American Journal of Scientific Research
ISSN 1450-223X Issue 7 (2010), pp.77-89

http://www.eurojournals.com/ajsr.htm

On the Generalisized Inverse of a Matrix

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Abstract

This paper seeks to find a generalized inverse of singular and rectangular matrices. It also looks at the applications of the generalized inverse to solution of systems of equations that are linearly dependent and unbalance.

Keywords: Generalized inverse of a matrix, singular rectangular matrices, system of equations, linearly dependent, unbalance system of equations.

1. Introduction

From the historical point of view, Fredholm (1903) was first the Mathematician to have mention the concept of a generalized inverse, he called a particular generalized inverse as pseudo inverse which serve as an integral operator. The class of all pseudo inverses was characterized in 1912 by Hurwitz who used the finite dimensionality of null operators of Fredholm operators to give a simple algebraic construction. Generalized inverse of differential operators, already implicit in Hilbert's discussions in 1904 of generalized Green functions were consequently studied by numerous authors, in particular, Myller in 1906, Bounitzky in 1909, Elliott in 1928 and Reid in 1931. However, Moore (1920), in his first publication on the subject, an abstract of a talk given at a meeting of American Mathematical Society appeared in 1920, his results are thought to have been obtained much earlier. He called the unique inverse matrix as a reciprocal finite matrix. Details were published only in 1935 after Moore's death. Little notice was taken of Moore's discovery for 30 years after his first publication, during which time generalized inverses were given for matrices by Siegel in 1937, and for operators by Tseng, Murray and Von Neumann in 1936 who defined a unique inverse called by him general reciprocal for every finite matrix (square or rectangular).

Alkinson, and others revival of interest in the subject in the 1950s centered on the least square properties (not mention by Moore) of certain generalized inverses. These properties were recognize in 1951 by Bjerhanmer who rediscovered Moore's inverse and also noted the relationship of generalized inverses to solutions of linear systems.

This matrix was defined by Penrose in 1955 as the generalized inverse, pseudoinverse, or Moore-Penrose inverse. It is a matrix 1 –inverse, and is implemented in Mathematica as PseudoInverse[m]. He sharpened and extended Bjerhammer's result on linear systems.

Generalized inverse is able to solve linearly dependent and unbalanced systems of equations; generalized inverse is of a great importance in its general applications to non-square and square singular matrices. In the case where A is non-singular, that is $G = A^{-1}$ and G is unique. The fact that A has a generalized inverse, even when it is singular or rectangular has particular applications in the problem of solving equation like:

$$AX = Y$$

Again generalized inverse are of great importance on the study of linear models where least square estimate often leads to equation

$$x^1 xb = x^1 y$$

This has to be expressed in the form

$$b = (x \ x)^{-1} = xy$$

But if x^1 x i.e. a singular then $(x^1 x)^{-1}$ does not exist, hence the use of generalized inverse is needed to solve such system of equations, which is one of the mean objectives of this paper.

In this paper, we also apply generalized inverse to models that are not all full rank.

2. Algrathm for the Generalized Inuerse of a Matrix

An algorithm for finding the generalized inverse of a matrix is as follows:

Step 1: in A of rank r, find any non-singular minor of order r call it M

Step 2: invert M and transpose the inverse $(M^{-1})^1$

Step 3: in A, replace each element of M by the corresponding element of $(m^{-1})^1$

That $a_{ij} = M_{st}$ = $(s,t)_{th}$ element of M, then replace a_{ij} by M^{ts}, the $(t,s)_{th}$ element of M^{-1} equivalent to the (s,t) element of the transpose of M^{-1}

Step 4: replace all the other elements of A by zero

Step 5: transpose the resulting matrix and the result is G a generalized inverse of A.

