On Simple And Bisimple Left Inverse Semi Groups

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ABSTRACT

This paper deals with Simple and bi-simple inverse semi-groups. The general properties and characteristics of simple and bi-simple semi-groups and inverse semi-groups were discussed. [Researcher. 2009;1(3):14-24]. (ISSN: 1553-9865).

Key words: Simple, bi-simple semigroup, inverse semigroup

INTRODUCTION


The importance of simple and bi-simple semi group in Algebra has developed into area of independent research of Mathematics, [1 - 13] have come to play a very significant role. In this paper, we look into simple and bi-simple left inverse semi groups.

According to Encyclopaedia of mathematics, A semi-group not containing proper ideals or congruences of some fixed type. Various kinds of simple semi-groups arise, depending on the type considered: ideal-simple semi-groups, not containing proper two-sided ideals (the term simple semi-group is often used for such semi-groups only); left (right) simple semi-groups, not containing proper left (right) ideals; (left, right) 0-simple semi-groups, semi-groups with a zero not containing proper non-zero two-sided (left, right) ideals and not being two-element semi-groups with zero multiplication; bi-simple semi-groups, consisting of one D-class (cf. Green equivalence relations); bi-simple semi-groups, consisting of two D-classes one of which is the null class; and congruence-free semi-groups, not having congruences other than the universal relation and the equality relation.
Every left or right simple semi-group is bi-simple; every bi-simple semi-group is ideal-simple, but there are ideal-simple semi-groups that are not bi-simple (and even ones for which all the $D$-classes consist of one element).

Various types of simple semi-groups often arise as "blocks" from which one can construct the semi-groups under consideration. For classical examples of simple semi-groups see Completely-simple semi-group; Brandt semi-group; Right group; for bi-simple inverse semi-groups (including structure theorems under certain restrictions on the semi-lattice of idempotents) see [1], [8], [9]. There are ideal-simple inverse semi-groups with an arbitrary number of $D$-classes. In the study of imbedding of semi-groups in simple semi-groups one usually either indicates conditions for the possibility of the corresponding imbedding, or establishes that any semi-group can be imbedded in a semi-group of the type considered. E.g., any semi-group can be imbedded in a bi-simple semi-group with an identity (cf. [1]), in a bi-simple semi-group generated by idempotents (cf. [10]), and in a semi-group that is simple relative to congruences (which may have some property given in advance: the presence or absence of a zero, completeness, having an empty Frattini sub-semigroup, etc., cf. [3]–[5]).

2. SIMPLE SEMIGROUP:- A semigroup $S$ is said to be simple if it contains only one $J$–class and BISIMPLE SEMIGROUP: A semigroup $S$ is said to be bisimple if it contains only one $D$–class.

**Theorem 1.0**

(i) If a $D$ – class of a semigroup $S$ contains a regular element , then every element of $D$ is regular. $J$ -

(ii) If $D$ is regular then every $L$ - class and $R$ – class contained in $D$ contain an idempotent.

**Proof:**

An element of a semigroup $S$ is regular if and only if $R_a [L_a]$ contains an idempotent.

It then follows that an $R$ – class $R$- ($L$ – class) contains a regular element ; then it contain an idempotent and every element of $R (L)$ is regular.

Since every $R$ – class of $S$ contained in $D$ meet every $L$ – class of $S$ contained in $D$, then (1) holds.

**Lemma 1.0**

If $a$ and $a^{-1}$ are inverse element of a semigroup $S$.

Then $e = aa^{-1}$ and $f = a'a$ are idempotent such that $ea = a'f$ and $a' = f a a' = a^{-1}$

Hence $e \in R_a \cap L_a^{-1}$ and $f \in R_a^{-1} \cap L_a$. The element $a$, $a^{-1}$, $e$ and $f$ all belong to the same $D$ – class.

**Theorem 1.1**

Let ‘$a$’ be a regular element of a semigroup $S$

Then (i) Every inverse of $a$ lies in $D_a$

(ii) An $H$- class, $H_b$ contains an inverse of $a$ if and only if both of the $H$ – classes $R_a \cap L_b$ and $R_b \cap L_a$ contains idempotent.

(iii) No $H$ – class contains more than one inverse of $a$

**COROLLARY 1.0**

A semigroup $S$ is an inverse semigroup if and only if each $L$ – class and each $R$ – class contains exactly one idempotent.

