-4	1.214E-5	4.082E-6	2.925E-6			
5	32	1.674E-2	1.252E-5	2.161E-7	2.315E-8	7.717E-9	7.717E-9		
6	64	4.187E-3	7.871E-7	7.717E-9	7.717E-9	7.717E-9	7.717E-9	7.717E-9	
7	128	1.047E-3	4.630E-8	0	0	0	0	0	0

Table 2b: Relative Errors for Example 2 using Method I

K	***n***	$E_{k,1}$	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$	$E_{k,6}$	$E_{k,7}$
0	2	4.195E-1						
1	4	3.976E-2	1.444E-2					
2	8	2.972E-3	5.199E-4	2.990E-4				
3	16	1.971E-4	1.213E-5	4.075E-6	2.917E-6			
4	32	1.252E-5	2.161E-7	2.315E-8	7.717E-9	7.717E-9		
5	64	7.871E-7	7.717E-9	7.717E-9	7.717E-9	7.717E-9	7.717E-9	
6	128	4.630E-8	0	0	0	0	0	0

Table 2c: Relative Errors for Example 2 using Method II

K	***n***	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$	$E_{k,6}$	$E_{k,7}$
0	4	1.444E-2					
1	8	5.199E-4	2.990E-4				
2	16	1.213E-5	4.075E-6	2.917E-6			
3	32	2.161E-7	2.315E-8	7.717E-9	7.717E-9		
4	64	7.717E-9	7.717E-9	7.717E-9	7.717E-9	7.717E-9	
5	128	0	0	0	0	0	0

Table 3a: Relative Errors for Example 3 using Romberg Integration Method

K	***n***	$E_{k,0}$	$E_{k,1}$	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$	$E_{k,6}$	$E_{k,7}$	$E_{k,8}$	$E_{k,9}$	$E_{k,10}$
0	1	2.500E1										
1	2	9.467E0	4.289E0									
2	4	3.508E0	1.521E0	1.337E0								
3	8	1.280E0	5.381E-1	4.726E-1	4.589E-1							
4	16	4.628E-1	1.903E-1	1.671E-1	1.622E-1	1.611E-1						
5	32	1.662E-1	6.727E-2	5.907E-2	5.736E-2	5.695E-2	5.685E-2					
6	64	5.938E-2	2.378E-2	2.089E-2	2.028E-2	2.013E-2	2.010E-2	2.009E-2				
7	128	2.115E-2	8.409E-3	7.384E-3	7.170E-3	7.118E-3	7.106E-3	7.102E-3	7.102E-3			
8	256	7.518E-3	2.973E-3	2.611E-3	2.535E-3	2.517E-3	2.512E-3	2.511E-3	2.511E-3	2.511E-3		
9	512	2.668E-3	1.051E-3	9.230E-4	8.962E-4	8.898E-4	8.882E-4	8.878E-4	8.877E-4	8.877E-4	8.877E-4	
10	1024	9.457E-4	3.716E-4	3.263E-4	3.169E-4	3.146E-4	3.140E-4	3.139E-4	3.138E-4	3.138E-4	3.138E-4	3.138E-4

Table 3b: Relative Errors for Example 3 using Method I

K	***N***	$E_{k,1}$	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$	$E_{k,6}$	$E_{k,7}$	$E_{k,8}$	$E_{k,9}$	$E_{k,10}$
0	2	4.289E0									
1	4	1.521E0	1.337E0								
2	8	5.381E-1	4.726E-1	4.589E-1							
3	16	1.903E-1	1.671E-1	1.622E-1	1.611E-1						
4	32	6.727E-2	5.907E-2	5.736E-2	5.695E-2	5.685E-2					
5	64	2.379E-2	2.089E-2	2.028E-2	2.013E-2	2.010E-2	2.009E-2				
6	128	8.409E-3	7.384E-3	7.170E-3	7.118E-3	7.106E-3	7.102E-3	7.102E-3			
7	256	2.973E-3	2.611E-3	2.535E-3	2.517E-3	2.512E-3	2.511E-3	2.511E-3	2.511E-3		
8	512	1.051E-3	9.230E-4	8.962E-4	8.898E-4	8.882E-4	8.878E-4	8.877E-4	8.877E-4	8.877E-4	
9	1024	3.716E-4	3.263E-4	3.169E-4	3.146E-4	3.140E-4	3.139E-4	3.139E-4	3.138E-4	3.138E-4	3.138E-4

Table 3c: Relative Errors for Example 3 using Method II

K	n	$E_{k,2}$	$E_{k,3}$	$E_{k,4}$	$E_{k,5}$	$E_{k,6}$	$E_{k,7}$	$E_{k,8}$	$E_{k,9}$	$E_{k,10}$
0	4	1.337E0								
1	8	4.726E-1	4.589E-1							
2	16	1.671E-1	1.622E-1	1.611E-1						
3	32	5.907E-2	5.736E-2	5.695E-2	5.685E-2					
4	64	2.089E-2	2.028E-2	2.013E-2	2.010E-2	2.009E-2				
5	128	7.384E-3	7.170E-3	7.118E-3	7.106E-3	7.102E-3	7.102E-3			
6	256	2.611E-3	2.535E-3	2.517E-3	2.512E-3	2.511E-3	2.511E-3	2.511E-3		
7	512	9.230E-4	8.962E-4	8.898E-4	8.882E-4	8.878E-4	8.877E-4	8.877E-4	8.877E-4	
8	1024	3.263E-4	3.169E-4	3.146E-4	3.140E-	3.139E-4	3.139E-4	3.138E-4	3.138E-4	3.138E-4

Results and Discussion

The examples were solved using Romberg Integration method and the two new methods developed above. The observed relative errors as percentage are given in Tables 1, 2 and 3.

From these Tables we can observe that the integrands converge to the same solutions for the two new methods as the Romberg integration, $T_{k,j}$. It is also interesting to note that $T_{k,0}, S_{k,1}$ and $P_{k,2}$ $(k = 2, 3, ...)$ require the same number of function evaluations. More number of preliminary estimates and extrapolates, which implies more computational cost, are needed for Romberg integration than for the new methods. For instance, for Example I, a total of 21 preliminary estimates and extrapolates is required for convergence by Romberg, 15 by Method I and only 10 by Method II such that a relative error of 1.668E-13 is obtained. Similarly for Example 2, to obtain a relative error of 0 (i.e. approximate solution being equal to the exact solution), we required 36 preliminary estimates and extrapolates for Romberg, 28 for Method I and 15 for Method II.

Conclusion

Two numerical methods were developed and tested for the numerical solution of definite integrals. The computing of these techniques is simple. The methods produce comparable results with Romberg integration but with reduced computational efforts involved.

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