2.1. Properties of Generalized Inverse of A Matrix

If G is a generalized inverse of A then

- AGA = A
- G is not unique
- G is of order $m \times n$ if A is of order $n \, x \, m$

If G is also a generalized inverse of $x^1 x$ then

- G^1 is also a generalized inverse of $x^1 x$
- XGX^1X =X; that is, GX^1 is a generalized inverse of X
- XG^1X is invariant to G
- XGX^1 is symmetric, whether G is or not

2.2. Apllication of Generlized Inverse to A Singular Matrix

Illustrative Example 1 on 3 x 3 matrix (Square matrix)

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$$

Matrix A is singular matrix, hence not of full rank. The rank of A is 2 and its generalized inverse is calculated as follows by the algorithm

Applying the Algorithms

Step 1

$$M = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ Find any non-singular minor of order r where r is the rank. Here the rank is 2; therefore we choose a 2 x 2 matrix from the original matrix which is not singular.

Step 2

Det $M = 4 - 1 = 3$

$$M^{-1} = \tfrac{1}{3}\begin{pmatrix} 1 - 1 \\ 1 - 4 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1/3 & -1/3 \\ -1/3 & 4/3 \end{pmatrix}$$

$$\left(M^{-1}\right)^{1} = \begin{pmatrix} 1/3 & -1/3 \\ -1/3 & 4/3 \end{pmatrix}$$

Step 3 / step 4 combined

$$\begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 5:

Transposing the matrix

$$G = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since matrix is symmetric, then

If G is a generalized inverse of A then AGA = A

$$AG = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 6 & 0 \end{pmatrix}$$

$$AGA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 6 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$$

Since AGA = A, then G is a generalized inverse of A

Considering another M from the given matrix,

Applying the Algorithms

Step 1

M can also be chosen as:

$$\begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$$

Step 2

Det M = 5 – 2 = 3

$$M^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix} \quad \left(M^{-1}\right)^{1} = \begin{pmatrix} \frac{5}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Step3 /Step 4

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{3} & -\frac{2}{3} & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Step 5:

Transposing the matrix

$$G = \begin{pmatrix} 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

Remark

Since AGA = A, G is a generalized inverse of A. from the two result it shows that generalized inverse of a matrix is not unique hence one can find a definite number of generalized inverses of a matrix. The number of generalized inverse of matrix will depend on the total number of non-singular minors of order 2 which implies that matrix A has 9 generalized inverses.

Illustrative Example 2 on 4 x 4 matrix (Square matrix)

$$B = \begin{pmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 3 \\ 3 & 2 & 1 & 7 \end{pmatrix}$$

Matrix B is of rank 2

Applying the Algorithms

Step 1:

$$M = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

Step 2:

Det M = 6 – 5 = 1

$$M = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \quad \left(M^{-1}\right) = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

Step3/step 4

$$\begin{pmatrix} 0 & 2 & -5 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 5

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If G is a generalized inverse of B then G must satisfy the equation BGB = B

$$BG = \begin{pmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 2 \\ 3 & 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$$BGB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 2 \\ 3 & 2 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 2 \\ 3 & 2 & 1 & 7 \end{pmatrix}$$

Since BGB =B

G is a generalized inverse of B. other generalized inverses of B can be found by inverting other non-singular minors of rank 2

Illustrative example 3 on 4 x 4 matrix (Square matrix)

$$C = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 10 & 7 \\ 2 & 1 & 1 & 6 \end{pmatrix}$$

Matrix C is of rank 3

Applying the Algorithms

Step 1

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

Step 2

$$\text{Det } M = 1\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} -2\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} +3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

= 1(50 – 48) – 2(40 -42) = 3(32- 35)

= 1(2) -2(-2) + 3(-3)

= 2 + 4 – 9

= -3

Det M = -3

$M^{-1} = (1/\det M)adj(M)$ = (1/det M) adj (M)

Adj (M) = adjoint of M which is the transpose of the cofactors of matrix M.

Cofactors of M is gotten below, thus

$$\text{Adj }(M) = \begin{pmatrix} 2 & 4 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix} \quad M^{-1} = -\tfrac{1}{3}\begin{pmatrix} 2 & 4 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} -\tfrac{2}{3} & -\tfrac{2}{3} & 1 \\ -\tfrac{4}{3} & \tfrac{11}{3} & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Step3/step 4

$$\begin{pmatrix} -\tfrac{2}{3} & -\tfrac{2}{3} & 1 & 0 \\ -\tfrac{4}{3} & \tfrac{11}{3} & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 5

$$G = \begin{pmatrix} -\tfrac{2}{3} & -\tfrac{2}{3} & 1 & 0 \\ -\tfrac{4}{3} & \tfrac{11}{3} & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