**Corollary 1.1**

A semigroup $S$ is simple if and only if

$S a S = S \ \forall \ a \in S$ i.e if and only if

$\forall a, b \in S \ \exists \ x, y$ in $S$ such that $xay = b$

**Theorem 1.2**

The following statements about a semigroup $S$ are equivalent.
(i) S is an inverse semigroup
(ii) S is regular and idempotent element commute.
(iii) Each \( \mathcal{L} \) – class and \( \mathcal{R} \) – class of S contain a unique idempotent.
(iv) Each principal left and right ideal of S contains a unique idempotent generator.

**Proof:**

It is clear by the definition of \( \mathcal{L} \) and \( \mathcal{R} \) that III and IV are equivalent.

To show that I \( \Rightarrow \) II

Let \( e, f \) be idempotent and let \( x = (ef)^{-1} \)

Then, \( ef \cdot x \cdot ef = ef \) and \( xefx = x \)

The element \( fxe \) is idempotent since

\[
(fxe)^2 = f(xefx) = fxe
\]

Also, \( (ef)(f)(ef) = ef^2 = ef \)

\[
(f)(ef)(fxe) = f(xefx) = fxe
\]

and \( ef \) is an inverse of \( fxe \).

But \( fxe \) being idempotent, it is its own unique inverse.

So \( fxe = ef \).

It is then follows that \( ef \) is idempotent and similarly we obtain that \( fe \) is also idempotent.

Hence, \( (ef)(fe)(ef) = (ef)^2 = ef \)

\[
(f)(ef)(fe) = (fe)^2 = fe
\]

\( \therefore \) \( fe \) is an inverse of \( ef \).

But \( ef \), being an idempotent is its own unique inverse and so., \( ef = fe \)

\( \therefore \) I \( \Rightarrow \) II.

To show that II \( \Rightarrow \) III

Since S is regular every \( \mathcal{L} \) – class contain at least one idempotent.

If \( e, f \) are \( \mathcal{L} \)- equivalent idempotent, then \( ef = e, fe = f \).

Since by hypothesis, \( ef = fe \), it follows that \( e = f \).

Similarly remark apply to \( \mathcal{R} \) – class we can express the property III of inverse semigroups as follows.

\( \mathcal{L} \cap (\text{EXE}) = \mathcal{R} \cap (\text{EXE}) = 1E \)

Where \( E \) is the set of idempotent of S.

\( \therefore \) II \( \Rightarrow \) III.

To show that III \( \Rightarrow \) I

Since a semigroup with the property III is necessarily regularly, then every \( \mathcal{D} \) – class contains an idempotent.

If \( a, a' \) are inverse of \( a \), then \( aa' \) and \( a'a' \) are idempotent in S that are \( \mathcal{R} \) equivalent to ‘a’ and hence to each other.

By property III, we have \( aa' = aa'' \)

Equally, \( a'a = a' \) and so

\[ a' = a'aa' = a'' \]

\( \Rightarrow \) \( a''a'' = a' \)

**Proposition 1.0**

Let S be an inverse semigroup with semilattice of idempotent E.

Then (i) \( (a')' = a \) \( \forall \ a \) in S.

\( (ii) \) \( e'' = e \) \( \forall \ e \) in E

\( (iii) \) \( (ab)' = b'a' \) \( \forall \ a, b \) in S

\( (iv) \) \( aea \in E, a'ea \in E, \forall a \in S \) and \( \forall e \) in E

\( (v) \) a \( \mathcal{R} \) b if and only if \( aa'' = bb'' \)

\( a \mathcal{L} \) b if and only if \( a'a = b'b \)

\( (vi) \) If \( e, f \in E \), then \( e \mathcal{D}, f \in S \) if and only if \( a \in S \) such that \( aa'' = e, a'a = f \).

**Proof:**

I and II are mutuality of the inverse property of a semigroup
To proof III
Since \( b^{-1}b \) and \( a^{-1}a \) are idempotent
\[
(ab)(b^{-1}a^{-1})(ab) = a(bb^{-1})(a^{-1}a)b
= aa^{-1}abb^{-1}b
= ab.
\]
Also,
\[
(ab)(b^{-1}a^{-1}) = b^{-1}(a^{-1}a)(bb^{-1})a^{-1}
= b^{-1}bb^{-1}a^{-1}aa^{-1}
= b^{-1}a^{-1}
\]
Thus \( b^{-1}a^{-1} \) is an inverse and hence the inverse of \( ab \).
i.e \( (ab)^{-1} = b^{-1}a^{-1} \)

To proof IV
\[
(aea^{-1})^2 = ae(a^{-1}a)ea^{-1}
= aea^{-1}a^2 e^{-1} = aea^{-1}
\]
Similarly, \( (a^{-1}ea)^2 = a^{-1}ea \).

Recall that a semigroup \( S \) is said to be simple if it contains only one \( \mathcal{I} \)-class. i.e \( S \) is simple if and only if \( \mathcal{I} = S \times S \)
i.e every element in \( \mathcal{I} \) is related to each other.

**Lemma 1.2**
An inverse semigroup \( S \) with semilattice of idempotent \( E \) is simple if and only if
\[
(\forall e, f \in E)(\exists g \in E)[g \leq f \text{ and } e \mathcal{D} g].
\]

**Proof:**
Let \( S \) be simple, if \( e, f \in E \), then \( e J f \) and so \( \exists x, y \in S \) such that \( e = xfy \).
Let \( g = fyex \), then
\[
g^2 = fyex(fyex)ex = fy^3x
\]
since \( g = fyex \), \( g \in E \)
Also,
\[
Fg = g \text{ and } g \leq f
\]
If \( z = x^{-1}e \), then \( xz = xx^{-1}e = xx^{-1}fy \)
\[
\Rightarrow xfy = e \text{ and so } e \mathcal{L} z
\]
Also
\[
Zx = x^{-1}ex = x^{-1}e^2 x
= x^{-1}xy = x^{-1}x y = gx^{-1}x
= fyex = fyex \cdot x = g
\]
\[
gx^{-1} = gx^{-1}xx^{-1} = x^{-1}xgx^{-1}
= x^{-1}xfyexx^{-1} = x^{-1}e^2 xx^{-1}
\]
\[
\Rightarrow x^{-1}xx^{-1}e = x^{-1}e = x^{-1}e = z \text{ and so } z \mathcal{R} g.
\]
Thus, \( e \mathcal{D} g \) as required.
Conversely if \( S \) has the property described above, considering any two idempotent \( e, f \) in \( S \) then \( \exists g \in E : g \leq f \) and \( e \mathcal{D} g \) and so, \( J_e = J_f \).
Equally, \( \exists h \in E : h \leq e \) and \( f \mathcal{D} h \) and so, \( J_f = J_h \leq J_e \).
Hence,
\( J_e = J_f \) and so all the idempotent of \( S \) fall in a single \( \mathcal{J} \)-class.

But every element of \( S \) in \( \mathcal{J} \) is \( \mathcal{D} \)-equivalent (indeed even \( \mathcal{R} \) or \( \mathcal{L} \)-equivalent) to some idempotent and so it follows that \( S \) is simple.
As a consequence, if \( S \) is a simple inverse semigroup with semilattice of idempotent \( E \), then \( E \) has the property
\[
(\forall e, f \in E)(\exists g \in E)[g \leq f \text{ and } Ee \sim Eg].
\]
Recall that a semigroup \( S \) is said to be Bisimple if it contains only one \( D \)-class. It is a semigroup in which \( D \) is the universal relation.
If \( S \) is a Bisimple inverse semigroup with semilattice of idempotent \( E \), then all the idempotent are mutually \( D \)-equivalent. i.e \( D \cap (E \times E) = E \times E \).
Hence it follows that $U = E \times E$, i.e. $E$ is a uniform semilattice.

Conversely, if we start with a uniform semilattice $E$, then we cannot expect that every inverse semigroup having $E$ as semilattice of idempotent will be Bisimple, $E$ itself is one such inverse semigroup and are assumed not Bisimple.

**DEFINITION**

If $(e, f) \in U$, let $T_{e,f}$ be the set of all isomorphism from $E_e$ onto $E_f$.

Let $T_E = \bigcup_{e, f \in U} T_{e,f}$.

Since all the element of $T_E$ are partial one-one mapping of $E$. We may therefore multiply element of $T_E$ as element of $J(E)$.

If $\alpha: E_e \to E_f$ and $\beta: E_g \to E_h$ are element of $T_E$, then the product of $\alpha$ and $\beta$ in $J(E)$ maps $(E_f \cap E_g)\alpha^{-1}$ onto $(E_g \cap E_h)$.