G satisfies the equation CGC = C

2.3. Application of Generalized Inverse to Rectangular Matrix

Illustrative example 1 on 3 x 4 matrix (Rectangular matrix)

$$A = \begin{pmatrix} 2 & 3 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 4 & 5 & 3 & 6 \end{pmatrix}$$

Rank of A = 3

Applying the Algorithms

Step 1

$$M = \begin{pmatrix} 3 & 6 & 7 \\ 3 & 4 & 5 \\ 5 & 3 & 6 \end{pmatrix}$$

Step 2

$$M = \begin{pmatrix} -\tfrac{9}{8} & \tfrac{15}{8} & -\tfrac{1}{4} \\ -\tfrac{7}{8} & \tfrac{17}{8} & -\tfrac{3}{4} \\ \tfrac{11}{8} & -\tfrac{21}{8} & \tfrac{3}{4} \end{pmatrix} \quad (M^{-1})^{1} = \begin{pmatrix} -\tfrac{9}{8} & -\tfrac{7}{8} & \tfrac{11}{8} \\ \tfrac{15}{8} & \tfrac{17}{8} & -\tfrac{21}{8} \\ -\tfrac{1}{4} & -\tfrac{3}{4} & \tfrac{1}{4} \end{pmatrix}$$

Step3/step 4

$$\begin{pmatrix} 0 & -\tfrac{9}{8} & -\tfrac{7}{8} & \tfrac{11}{8} \\ 0 & \tfrac{15}{8} & \tfrac{17}{8} & -\tfrac{21}{8} \\ 0 & -\tfrac{1}{4} & -\tfrac{3}{4} & \tfrac{1}{4} \end{pmatrix}$$

Step 5

Transposing the Matrix

$$G = \begin{pmatrix} 0 & 0 & 0 \\ -9/8 & 15/8 & -1/4 \\ -7/8 & 17/8 & -3/4 \\ 11/8 & -21/8 & 3/4 \end{pmatrix}$$

If G is a generalized inverse of A then G must satisfy the equation AGA = A

$$AG = \begin{pmatrix} 2 & 3 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 4 & 5 & 3 & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -9/8 & 15/8 & -1/4 \\ -7/8 & 17/8 & -3/4 \\ 11/8 & -21/8 & 3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 4 & 5 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 4 & 5 & 3 & 6 \end{pmatrix}$$

For matrix A, AG = I (Identity matrix) but G is not unique in the case of square non-singular matrix where G is unique and it is denoted as A^{-1}. For matrix A we can find G

Illustrative example 2 on 3 x 4 matrix (Rectangular matrix)

$$B = \begin{pmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{pmatrix}$$

Matrix B is of rank 2

Applying the Algorithms

Step 1:

$$M = \begin{pmatrix} 2 & 5 \\ 7 & 12 \end{pmatrix}$$

Step 2:

$$M^{-1}\begin{pmatrix} -12/11 & 5/11 \\ 7/11 & -2/11 \end{pmatrix} \quad (M^{-1})^{1} = \begin{pmatrix} -12/11 & 5/11 \\ 7/11 & -2/11 \end{pmatrix}$$

Step3/step 4

$$\begin{pmatrix} 0 & -12/11 & 7/11 & 0 \\ 0 & 5/11 & -2/11 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 5

$$\begin{pmatrix} 0 & 0 & 0 \\ -12/11 & 5/11 & 0 \\ 7/11 & -2/11 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Illustrative example 3

$$C = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 8 & 7 & 2 \\ 3 & 5 & 4 \end{pmatrix}$$

Rank c = 3

Applying the Algorithms

Step 1

$$M = \begin{pmatrix} 1 & 3 & 6 \\ 8 & 7 & 2 \\ 3 & 5 & 4 \end{pmatrix}$$

Step2

$$M^{-1} = \begin{pmatrix} 18/46 & 14/46 & -34/46 \\ -26/46 & -10/46 & 44/46 \\ 19/46 & 2/46 & -18/46 \end{pmatrix} \quad (M^{-1})^1 = \begin{pmatrix} 18/46 & -26/46 & 19/46 \\ 14/46 & -10/46 & 2/46 \\ -34/46 & 44/46 & -18/46 \end{pmatrix}$$