Then $x \in (E_f \cap E_g)\alpha^{-1}$ if and only if $x \in (E_g \cap E_h)$.

Similarly, $x \in E (E_{ig})\beta$.

Thus $\alpha \beta$ maps the principal ideal $E_i$ onto the principal ideal $E_j$.

Since it is clearly an isomorphism, we have that $\alpha \beta \in T_E$. Thus $T_E$ is a subsemigroup of $J(E)$.

**Proposition 1.2**

If $E$ is a uniform semilattice, then $T_E$ is a Bisimple inverse semigroup.

**Proof:**

This proves more generally that if $E$ is any semilattice whatever, then in $T_E D \cap (E \times E) = U$.

Since $T_E$ is an inverse semigroup whose semilattice of idempotent is (effectually) $E$, one half of this result is obvious.

Suppose that $(e, f) \in U$. Then $E_e \sim E_f$ and so there exist at least one $\alpha$ in $T_E$ such that $\text{dom}(\alpha) = E_e$ and $\text{ran}(\alpha) = E_f$.

Then $x \in (E_f \cap E_g)\alpha^{-1}$ if and only if $x \in (E_g \cap E_h)\alpha^{-1}$.

Thus $x \in E (E_{ig})\beta$.

Thus $\alpha \beta$ maps the principal ideal $E_i$ onto the principal ideal $E_j$.

Since it is clearly an isomorphism, we have that $\alpha \beta \in T_E$. Thus $T_E$ is a subsemigroup of $J(E)$.

**DEFINITION:**

Let $T$ be a semigroup with identify $I$ and $\theta$ be a homomorphism from $T$ into $H$, the H-class containing the identity of $T$ (what is often called the group of units of $T$).

Let $N = \{0, 1, 2, \ldots\}$.

We can make $N \times T \times N$ into a semigroup by defining:

$$(m, a, n) (p, b, q) = (m - n + t, a^p b^q, q - p + t)$$

where $t = \max(n, p)$ and $\theta^0$ is interpreted as the identity map of $T$.

To check that the given composition is associative, we observe that:

$$(m, a, n) (p, b, q) (r, c, s) = (m - n + w, a^{r - n} b^{s - n} b^q, s - r + q + w)$$

where

$$U = \max(q - p + \max(n, p) r)$$

$$W = \max(n, p - q \max(q - r))$$

The outer coordinates in multiplication (***) combining exactly as in the bicyclic semigroup which associative since it is isomorphism to $T_{cs}$.

Hence by equating the first coordinates or (equivalently third coordinates) we obtain $W = u + p - q$. 

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It is then clear that this result implies the quality of the two middle coordinates and so the composition (***) is indeed associative and shall be denoted by the semigroup obtained in this way by $S = BR(T, \theta)$ which refers to as the BRUCK – Reilly Extension of $T$ determined by $\theta$.

**Proposition 1.3**

If $T$ is a semigroup with identity $1$ and $S = BR(T, \theta)$.

Then,

(i) $S$ is a simple semigroup with identify $(0, 1, 0)$

(ii) Two element $(m, a, n)$ and $(p, b, q)$ of $S$ are D- equivalent in $S$ if and only if $a$ and $b$ are D – equivalent in $T$.

(iii) The element $(m, a, n)$ of $S$ is idempotent if and only if $m = n$ and $a^2 = a$.

(iv) $S$ is an inverse semmigroup if and only if $T$ is an inverse semigroup.

**Proof:**

1. We show that if $(m, a, n)$ and $(p, b, q)$ are arbitrary element of $S$ the $3$ $(r, x, s)$ and $(t, y, u)$ such that $(r, x, s) (m, a, n) (t, y, u) = (p, b, q)$.

Let $(r, u, s) = (p(a^{\theta}^{-1}m + 1))$ and $(t, y, u) = (n + 1, b, q)$ where $(a, \theta)^{-1}$ is the inverse of $a\theta$ in the group $H_i$

Then it is easy to check that the desired equality holds.

That $(0, 1, 0)$ is the identify of $S$ is a matter of routine verification.

2. Let us use superscripts $S$ and $T$ to distriquished between the green equivalent on $S$ and those on $T$. if $(m, a, n) R^T (p, b, q)$ for some $(r, x, s)$ in $S$

Hence $P = m – n + \max (n, r) \geq m$

Equally, we show that $m \geq p$ and so infect $m = p$ it follows that $m – n + \max (n, r) = m$ and

Hence that $n \geq r$.

By equating the middle coordinate, we have

$A(x^{\theta}m = b$, so $R_0 \leq R_0$ in $T$.

Similarly, we show that $R_0 \leq R_0$ and so $aR^Tb$.

Conversely, if $aR^Tb$ then $ax = b$, $bx = a$ for some $x, x^T$ in $T^T$).

Hence,

$(m, a, n) (n, x, q) = (m, b, a)$

$(m, b, q) (q, x, n) = (m, a, n) \in S$ and so $(m, a, n) R^T (m, b, q)$

$\Rightarrow (m, a, n) R^T (p, b, q) \iff m = p$ and $aR^Tb$

A dual argument establishes that $(m, a, n) L^T (p, b, q) \iff n = q$ and a $L^Tb$.

Suppose that $(m, a, n) p b, q) \in S$ are such that

$(m, a, n) D^T (p, b, q)$. Then there exist $(r, c, s)$ in $S$ for which $(m, a, n) R^T (r, c, s) L^T (p, b, q)$.

It then follows that $aR^Tc$ and $CL^Tb$ (and $r = m, s = q$)

Hence $aD^Tb$.

Conversely, if $aD^Tb$, then for some $c$ in $T$ we have $aR^Tc$ and $CL^Tb$.

Therefore for every $m, n, p, q$, in $N$

$(m, a, n) R^T (m, c, q) \iff (m, a, n) L^T (p, b, q)$ and so $(m, a, n) D (p, b, q)$.

$(m, a, n)^2 = (m – n + t, a\theta^{m}b\theta^{m}, n – m + t)$

where $t = \max (m, n)$

Hence $(m, a, n)$ can be idempotent only if $m = n$.

Since $(m, a, m)^2 = (m, a^2, m)$, the element $(m, a, m)$ is idempotent if and only if $a^2 = a$.

(iii) If $T$ is an inverse semigroup, then each element $(m, a, n)$ of $S$ has an inverse $(n, a^T, m)$

Thus $S$ is regular.

To show that it is an inverse seigroup.

Let $(m, e, m) (n, f, n)$ be idempotent in $S$ (with $m \geq n$ say)

Then

$(m, e, m)(n, f, n) = (m, e(f\theta^{m-n})m)\bigg[$

$(n, f, n) (m, e, m) = (m(f\theta)^{M-n} e, m)\bigg[$

Now $f\theta^{m-n}$ is an idempotent in $T$.

(Indeed if $m \neq n$, we must have $f\theta^{m-n} = 1$ the only idempotent in $H_i$)
Hence \( e (f^{m-n}) = (f^{m-n}) e \) and so idempotent commutes in \( S \).

Conversely if \( S \) is an inverse semigroup and if \((p, b, q)\) is the inverse of \((m, a, n)\), then \((m, a, n) (p, b, q) = (m - n + t, a \theta t - n b \theta t - p, q - p + t)\) with \( t = \max(n, p) \) is an idempotent \( R^t \) – equivalent to \((m, a, n)\) and \( L^s \) – equivalent to \((p, b, q)\).

Therefore \( m = m - n + t = q - p + t = q \) and so \( n = p \) (= \( t \)) , \( m, = q \).

The inverse property now gives

\[
(m, a, n) = (m, a, n)(n, b m) \quad (m, a, n) = (m aba, n)
\]

\[
(n, b, m) = (n, b, m) (m, a, n)(n, b, m) = (n, bab, n)
\]

Thus \( aba = a bab = a \) and so is an inverse of \( a \) in \( T \). Thus \( T \) is regular.

If \( e, f \) are idempotent in \( T \), then the commuting of the idempotent \((o, e, o), (o, f, o)\) of \( S \) implies that \( ef = fe \) in \( T \).

2.1 A semigroup \( S \) is called left inverse if every principal right ideal of \( S \) has a unique idempotent generator. Many authors and scholars have laid their hands in solving problems relating to simple and bisimple semigroup. Here in this chapter, we investigate the D- class of regular semigroups and of left inverse semigroups.

Lemma, proposition and Theorems were also considered to support each statement.