Step3/ step 4

$$\begin{pmatrix} 0 & 0 & 0 \\ 18/46 & -26/46 & 19/46 \\ 14/46 & -10/46 & 2/46 \\ -34/46 & 44/46 & -18/46 \end{pmatrix}$$

Step 5

Transposing to obtain G

$$G = \begin{pmatrix} 0 & 18/46 & 14/46 & -34/46 \\ 0 & -26/46 & -10/46 & 44/46 \\ 0 & 19/46 & 2/46 & -18/46 \end{pmatrix}$$

3. Generalized Inverse of A Matrix with Rank 1

A generalized inverse of a matrix with rank 1 is calculated by the following algorithm.

Step 1: find the reciprocal of any element of matrix A call it M

Step 2: substitute the reciprocal element back into the matrix where it was chosen

Step 3: replace all other element of the matrix A by zeroes

Step 4: transpose the matrix and the result is a generalized inverse of A

Illustrative example 1 on 2x2 matrix

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

Matrix A is of rank 1

Applying the Algorithms

Step 1

M = ½

Step 2 / Step 3:

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$$

Step 4

$$G = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$$

If G is a generalized inverse of A, then G must satisfy the equation AGA =A

$$AG = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$

$$AGA = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

Since AGA = A G is a generalized inverse of A. other generalized inverse of A is given as:

$$\begin{pmatrix} 0 & 1/4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1/4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/8 \end{pmatrix}$$

Illustrative example 2 on a 3 x 4 matrix (rectangular matrix)

$$B = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \\ 8 & 16 & 24 \end{pmatrix}$$

Matrix B is of rank 1

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \end{pmatrix}$$

Since B is a 3 x 4 matrix we can find 12 generalized inverses of matrix B.

In general, for an n x m matrix of rank 1 the number of generalized inverses that can be found is equal to nm, and its generalized inverse is found by taking the reciprocal of any element of the matrix and the rest being zeros.

If

$$B = \begin{pmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \vdots b_{2n} \\ \vdots & \vdots \ddots \vdots \\ b_{m1} & b_{m2} \cdots b_{mn} \end{pmatrix}_{m\times n}$$

And let B be of rank = 1

Then we can find nm, generalized inverses of A as follows

1. $$\begin{pmatrix} (b_{11})^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

2 $$\begin{pmatrix} 0 & (b_{12})^{-1} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$(mn).\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (b_{mn})^{-1} \end{pmatrix}$$

4. Application of Generalized Inverse to Linearly Department Unblananced Equations and Models Not of Full Rank

4.1. Linearly Dependent Equation

Example 1

Consider the system of linear equation given below

$4x_1 + x_2 + 2x_3 = 3$

$x_1 + x_2 + 5x_3 = 9$

$3x_1 + x_2 + 3x_3 = 5$

The above equation can be written in matrix form as follows

$$\underset{A}{\begin{pmatrix} 4 & 1 & 2 \\ 1 & 1 & 5 \\ 3 & 1 & 3 \end{pmatrix}} \quad \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{Y}{\begin{pmatrix} 3 \\ 9 \\ 5 \end{pmatrix}}$$

Consistent equation has a solution X = GY if and only if AGA = A

A generalized inverse of A is given as

$$G = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

X = GY becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \\ 0 \end{pmatrix}$$

$x_1 = -2, x_2 = 0$ and $x_3 = 0$ satisfies the system of equation. If a generalized inverse of A is chosen as

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 5/3 & -2/3 & 0 \\ -1/3 & 1/3 & 0 \end{pmatrix}$$

Then, X = GY becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$x_1 = 0, \quad x_2 = -1$ and x_3 satisfies the system of equation. This clearly shows that the solutions is not unique and every generalized inverse has its own solution to the system of equations which is given by

X = GY + (GA – I) Z

Where

G = any generalized inverse of A

A = the matrix of the coefficient of the x

I = the identity matrix

Z = a column vector

Applying the general formula to

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 5/3 & -2/3 & 0 \\ -1/3 & 1/3 & 0 \end{pmatrix}$$