A description of a bisimple let inverse semigroup \( S \) with identity element \( e \) as a quotient of the contesian product \( L_{x} x L_{y} \) of \( L \) – class \( L_{x} \) of and the \( R \) – class \( R_{y} \) of \( S \) containing \( e \).

We also describe the maximal inverse semigroup homomorphism of \( S \).

3. \( D \) – CLASSES IN REGULAR SEMIGROUPS

Let \( S \) be a regular semigroup and \( a \in S \). The \( L \) – class of \( S \) containing the element \( a \) is denoted by \( L_{a} \).

Let \( A \) be a subset. Throughout \( a' \) denotes an inverse of \( a \) and \( A' \) denotes the set of all inverse of elements of \( A \)

Lemma 1

Let \( S \) be a regular semigroup. Then \( S \) is Bisimple if and only if for any two idempotent \( e, f \) in \( S \) there exist an element \( a \) of \( S \) and \( a' \) of \( a \) such that \( aa' = f \).

Lemma 2

Let \( S \) be a regular semigroup and \( e \) be an idempotent of \( S \) write \( L = L_{e}, R = R_{e}, H = H_{e}, \) and \( D = D_{e} \), then

i. \( L \subseteq R^{1} \) and \( R \subseteq L^{1} \)
ii. \( LR = D \)
iii. Let \( m, n, b, d \in R \). then \( mb = nd \) if and only if \( \exists u \in H \) such that \( mu = n \) end \( ud = b \).

Proof : Let \( x \in L \), then \( \exists x' \) of \( x \) such that \( x' \in R \). so \( x \in R' \) and Hence \( L \subseteq R' \)

Similarly, Let \( m, n, b, d \in R \) then there exist inverse \( m', n', b' \) and \( d' \) of \( m, n, b, d \) respectively such that \( m'm = n'n = bb' = dd' = e \).

Let \( md = nd \).

Then \( m'md = d' \)

And so \( mu'n = n \). Let \( u = m'n \).

Now \( mu = n \) and \( ud = m'nd = m'mb = b \).

Further, \( eu = ue = u \) and \( ud(b'd') = bb' = e = n'n = (n'm)u \).

Thus \( u \in H \).

Remark The element \( u \) above is unique.

If \( X \) is a subset of a semigroup \( S \), then \( E(X) \) denote the set of all idempotent in \( x \).

Let \( S \) be a regular semigroup for any \( a \in S \).

Lemma 3

Let \( D \) be a \( D \) – class of regular semigroup \( S \)

Let \( E(D) \) be a subsemigroup of \( S \). Then \( D \) is a bisimple subsemigroup of \( S \)

Proof: Let \( a, b, \in D \) and let \( f = a'a \) and \( g = bb' \).
Then \( f \in L_a \) and \( g \in R_b \), so \( fg \in L_a R_b \).

But \( L_a R_b \) is contained in some \( D \)–class \( D' \) of \( S \).

Since \( fg \in D \). By hypothesis we then conclude that \( D' = D \) and \( a b \in L_a R_b \).

Hence, \( D \) is a subsemigroup of \( S \).

**Lemma 4**

Let \( S \) be a regular semigroup and \( e \) be an idempotent of \( S \). Write \( L = L_e \), \( R = R_e \) and \( D = D_e \).

Let \( E(S) \) be a subsemigroup of \( S \). Then the following condition on \( S \) are equivalent

i. \( f e f = f \) for any idempotent \( f \in D \)

ii. \( R \) is a subsemigroup of \( S \)

iii. \( L \) is a subsemigroup of \( S \)

**Proof:**

Assume (1). Let \( a, b \in R \). then there exist inverse \( a' \) of \( a \) and \( b' \) of \( b \) such that \( aa' = bb' = e \)

By (1) we have \( a' a e a' = a'a \) and \( aea' = aa'a' = e \)

That is \( abb'a' = e \), now \( b'a' \) is an inverse of \( ab \) and therefore \( ab \in R \).

Conversely Assume (ii)

Let \( f^2 = f = D \). then exist a \( R \) and an inverse \( a' \) of \( a \) such that \( aa' = e \) and \( a'a = f \).

By Hypothesis \( ef \) and \( fe \) are idempotent.

Therefore \( ea' \) is an inverse of \( ae \).

By (ii) we have \( a e \in R \)

Hence \( aea' \in R \). Now \( fe = a'aea = a' (aea')a = aea = a'a = f \) given \( A \).

Thus (1) and (II) are equivalent

Similarly (1) and (III) are also equivalent

therefore \( 1 \Rightarrow (II) \Rightarrow (III) \) in an arbitrary regular semigroup \( S \)

**Lemma 5**

Let \( S \) be a regular semigroup and \( e \) be an idempotent of \( S \) write \( L = L_e \), \( R = R_e \) and \( D = D_e \).

Let \( e \) be a left or right identity element for \( D \).

Then \( R \) and \( L \) are subsemigroup of \( S \).

**Proof :**

Let \( a, b, \in R \), then there exist inverse \( a' \) of \( a \) and \( b' \) of \( b \) such that \( aa' = bb' = e \). As \( e \) is a left or right identity for \( D \) we get \( peq = pq \) or any \( p, q, \in D \).

Now \( abb'a = aea = aa' = e \) and so \( b'a' \) is an inverse of \( ab \). Hence \( ab, \in R \) and \( R \) is a subsemigroup of \( S \).

Similarly \( L \) is a subsemigroup of \( S \).

**Lemma 6**

Let \( S \) be a regular semigroup and \( e \) be an idempotent of \( S \).

Write \( L = L_e \), \( R = R_e \), \( D = D_e \). The following conditions on \( S \) are equivalent.

(I) \( e \) is a right \( [ \text{ left} \] identity element for \( D \).

(II) \( e \) is an identity element for \( R \) \([ \text{L} \])

(III) \( R \) \([ \text{L} \]) is a right \( [ \text{left} \] cancellative subsemigroup of \( S \).

**Proof:**

Clearly (I) implies (II). Conversely assume (II).

Let \( f^2 = f \in D \). let \( a \in R \cap L_e \).

Then there exist an inverse \( a' \) of \( a \) such that \( a'a = f \).

Now by (II) we get \( fe = a'ae = a'a = f \)

So we get (I). Hence (I) and (II) are equivalent.

Now assume (I). Then by lemma 5.3 \( R \) is a subsemigroup of \( S \).

Let \( ax = bx \) where \( a, b, x \in R \). As \( xx' = e \)

For some inverse \( x' \) of \( x \), by (I) we get \( a = b \). and hence (III)

Assume (III) let \( a \in R \), then \( a e \in R \) now \( aee = ae \) and by right cancellative we get \( ae = a \) so we get (II) and hence (I).

**Lemma 7**
Let $S$ be a regular semigroup and $e$ be an idempotent of $S$. Write $R = R_e$ and $D = D_e$.

(I) \[ Sa \cap R \subseteq Ra \] for any $a \in R$

(II) if $R$ is a subsemigroup of $S$ then $Sa \cap R \subseteq Ra$ for any $a \in R$

(III) Let $e$ be an identity element for $D$ and $a \in R$, then $S_a \cap R = Ra$ if and only if $a \in R$

**Proof:**

(I) Let $a \in R$ and $x \in Sa \cap R$, then $x = ta \in R$ for some $t \in S$. Now there exist inverse $a'$ of $a$ and $(ta)'$ of $t$ such that $aa' = ta (ta)' = e$.

So $e = t (ea) (ta)' \in E e S$.

Again $ta = (te) a$ given $te = e t e \in S$.

Hence $te \in R$. now $x = ta = (te) a \in Re$, proving (I).

(II) If $R$ is a subsemigroup of $S$, then $R_a \subseteq R$ for any $a \in R$.

now from (I) we get (II)

(III) let $e$ be an identity for $D$, then from lemma (5.1 and (2) above we get $S_a \cap R = Ra$ for any $a \in R$.

The converge is obvious.

### 4 D - CLASSES IN LEFT INVERSE SEMIGROUP

Recall that a semigroup $S$ is called a left (right) inverse semigroup if every principal right (left) ideal of $S$ has unique idempotent generator.

A left (right) inverse semigroup is clearly a regular semigroup.

**Lemma 1**

Let $S$ be a regular semigroup. Then the following condition on $S$ are equivalent.

(I) \[ Se \cap Sf = Sef (=Sef) \] for any two idempotent $e, f$ in $S$.

(II) \[ ef = ef \] for any two idempotent $e, f$ in $S$.