We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 5/3 & -2/3 & 0 \\ -1/3 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 9 \\ 5 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 5/3 & -2/3 & 0 \\ -1/3 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 1 & 1 & 5 \\ 3 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right. \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 6 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -z_1 \\ 6z_2 \\ -z_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ 6z_2 - 1 \\ -z_1 + 1 \end{pmatrix}$$

$x_1 = z_1, x_2 = z \;\; -1 \;\; and \;\; x_3 = -z_1 + 2$

Where z_1 is arbitrary. If $z_1 = 0$

Then x_1 = o, x_3 = -1 and x = 2 which is the solution when

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 5/3 & -2/3 & 0 \\ -1/3 & 1/3 & 0 \end{pmatrix}$$

When $z_1 = 2$

then $x_1 = -2, \; x_2 = 11 \, and \;\; x_3 = 0$. Which is the solution when

$$G = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The expression $x_1 = -z_1$, $x_2 = 6z_1 - 1$ and $x_3 = -z_1 + 2$ links the solution of the various generalized inverses, with z_1 being arbitrary we can find an infinite number of solutions for the system of equations even though only a finite number of generalized inverses exit for matrix A.

4.2. Unbalanced Equation

Illustrative example

Consider the unbalance equation

2x + 3x + 6x + 7x = 8

X + 3x + 4x + 5x = 24

4x + 5x + 3x + 6 = 16

The set of equation can be written in matrix form as follows

$$\underset{A}{\begin{pmatrix} 2 & 3 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 4 & 5 & 3 & 6 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \underset{Y}{\begin{pmatrix} 8 \\ 24 \\ 16 \end{pmatrix}}$$

X = GY

A generalized inverse of A is given as

$$G = \begin{pmatrix} 0 & 0 & 0 \\ -9/8 & 15/8 & -1/4 \\ -7/8 & 17/8 & -3/4 \\ 11/8 & -21/8 & 3/4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -9/8 & 15/8 & -1/4 \\ -7/8 & 17/8 & -3/4 \\ 11/8 & -21/8 & 3/4 \end{pmatrix} \begin{pmatrix} 8 \\ 24 \\ 16 \end{pmatrix} = \begin{pmatrix} 0 \\ 32 \\ 32 \\ -40 \end{pmatrix}$$

The solution $x_1 = 0$, $x_2 = 32$, $x_3 = 32$ and $x_4 = -40$ satisfies the equations. To find the general solution we use the equation

X = GY + (GA –I) Z

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 32 \\ 32 \\ -40 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & 0 & 0 \\ -9/8 & 15/8 & -1/4 \\ -7/8 & 17/8 & -3/4 \\ 11/8 & -21/8 & 3/4 \end{pmatrix} \begin{pmatrix} 2 & 3 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 4 & 5 & 3 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 32 \\ 32 \\ -40 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -11/8 & 1 & 0 & 0 \\ -21/8 & 0 & 1 & 0 \\ 25/8 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 32 \\ 32 \\ -40 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -11/8 & 0 & 0 & 0 & 0 \\ -21/8 & 0 & 0 & 0 & 0 \\ 25/8 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 32 \\ 32 \\ -40 \end{pmatrix} + \begin{pmatrix} -z_1 \\ -11/8\, z_1 \\ -21/8\, z_1 \\ 15/8\, z_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -z_1 \\ -11/8\, z_1 + 32 \\ -21/8\, z_1 + 32 \\ 15/8\, z_1 - 40 \end{pmatrix}$$

$X_1 = -z$, $x_2 = -11/8z_1 + 32$, $x_3 = -21/8z_1 + 32$ and $x_4 = 25/8z_1 – 40z_1$ are arbitrary and any value of z_1 gives a solution to the system of equations, this general approach shows that only one generalized inverse is enough to link all the other solution of the different generalized inverses.

Conclusion

The method of generalized inverse and linear models cannot be overlooked since it plays a very important role in models not of full rank. The use of generalized inverse of a matrix enables us to solve systems of linear equations that are unbalance and linearly dependent easily, examples of these have been shown in the paper.

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