(III) If $a'$ and $a''$ are inverse of $a$ in $S$, then $aa' = aa''$

(IV) $S$ is a left inverse semigroup

**COROLLARY 1**

Let $S$ be a left inverse semigroup and $e$ be an idempotent of $S$. Then

(I) \[ aa' = a \] for any inverse $a'$ of $a$ in $Re$.

(II) $E(S)$ is a subsemigroup of $S$.

(III) If $a', b'$ are inverse of $a, b$ in $S$ then $b'a'$ is an inverse of $ab$.

**Theorem 1**

Let $S$ be a regular semigroup. Then $S$ is a left inverse semigroup if and only if $L_e = (Re)'$ for any idempotent $e$ in $S$.

**Proof:**

Let $e$ be any idempotent in $S$ write $L = L_e$ and $R = R_e$

Let $S$ be a left inverse semigroup and $P \in R_1$

Then $p = X'$ is an inverse of some $X \in R$.

Now $xx'$ is an idempotent in $Re$.

Hence $xx' = e$ since $S$ is left inverse.

Consequently $x' \in L$ and $x' = L_e$ and hence $R' \in L$.

Conversely, let $L = R'$ for any idempotent $e$ in $S$.

Let $f$ and $g$ be idempotent of $S$ and let $fs = gs$. then $gf = f, fg = g$ and $f$ is an inverse of $g$.

Now by hypothesis we get $f \in Lg$.

So $fg = f$ an hence $f = g$.

Thus $S$ is a left inverse semigroup.

**Lemma 2**

Let $S$ be a left inverse semigroup and $e$ be an idempotent of $S$. Let $a, c, u, \in, R_e$.

Let $a', c'$ be inverse of $a, c$ respectively, then

(I) if $a'u = c'$, then $a = uc$

(II) If $a' = c'$, then $a = c$.

**Proof:**

(i) let $a'u = c$, Let $u'$ be an inverse of $u$.  

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Then \(aa' = uu' = e\) and \(a' = uc'\).
Therefore \(a'a = a'(uc)\)
This implies that \(a = uc\).
(ii) Let \(a' = c'\), then \(a\) and \(c\) both are inverse of \(a'\)
\[\therefore a'a = a'(uc)\]
Hence \(a = c\).

**Lemma 3**
Let \(S\) be a left inverse semigroup and \(e\) be an idempotent of \(S\).
Let \(e\) be an identity element for \(D_e\).
Let \(c, d \in R = R_e\) then \(Re = Rd\) if and only if for any given inverse \(c'\) of \(c\), there exist an inverse \(d'\) of \(d\) such that \(c'c = d'd\).

**Proof:**
Let \(Re = Rd\) and \(Let c'\) be the given inverse of \(c\).
Now \(e = c c' = (i, j, e) c' = i j\) and 
Similarly, \(e = j i\).
Now \(c' i\) is an inverse of \(d = j c\) and \(d' = c' d = c' c\).

**Theorem 2**
Let \(D\) be a \(D\) – class of the left inverse semigroup \(S\).
Let \(R\) be an \(R\)- class of \(S\) contained in \(D\) then.
The following condition on \(S\) are equivalent.
(I) \(E(D)\) is a subsemigroup of \(S\).
(II) \(D\) is a (Bisimple) subsemigroup of \(S\).
(III) For any \(a, b, \in R\) there exist \(c \in R\) such that \(Sa \cap Sb = Sc\).

**Proof:**
(i) Implies (II) by lemma 5.3.
Now Assume (II), Let \(a, b, \in R\).
Let \(a'\) be an inverse of \(a\) and \(b'\) an inverse of \(b\).
Let \(a'a = f\) and \(b'b = g\). then \(f, g\) and \(f g \in D\)
\[\therefore Sa \cap Sb = Sf \cap Sg = Sfg\] by lemma 5.6
Let \(c \in R \cap Lfg\). Then, \(Sfg = Sc\).
Assume (III):
Let \(f, g \in E (D)\).
Let \(a \in R \cap Lf\) and \(b \in R \cap Lg\).
Then by lemma 5.6, \(Sa \cap Sb = Sfg\).
But there exist \(c \in R\) such that \(Sfg = Sc\)
\[\therefore fg \in D, so fg \in E (D)\].

**Conclusion**
We only focused on the \(D\)-classes of left inverse semigroup whereby we established that a left inverse semigroup is clearly a regular semigroup.
References


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